Mathematics. — *"Series of analytical functions"*. By Prof. J. WOLFF. (Communicated by Prof. L. E. J. BROUWER).

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OSGOOD's theorem: *"If the series $f_1 + f_2 + \ldots$, all the terms whereof are analytical functions within a region T of the complex plane, is convergent at the points of a set (β) everywhere dense in T, and if besides, $|f_1 + f_2 + \ldots + f_n| < G$ at every point of T (G being a constant), the series converges everywhere in T and there represents an analytical function"* [1]) has been again demonstrated by ARZELA [2]).

VITALI [3]) and PORTER [4]) have extended the theorem by proving that it is sufficient if only the set (β) of the points where the series converges has an internal point of T for a limiting point.

Of the thus extended theorem a simple demonstration shall be given in the sequel.

1. To this end we suppose that the f_i are analytical in T, that in T everywhere $|S_n| < G$ for every n, G being a constant, and that the series is convergent at the points $\beta_1, \beta_2, \ldots$, having the internal point z_0 of T for limiting point, and we shall prove that the series converges uniformly in every region lying with its boundary within T.

Now describe a circle (R) with centre z_0 and radius R, lying in T. Let β_i be a point of (β) inside a circle $(\frac{1}{2}R)$ with centre z_0, and let $f(\beta_i)$ denote the sum of the series at (β_i), $S_n(\beta_i)$ the sum of n terms, then

$$S_n(\beta_i) = \frac{1}{2\pi i}\int\limits_{(R)} \frac{S_n(t)dt}{t-\beta_i} \quad \text{and} \quad f(\beta_i) = \frac{1}{2\pi i} \lim_{n=\infty} \int\limits_{(R)} \frac{S_n(t)dt}{t-\beta_i}$$

If β_k denotes another point of (β) inside $(\frac{1}{2}R)$, then

$$|S_n(\beta_i) - S_n(\beta_k)| < \frac{4G}{R}\,|\beta_i - \beta_k|,$$

1) W. F. OSGOOD. *Functions defined by infinite series.* Annals of Mathematics, Series 2, Vol. 3, Oct. 1901, p. 26.

2) V. C. ARZELA. Annals of Mathematics, Series 2, Vol. 5, 1904, p. 51.

3) G. VITALI. *Sopra le serie di funzioni analitiche.* Annali di Matematica, Serie 3a, tomo 10, 1904, p. 65.

4) M. B. PORTER. Annals of Math. Series 2, Vol. 6, 1904—5, p. 45 and p. 190.

for every n, so that

$$|f(\beta_i)-f(\beta_k)| \leq \varepsilon$$

as soon as

$$|\beta_i-\beta_k| < \frac{R\varepsilon}{4G},$$

whence it follows that $f(\beta_i)$ tends to a limit $f(z_0)$ as β_i tends to z_0.

Let η denote an arbitrary positive number. For all sufficiently small values of $|\beta_i-z_0|$ we then have:

$$|f(\beta_i)-f(z_0)| < \frac{\eta}{3}.$$

Also $|S_n(\beta_i)-S_n(z_0)| < \frac{2G}{R}|\beta_i-z_0|$ for every n, so that for all sufficiently small values of $|\beta_i-z_0|$ the relation

$$|S_n(\beta_i)-S_n(z_0)| < \frac{\eta}{3}$$

holds, where n is arbitrary.

Now choose a β_i satisfying these two conditions, then from a certain n onwards we have:

$$|S_n(\beta_i)-f(\beta_i)| < \frac{\eta}{3}.$$

It follows thence that from this value of n onwards we have continually:

$$|S_n(z_0)-f(z_0)| < \eta,$$

from which we conclude to the convergence of the series at z_0. The sum there is $f(z_0)$.

At the same time it has become evident that

$$\lim_{\beta_i=z_0} f(\beta_i) = f(z_0)$$

2. Provisorily we consider n constant. Then for $|z-z_0| < \frac{1}{2}R$ we have:

$$S'_n(z_0) = \frac{S_n(z)-S_n(z_0)}{z-z_0} + \psi_n(z),$$

where $\lim_{z=z_0} \psi_n(z) = 0$. The function $\frac{S_n(z)-S_n(z_0)}{z-z_0}$ is analytical inside $(\frac{1}{2}R)$, its absolute value being less than $\frac{2G}{R}$. If n now is made to increase infinitely, this function tends to a limiting value at the points β_i and therefore, according to § 1, also at z_0. At z_0 it has for every n the value $S_n'(z_0)$. It follows thence that $S_n'(z_0)$ tends to a limit $f^{(1)}(z_0)$.

Similarly

$$S''_n(z_0) = 2!\,\frac{S_n(z)-S_n(z_0) - \frac{z-z_0}{1}S'_n(z_0)}{(z-z_0)^2} + \Phi_n(z),$$

where $\lim\limits_{z=z_0} \Phi_n(z) = 0$. The first term of the right-hand side is analytical inside $(\frac{1}{2}R)$, its absolute value being less than $\frac{4G}{R^2}$, since $\left|\frac{S_n^{(k)}(z_0)}{k!}\right|$, the absolute value of the coefficient of $(z-z_0)^k$ in the development $S_n(z) = S_n(z_0) + \sum\limits_1^\infty a_k(z-z_0)^k$, is less than $\frac{G}{R^k}$. As n increases infinitely this function tends to a limit at the points β_i and therefore at z_0 too, from which it follows that a limit

$$\lim_{n=\infty} S''_n(z_0) = f^{(2)}(z_0)$$

exists.

Thus pursuing we find that for every k a limit

$$\lim_{n=\infty} S^{(k)}(z_0) = f^{(k)}(z_0).$$

exists.

