Mathematics. — *"Note on linear homogeneous sets of points".* By Dr. B. P. HAALMEIJER. (Communicated by Prof. L. E. J. BROUWER).

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We shall call a linear set of points π *homogeneous* in the interval AB, if its subset, interior to an arbitrary sub-interval, allows of a uniformly continuous one-one representation on the subset of π interior to AB [1]).

If the set π is everywhere dense in the interval AB [2]), each of these representations determines a continuous one-one correspondence between the entire linesegments. As will be shown, we may in this case, assume the correspondences, postulated for a homogeneous set of points, to leave relations of order unaltered.

Let CD be a sub-interval of AB (possibly identical to AB) and E a point between C and D. We consider the following possibilities:

1. For every system of points C, D, and E the representation of the interval CD on CE leaves relations of order unaltered.
2. This is not the case.

First case. Suppose a representation of AB on FH has to be effected (order from left to right A, F, H, B). According to the assumption both AB and FH can be represented on AH with unaltered relations of order, hence AB on FH in the same way.

Second case. The assumption postulates the existence of an interval CD which can be represented on its sub-interval CE with inverted relations of order. Considering this representation is a continuous one-one correspondence between entire linesegments, it follows from the DEDEKIND axiom that a point P exists (not necessarily belonging to the set π), which corresponds to itself. This however establishes the possibility of representing the part of π interior to CD on itself with inversion of order-relations. It follows that the part of π interior to an arbitrary sub-interval of AB, possesses this same

1) An analogous definition has been given by HAUSDORFF for ordered sets, Grundz. der Mengenlehre p. 173. For linear sets of points BROUWER has introduced the following more extensive definition: a linear set of points π is homogeneous in the interval AB if for each couple PQ of its points interior to AB, there exists a continuous one-one transformation of the interval AB in itself, such that π passes into itself and the point P into the point Q. These *Proceedings* XX, p. 1194.

2) Which obviously is the case if π has any points inside AB.

property, hence all correspondences, postulated for the homogeneous set π, can be effected in such a way as to leave relations of order unaltered.

We now formulate the following theorem: *The linear continuum cannot be composed of two homogeneous sets of points, possessing the same geometric type.*

Our demonstration is to be an indirect one. Let the open line-segment AB consist of two sets of points π and π' of the kind mentioned. These sets π and π' possess the same geometric type, that is there exists uniformly continuous one-one correspondence between them. Evidently π and π' are both everywhere dense on AB.

To begin with, we assume that this correspondence inverts relations of order. Then π can be divided into two subsets π_1 and π_2 such that every point of π_1 is situated on the left, and every point of π_2 on the right of the corresponding point of π'. Besides, every point of π_1 lies on the left of every point of π_2. Hence, as $\pi_1 + \pi_2$ is everywhere dense, the DEDEKIND axiom postulates the existence of a separating point R. This point R however can belong to neither π_1 nor π_2. For instance let us assume R to be a point of π_1, then it is situated on the left of the corresponding point of π' and the continuity of the correspondence makes that this is also the case for all points of π inside a certain finite neighbourhood of R, which means a contradiction. Hence R belongs to π', but this also leads to a contradiction as the fact that R is situated either on the left or on the right of its corresponding point cannot be made to agree with the circumstance that all points of π' on the left (right) of R are also situated on the left (right) of their corresponding points.

We now come to the second possibility, namely that the correspondence between π and π' leaves relations of order unaltered. We distinguish two cases:

1. The set π contains both points situated on the left, and points situated on the right of the corresponding points or π'.

2. All points of π lie on the same side of the corresponding points.

First case. Let the point P_1 of π be situated on the left of its corresponding point P_1' and P_2 on the right of P_2'. The subset of π between P_1 and P_2, including the endpoints shall be called π_1. Let ${}_1\pi_1$ be the subset of π_1 consisting of those points, which, together with all points of π_1 situated more to the left, precede their corresponding points[1]), and let R be the last limiting point of ${}_1\pi_1$ on the right hand side. Then the assumption that R precedes its corresponding point, as well as the assumption according to which R follows on its corresponding point, leads immediately to a contradiction (we here consider the transformation of the *entire* segment AB in itself, which is determined by the correspondence between π and π'). Hence the point R must correspond to itself, but this is out of the question, both if R belongs to π or to π'.

Second case. All points of π lie on the left of the corresponding points. Let the points P_1' and P_2' of π' correspond to P_1 and P_2 of π respectively and let the order from left to right be P_1, P_1', P_2, P_2'. Of course, such a system of points can always be found.

We choose a point C of π' on the left of P_1 and we subject π' to a uniformly continuous one-one transformation in itself, such that P_1' passes into C and P_2' remains in its place. A transformation of this kind can certainly be found as π' is homogeneous. Let π'' be the transformed set, then a uniformly continuous one-one correspondence exists between π'' and π, such that π'' contains both points preceding and points coming after the corresponding points, and the reasoning used for the *first case* can now be applied.

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[1]) "Precede" here stands for "are situated on the left of".