Mathematics. — "*Quadratic involutions among the rays of space.*" By Prof. JAN DE VRIES.

(Communicated at the meeting of December 28, 1918).

In a communication which is to be found in part Vol. XXII, p. 478 of these *Proceedings* I have dealt with an involution, the pairs of which consist of the transversals to quadruplets of straight lines belonging one to each of four given arbitrary plane pencils of rays. In the sequel I shall consider a few involutions related to the above mentioned.

1. In the first place we assume two plane pencils of lines $(A, \alpha) \equiv (a)$, $(B, \beta) \equiv (b)$ and a quadric regulus $(c)^2$ *i.e.* one set of generators of an hyperboloid Γ^2. An arbitrary line t meets one ray a, one ray b and two rays c. If we conjugate to t the second transversal t' of these four lines, a *quadratic involution among the rays of space* is thereby defined.

If t describes a plane pencil, an involution is thereby determined in $(c)^2$, the pairs of which correspond projectively to the rays of the pencils (a) and (b).

Now consider the more general case where a quadratic involution in $(c)^2$ is brought into a projective correspondence to the pencils (a) and (b) in an arbitrary way. The transversals t, t' of the quadruplets of rays a, b, c, c' will constitute a ruled surface, the order of which we shall determine by an investigation after the number of lines t which rest on the line of intersection of the planes α and β.

On the line $\alpha\beta$ the projective pencils (a), (b) determine two projective point-ranges. Through each of the two united points (coincidences) passes a line t. The remaining rays t which meet $\alpha\beta$, lie in α or in β.

On the intersection of Γ^2 and α the points of transit of the pairs c, c' constitute an involution; the joins of the pairs of this involution form a pencil (C, α), which is projective to the point-range cut out on α by the pencil (b) and therefore also projective to the line-pencil which projects this point-range from C. Since each of the two united-rays (coincidences), rests on four corresponding rays a, b, c, c' there are in α (and in β too) two rays of (t). Hence *the ruled surface* (t) *is of degree six.*

The plane α intersects (t^2) still along an additional curve α^4, which must needs have a double point at A, as an arbitrary ray a is met by two transversals t, t' only. Since the line AB outside A and B meets two lines t [1]) and therefore at A has two points in common with $(t)^6$, it is necessary that A, and B too, is a *double point* of the ruled surface.

The curve α^4 has six tangents passing through A; hence $(t)^6$ contains six united-rays (double rays) of the involution (t, t').

The transversals of the pairs a, b form a quadratic line-complex; for, in an arbitrary plane (a) and (b) determine two projective point-ranges and the joins of corresponding points envelop a conic. This complex has four rays in common with the second regulus (set of generators) $(\gamma)^2$ of Γ^2. Each of these four rays meets two corresponding rays a, b and at the same time the rays c, c' conjugated thereto. Hence the ruled surface $(t)^6$ has four lines in common with the hyperboloid Γ^2.

2 If t is caused to describe the pencil (T, τ) the ruled surface $(t)^6$ breaks up into this pencil (t) and a ruled surface $(t')^5$. *Thus the transformation (t, t') converts a pencil into a ruled surface of degree five.*

Of the two united-points of the projective point-ranges on $\alpha\beta$ one now lies at $\alpha\beta\tau$; through the other passes a ray t'. Thus in α (and in β) there lie again two rays t'. The remainder of the intersection of $(t')^5$ and α is a nodal α^3 with double point at A. Each point of intersection of α^3 and τ is the transit of a ray t' which coincides with its conjugated ray t. Hence *the double rays of the involution (t, t') form a cubic complex.*

A confirmation of this enunciation can be obtained as follows. With Γ^2 $(t')^5$ has four rays γ in common (§ 1) and in addition thereto a twisted curve γ^6. At a point of intersection, C, of γ^6 and τ a ray t is intersected by the corresponding ray t'; hence C lies on a double ray $t \equiv t'$ and the second line of $(c)^2$ resting on this double ray meets τ in a point C', which must lie also on γ^6. Thus the six points of transit of γ^6 lie in pairs on three double rays belonging to (T, τ).