3. For an arbitrary z inside $(\frac{1}{2}R)$ we have:

$$S_n(z) = S_n(z_0) + (z-z_0)\,S'_n(z_0) + \dots + \frac{(z-z_0)^k}{k!}S_n^{(k)}(z_0) + \dots \quad (1)$$

For a fixed value of z the terms of this series, if n increases infinitely, tend to those of the series

$$f(z) = f(z_0) + (z-z_0)f^{(1)}(z_0) + \dots + \frac{(z-z_0)^k}{k!}f^{(k)}(z_0) + \dots\dots \quad (2)$$

which represents a function, analytical in $(\frac{1}{2}R)$, since $\left|\frac{f^{(k)}(z)}{k!}\right| \leq \frac{G}{R^k}$.

From $\left|\frac{S_n^{(k)}(z_0)}{k!}\right| < \frac{G}{R^k}$ it follows that the series (1) converges *uniformly*, if the terms are considered as functions of the two independent variables z and n, at the points of the set $|z-z_0| \leq \frac{1}{2}R, n=1, 2, \dots$

It follows from this that S_n converges uniformly to f inside $(\frac{1}{2}R)$.

4. Instead of $\frac{1}{2}R$ as well λR could have been chosen, where λ is an arbitrary positive number < 1. Hence S_n converges uniformly to an analytical function in the interior of every circle (R) lying wholly inside T. Let z be an arbitrary point in T, then z can be

enclosed within the last of a chain of circles, all lying within T, the first being (R) with centre z_0 and every circle having its centre within the preceding one. Since the points where S_n converges condense towards the second centre, S_n converges uniformly throughout the second circle; similarly within all the following, hence in any circle with centre z which lies in T. Every region lying with its boundary within T can be covered by a *finite* number of such circles which involves the uniform convergence of S_n to an analytical function throughout τ.

5. Lastly we shall give a simple proof of OSGOOD's original theorem.

According to § 1, if S_n is convergent at the points of the set (β) which is everywhere dense in T, it converges everywhere throughout T and the limiting function f is continuous in T, whilst $|f| \leq G$. Now draw a circle (R) with centre z_0, lying altogether in T. If $|z-z_0|$ is again $< \frac{1}{2} R$, then

$$f(z) = \frac{1}{2\pi i} \lim_{n=\infty} \int_{(R)} \frac{S_n(t)dt}{t-z} = \frac{R}{2\pi} \lim_{n=\infty} \int_0^{2\pi} \frac{S_n(t)\, e^{\theta i}\, d\theta}{t-z}, \quad t = z_0 + Re^{\theta i}.$$

We now make use of the self-evident extension to complex functions of the following theorem of OSGOOD [1]):

If a function $\varphi_n(\theta)$, continuous in the interval $a \leq \theta \leq b$ for every n, converges to a function $\varphi(\theta)$ which is continuous throughout this interval, and if, besides, $|\varphi_n(\theta)| < G$ throughout the interval and for every n, G being a constant, then

$$\lim_{n=\infty} \int_a^b \varphi_n(\theta)\, d\theta = \int_a^b \varphi(\theta)\, d\theta.$$

If we put $\varphi_n(\theta) = \dfrac{S_n(t)\, e^{\theta i}}{t-z}$, then φ_n is continous in $0 \leq \theta \leq 2\pi$, $|\varphi_n| < \dfrac{2G}{R}$, and $\varphi(\theta) = \dfrac{f(t)\, e^{\theta i}}{t-z}$ is continous in $0 \leq \theta \leq 2\pi$. Hence

$$f(z) = \frac{1}{2\pi i} \int_{(R)} \frac{f(t)\, dt}{t-z}.$$

Since f is continuous on (R), it follows from this that f is analytical inside $(\frac{1}{2} R)$.

The same lemma can be used to prove in a simple way that S_n converges uniformly to f inside $(\frac{1}{2} R)$.

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1) W. F. OSGOOD. *On the non uniform convergence.* Am. Journal of Math. 1897. For an extension *vide* H. LEBESGUE, *Leçons sur l'Intégration,* p. 114.

For $|z-z_0| \leq \frac{1}{2}R$ *viz.* we have

$$|f(z)-S_n(z)| \leq \frac{1}{\pi}\int_0^{2\pi}|f(t)-S_n(t)|\,d\theta, \quad t = z_0 + Re^{\theta i}.$$

Here $\varphi_n = |f(t) - S_n(t)|$ is continuous in $0 \leq \theta \leq 2\pi$ and $2G$; φ is zero. Hence

$$\lim_{n=\infty}\int_0^{2\pi}|f(t)-S_n(t)|\,d\theta = 0.$$

For all sufficiently large values of n therefore $|f-S_n|$ is less than a given number everywhere inside $(\frac{1}{2}R)$, that is to say, S_n converges uniformly to f inside $(\frac{1}{2}R)$.

By virtue of the same lemma we have

$$\lim_{n=\infty} S_n^{(k)}(z) = \frac{k!}{2\pi i}\lim_{n=\infty}\int_{(R)}\frac{S_n(t)dt}{(t-z)^{k+1}} = \frac{k!}{2\pi i}\int_{(R)}\frac{f(t)dt}{(t-z)^{k+1}} = f^{(k)}(z),$$

whence it follows that the series may be differentiated termwise infinitely often.

Let C be a regular curve in T then, again by virtue of the above-mentioned lemma, we have

$$\int_C f(z)dz = \lim_{n=\infty}\int_C S_n(z)dz.$$

Hence, if C is closed and its points, as well as the points enclosed, are all internal points of T, then

$$\int_C f(z)dz = 0.$$

From this also it may be concluded, according to a theorem enunciated by MORERA [1]), that f is analytical throughout T.

[1]) Reale Istituto Lombardo di Sc e lettere, Rendic., 2nd series, Vol. 19, 1886.