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Mathematics. — *"A Congruence of Conics"*. By Prof. JAN DE VRIES.

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1. We shall suppose, that a trilinear correspondence [1]) exists between the ranges of points (A_1), (A_2), (A_3) lying on the crossing straight lines a_1, a_2, a_3. Through each triplet of corresponding points A_1, A_2, A_3 let a conic λ^2 be passed which intersects the fixed conic β^2 twice. The congruence $[\lambda^2]$, arising in this way, will be examined more closely; it passes into a congruence of circles, if β^2 becomes the imaginary circle at infinity.

2. PAIRS OF LINES. In four different ways λ^2 can degenerate into a pair of straight lines.

1. One of the lines, g, rests on a_1, a_2, a_3, the other, h, lies in the plane β of β^2.

If we keep the point A_1 fixed, A_2 and A_3 describe projective ranges, so that $g_{23} \equiv A_2 A_3$ describes a quadratic scroll. There are therefore two lines g_{23} resting on a_1; the two supporting points A'_1 will be associated to A_1. Each point A'_1 belongs to one point A_1; for the transversal through A'_1 of a_2 and a_3 determines two points A_2,A_3, hence one point A_1. Three times A_1 coincides with A'_1; there are therefore three lines g_{123}, each containing a group A_1, A_2, A_3. Each line h_{123} in β, intersecting g_{123}, forms together with this line a pair of lines belonging to the congruence. To group 1 belong accordingly *three systems*, each consisting of a fixed straight line and a ray of a plane pencil.

2. One of the lines, g_{23}, rests on a_2 and a_3, the other lies in β.

To the intersection A_1^* of a_1 with β a scroll (g_{23}^*) is associated, which intersects β in a conic γ_{23}^2. Each ray of the plane pencil (A_1^*, β) intersects on γ_{23}^2 two lines g_{23}, and forms with each of them a pair of lines. Group 2 contains therefore *three systems*, each consisting of a ray of a plane pencil and a straight line of a quadratic scroll.

3. Let us denote the point of a_3 associated to A_2^*, A_3^*, by A_1^{**}. Each line through A_1^{**} resting on $A_2^* A_3^*$, forms with the latter a

[1]) R. STURM, Die Lehre von den geometrischen Verwandtschaften, I, 320.

pair of lines. Also here we find *three systems*, each consisting of a fixed line and a ray of a plane pencil.

4. The line g rests on a_2, a_3 and β^2; the line h cuts a_1 and β^2. Through the point B of β^2 passes one transversal $g_{23} \equiv A_2 A_3$; the corresponding point A_1 determines the plane of λ^2 and in this way the point B' of β^2; $h_1 \equiv A_1 B'$ forms with g_{23} the pair of lines. We find therefore *three systems* of pairs of lines in group 4.

Let us consider the correspondence (B, B'). Any ray h_1 of the plane pencil $(B'A_1)$ is cut by two rays g_{23} of the scroll corresponding to A_1; the transversal through B' of a_2 and a_3 is associated to a definite point of a_1, and intersects the corresponding ray h_1 in B'. Hence the ruled surface of the pairs of lines g_{23} which we have associated to the rays h_1, intersects the plane $(B'a_1)$ along a cubic passing through B'. But in this plane lies a line g_{23} connecting the points A_2, A_3 in $(B'a_1)$. The ruled surface (g_{23}) is therefore of order four; it intersects β^2 besides in B' in seven points B, which in the correspondence in question are associated to B'. Each of the eight coincidences is the double point D_1 of a pair of lines; the locus of D_1 is for this reason a twisted curve of order eight, δ_1^8.

The lines g_{23} form a ruled surface of order *four* with nodal lines a_2, a_3 and directrix β^2. To each point A_1 are associated four points D_1, while to a point D_1 there corresponds one point A_1. From this follows, that the order of the ruled surface (h_1) with director lines a_1 and δ_1^8, is *twelve*.

3. Order and class. With a view to defining the order of the congruence, we consider the conics λ^2 through a point P in β. To them belong in the first place the three pairs of lines of group 1, each formed by one of the lines g_{123} together with the line through P and the point (g_{123}, β). Further the six pairs of lines of group 2, defined by the three rays PA_k^*. As each of these three rays belongs to two pairs, we come to the conclusion, that the order of $[\lambda^2]$ is *nine*.