3. A ray t_A through A is intersected by a ray b and by two rays c, c' of $(c)^2$. Each ray t' which meets b, c and c' intersects on α a certain ray a and is therefore conjugated to t_A; hence the ray t_A is *singular.*

1) Lying in the united-planes (coincidences) of the projective pencils of planes which project (a) and (b) from AB.

The tangent plane of the hyperboloid (b, c, c') at A intersects α along a line a which touches (b, c, c'). The transversals of the four rays a, b, c, c' therefore coincide. Hence every ray t_A is also to be regarded as a double ray; thus *the cubic complex of double rays* has *principal points* at A and B and, accordingly, α and β as *principal planes.*

It follows from the above, that the *sheaves of lines A* and *B* and *the planes α* and *β* consist of *singular rays of the involution* (t, t').

Together with A a ray b determines a pencil (t_A) and thereby at the same time a quadratic involution I^2 among the rays of the regulus $(c)^2$. Now, let there be given in the plane λ a pencil (l) with vertex L; then each point of $\beta\lambda$ determines, by means of I^2, an involution I^2 on the conic (Γ^2, λ). Through L therefore passes a ray l joining the points of transit of two rays c, c', which in combination with b determine a transversal t_A. If this ray l is conjugated to the ray l', which meets b, a projectivity is established in (l). Each of the two united-rays is then a ray t' which is conjugated to a ray t_A. It follows from this that the reguli $(t')^2$ which are conjugated to the *singular rays t_A*, constitute a *quadratic line-complex.*

Three other quadratic complexes $\{t'\}^2$ correspond to the sheaf of lines $[t_B]$ and to the plane systems of rays $[t_\alpha]$ and $[t_\beta]$.

The pencil (T, τ) contains two rays of each of these complexes; accordingly A, B, α, and β each carry two rays t' of the ruled surface $(t')^6$ into which (t) is transformed by the involution (t, t'). Thus it appears again that $(t')^6$ has A and B as double points, α and β as double tangent planes.

The ray AB meets two definite rays c, c', but *all* rays a and b. To $t \equiv AB$ therefore are conjugated all the rays of the bilinear congruence which has c and c' as directrices[1]). Similarly $t \equiv \alpha\beta$ is conjugated to ∞^2 rays t'. Thus the involution (t, t') has *two principal rays,* AB and $\alpha\beta$.

4. The lines of the regulus $(\gamma)^2$ too are *principal rays,* for a line γ meets two definite rays a and b, but *all* rays c; each transversal, t', of a and b rests on two rays c and is therefore conjugated to $t \equiv \gamma$.

The involution (t, t') has still other *singular rays.* If the point of intersection, S, of two rays a and c lies in the plane σ passing through two rays b and c', then the pencil (S, σ) consists of rays s

1) The congruence $[t']$, conjugated to AB belongs to the intersection of the line complexes which correspond to the sheaves $[t_A]$ and $[t_B]$.

each conjugated to all the other, hence of singular rays. Now a plane σ is intersected at two points S by the conic α^2, which Γ^2 has in common with α; every plane tangent to Γ^2 therefore contains two pencils (s).

In any arbitrary plane lie two points S, and therefore two rays s; through an arbitrary point pass two planes σ and consequently four rays s. Since a second system of singular rays is obtained by interchanging a and b in the foregoing reasoning *the pencils of singular rays form two congruences* (4,2).

The vertices of the pencils (s) lie on the conics α^2 and β^2, their planes envelop the hyperboloid Γ^2.

5. In order to obtain *another involution* among the rays of space, we consider two reguli $(c)^2$ and $(d)^2$, of the hyperboloids Γ^2 and Δ^2 respectively. Any two rays c, c' determine in combination with any two rays d, d' a pair of transversals (t, t') constituting one pair of the involution which will here be considered.

Now suppose that on Γ^2 an involution (c, c') be given which in some way is projectively related to an involution (d, d') assumed on Δ^2.

The transversals of the pairs d, d' form a linear line-complex, for, in a plane λ the points of transit D, D' of these pairs determine an involution on the transit (conic) of Δ^2, so that the joins of the point-couples D, D' form a pencil. This complex contains two lines l of the second regulus of Γ^2. There are therefore two transversals of pairs d, d' which meet *all* the rays c. In addition to these two an arbitrary ray c meets the two transversals of the pairs in $(c)^2$ and $(d)^2$ which are determined by c. Hence the transversals t, t' of the pairs c, c' and d, d' form a *ruled surface of degree four*, denoted by $(t)^4$.

Evidently $(t)^4$ contains also two rays of the second set of generators, $(\delta)^2$ of Δ^2.

6. Thus to the rays t of a pencil (T, τ) corresponds *a ruled surface* $(t')^3$, which contains two lines γ and two lines δ. This surface meets the intersection ϱ^4 of Γ^2 and Δ^2 at 12 points, eight of which lie on the last mentioned four lines; the remaining four carry each one ray c and one ray d intersecting τ at two points which are collinear with T.

This statement may be corroborated as follows. Through each point of ϱ^4 pass a line c and a line d. Their points of transit, C and D, through τ determine two point-ranges related by a 2,2-correspondence on the curves of transit γ^2 and δ^2 of Γ^2 and Δ^2.

The lines TC and TD are therefore reciprocally conjugated in a correspondence (4, 4). Of the 8 united-rays of this correspondence four pass through the points of intersection of γ^2 and δ^2; the remaining four each meet a pair c, d the point of intersection of which lies on ϱ^4 and therefore carries a ray t' conjugated to a ray t.

In addition to the two lines γ already mentioned the ruled surface $(t')^3$ has a twisted quartic γ^4 in common with Γ^2. This curve intersects τ at four points, which are two and two collinear with T (§ 2). It follows from this that *the double rays of the involution* (t, t') *form a quadratic complex.*

The single directrix of $(t')^3$ lies in τ, the double one passes through T.

7. The rays of the reguli $(\gamma)^2$ and $(\delta)^2$ are evidently (§ 4) *principal rays* of (t, t'). To each of these rays a *bilinear line-congruence* is conjugated having two lines c or two lines d for directrices. As each line c acts as directrix to two congruences (1,1), there emanate two pencils (t') from each of its points. The congruences (1,1) corresponding to the principal rays therefore constitute *two quadratic complexes.*

In a similar way as in § 4 we find a congruence of singular rays. Of the intersection ϱ^4 of the hyperboloids Γ^2 and Δ^2 each point is the vertex of a pencil (S, σ) consisting of rays s which are each conjugated to all the others, hence singular. For, in fact, the plane σ through the lines γ and δ, which are concurrent at S, intersects ϱ^4 still in the additional points C of γ, D of δ and E. Evidently CE belongs to $(c)^2$, DE to $(d)^2$. Each ray of (S,σ) meets two rays c', d' at S and intersects the lines $c \equiv CE$ and $d \equiv DE$; therefore (S,σ) consists of reciprocally conjugated rays s of the involution (t, t').

Since the vertices of the pencils S lie on ϱ^4 and the planes σ envelop a developable of the fourth class, *the pencils of singular rays form a congruence* (4,4).

8. Any three rays c of a cubic regulus $(c)^3$ determine in combination with each ray a of a pencil (A,a) two transversals, which form a pair of an involution of rays in space.

By the rays of a pencil (t) the rays of $(c)^3$ are ordered in an I^3, the sets of which are projectively correlated to the rays a. To begin with we again suppose that this correspondence is established in an arbitrary manner; then the transversals t, t' of the quadruplets of rays constitute a ruled surface which will here be investigated.