A plane through an arbitrary line k intersects a_1 and a_2 in the points A_1, A_2, and a_3 in a point A'_3, which we associate to the point A_3 corresponding to A_1, A_2. Of the scroll (g_{12}) defined by A_3, two lines rest on k; hence two points A'_3 are associated to A_3. As A'_3 coincides three times with A_3, three planes $A_1A_2A_3$ pass through k, which is consequently a chord of three conics λ^2. The class of $[\lambda^2]$ is therefore *three*.

4. Singular chords. According to a well known property of the

trilinear correspondence there are two neutral pairs A_1^n, A_2^n, which form a group with any point A_3. The line $A_1^n A_2^n$ is therefore a *singular chord.*

One of the conics λ^2 consists of this chord and the line in β resting on it and on a_3. From this follows, that the locus of the λ^2 which pass through A_1^n and A_2^n, is a *cubic dimonoid,* containing a_3.

The conical points of the six dimonoids can be indicated by $A_1^n, {}^nA_2, A_3^n, {}^nA_1, A_2^n, {}^nA_3$; in this order the six neutral chords are each time defined by two successive symbols. They form a hexagon, inscribed in a_1, a_2, a_3.

To the *singular chords* belong apparently also the three lines g_{123} and the three lines $A_k^* A_l^*$ in β.

Also the three lines a_k are *singular.* For each plane through a_1 contains the conic determined by the intersections with a_2 and a_3. Let us consider the intersection of the surface $\mathfrak{A}_1$, formed by these conics, with the plane β. To this belongs the conic β^2; the rest consists of straight lines. On a_1 rest two lines g_{23}; their intersections with β determine together with the point A_1^* two straight lines belonging to $\mathfrak{A}_1$.

The line $A_1^* A_2^*$ is cut by a line $A_1 A_3$ of the scroll corresponding to A_2^*; it lies therefore on $\mathfrak{A}_1$, as well as the line $A_1^* A_3^*$. Each of the three lines g_{123} forms a pair of lines with a straight line in β through A_1^*. The intersection of $\mathfrak{A}_1$ with β is therefore of order *nine.*

The locus of the conics λ^2 which intersect a_1 twice, is accordingly a surface $\mathfrak{A}_1^9$ with a sevenfold line a_1. containing the lines a_2, a_3 and the conic β^2.

5. SINGULAR POINTS. All points A_k of the lines a_k are *singular.* A straight line k through a point A_1 is intersected by two lines g_{23}, is therefore a chord of two λ^2 passing through A_1. The planes of the λ^2 through A_1 envelop consequently a quadratic cone; from this follows, that through any point of β^2 two of these λ^2 pass. Hence the locus of the λ^2 through A_1 is a surface $(A_1)^4$ with double curve β^2 and conical point A_1.

Also the points B of β^2 are *singular.* Through two points B, B' pass three λ^2; hence β^2 counts three times in the locus $\mathfrak{B}$ of the λ^2 through B. Moreover β and $\mathfrak{B}$ have in common the three lines through B meeting the lines g_{123}, and the lines joining B and the points A_k^*, which have to be counted twice. We conclude from this, that $\mathfrak{B}$ is a surface of order *fifteen* with threefold curve β^2 and three nodal lines a_k; the point B is twelvefold.

6. Surface of the conics resting on a given line l. Let us consider the intersection of this surface with the plane β. To this β^2 belongs *fifteen* times. Further three rays h_{123} which intersect l and each of which forms with one of the straight lines g_{123} a λ^2. Also the three lines joining the points A_k^* with the point (l,β) and each belonging to two pairs of lines. Then the two rays of the plane pencil (A_k^*, β), each forming a λ^2 with a straight line $A_l A_m$ resting on l; in all six rays. Finally the three lines $A_k^* A_l^*$, each of which belongs to a λ^2 of which the second component is the ray through A_m^{**} intersecting l. The complete intersection is therefore of *order* 48.

The surface in question is accordingly a A^{48} with fourfold lines a_1, a_2, a_3, fifteenfold curve β^2 and three double conics λ^2; these are the conics which have l for a chord and therefore intersect it twice.

Besides the lines mentioned lying in the plane β, A contains the three lines g_{123}, two lines g_{12}, two lines g_{23} and two lines g_{13}, all crossing the line l; further two lines g_{12}, two lines g_{23} and two lines g_{13}, intersecting l; then three lines resting on l, successively directed to the three points A_k^{**}; finally 3×16 pairs of lines, the components of which each contain one point of β^2; in 3×4 of them the line g_{kl} and in 3×12 the line h rests on l.