On the nodal curve γ^3, along which the ruled cubic Γ^3 is intersected by the plane α, the triplets of rays c determine an I^3. The conics joining two sets of this I^3 with the double point D and another point B of γ^3 have in addition to these points two points B', B'' in common, not lying on γ^3. The sets of the I^3 are therefore cut out on γ^3 by the system of conics with basal points D, B, B', B''. Only the pair of lines $DB, B'B''$ furnishes a set consisting of three collinear points. It appears from this that the plane α contains *one* line of the ruled surface (t), for the line $t \equiv B'B''$ does not rest on the three rays c of a triplet only, but also on the ray a conjugated thereto.

Through A passes similarly one ray of (t). Since a is still intersected by two additional transversals t, t', *the ruled surface (t) is of the fourth degree.*

The remaining curve α^3 which $(t)^4$ has in common with α, sends four tangents through A. Hence $(t)^4$ contains four double rays of the involution (t, t').

If t is caused to describe a pencil (T, τ) then $(t)^4$ breaks up into (t) and a cubic regulus $(t')^3$. Now again α contains one of the rays t'; the points of transit of the remaining lines t' constitute a conic α^2, which passes through A and intersects τ on the double rays which belong to the pencil. Hence *the double rays of the involution (t, t') form a quadratic complex.*

9. Let a_e be the particular ray of (A, α) which is intersected by the single directrix e of $(c)^3$. Every line t' which rests on a_e, is in (t, t') conjugated to e. To the line $t \equiv e$ therefore correspond all the rays of a *special linear complex.*

Similarly the double ray d of $(c)^3$ is conjugated to all the rays of the special linear complex having the ray a_d which rests on d for its axis.

In this involution (t, t') also the rays t_A through A are singular and each conjugated to the rays of a regulus having three lines c for its directrices and containing the lines d and e.

Similarly the rays t_α, lying in the plane α, are singular too and each correlated to the rays of a regulus which contains d and e.

Now consider the system of the hyperboloids (H), which are each determined by three lines c. The specimens which pass through a given point P arrange the lines c into the sets of a cubic involution of the second order. The involutions I^2_3 which thus belong to the points P, P', P'', have *one* set in common; the hyperboloids H therefore form a *complex* (triply infinite system). The hyperboloids corresponding to the rays t_A and therefore passing through A then constitute a *net* (twofold infinite system) all the specimens whereof have the lines d, e and the transversal t_0 through A of d and e in common. Through a point P therefore passes a single infinity of hyperboloids and these still have the transversal through P of d and e in common. Hence the lines t' through P which are conjugated to the rays of the sheaf $[A]$ form a pencil in the plane $(P\,t_0)$.

10. There are still other singular rays. Each plane ε through e contains two lines c. In ε lies a pencil of rays t, which has the point of intersection E of e and a_e for its vertex; these rays are *singular*, since they rest at E on a third line c and are therefore all conjugated to each other.

The sheaf $[E]$ *is therefore composed of* ∞ *pencils of singular rays.*

The plane δ passing through d and a_d contains a line c_0; through each point D of d pass two lines c, hence ∞^1 lines t, which rest at the same time on c_0 and a_d. It follows from this that *the plane of rays* $[\delta]$ *is composed of* ∞^1 *pencils of singular rays.* These have their vertices on the line d.

11. Lastly we consider a ruled surface Γ^4 with a double curve ϱ^3. The linear complex which can be laid through five generators c of Γ^4 contains all the lines c. The four rays c which rest on a line t meet besides the line t', which by the complex is conjugated to t. The involution (t, t') then consists of the pairs of conjugated directrices of a linear complex; its double rays are the rays of this complex.

Another well-known involution (t, t') is originated by the pairs of reciprocal polar lines of a hyperboloid. Its double rays are the two sets of generators of the hyperboloid.