Mathematics. — *"On n-uple orthogonal systems of n—1-dimensional manifolds in a general manifold of n dimensions."* By Prof. J. A. SCHOUTEN and D. J. STRUIK. (Communicated by Prof. J. CARDINAAL).

(Communicated at the meeting of October 25, 1919).

II.

7. DUPIN's *theorem and an inversion.* From theorem I we conclude that DUPIN's theorem also holds for a general manifold:

The V_{n-1} of an n-uple orthogonal system intersect along the lines of curvature.

This theorem may be inverted in the following way:

When $n-1$ mutually orthogonal V_{n-1}-systems, determined by the congruences $\mathbf{i}_1, \ldots, \mathbf{i}_{n-1}$ perpendicular to them, intersect along a congruence $\mathbf{i}_n$, and when we can choose the arrangement of the first congruences in such a way that the congruence $\mathbf{i}_n$ in each $V_{n-k+1} \perp \mathbf{i}_1, \ldots, \mathbf{i}_{k-1}$ is a congruence of lines of curvature for the V_{n-k} being the intersection of this V_{n-k+1} with the $V_{n-1} \perp \mathbf{i}_k$, $k = 1, \ldots, n-1$, then $\mathbf{i}_n$ is perpendicular to a V_{n-1}-system, orthogonal to the $n-1$ given systems, and $\mathbf{i}_1, \ldots, \mathbf{i}_n$ are the congruences of the lines of curvature for each of the n systems.

Proof. When the fundamental tensor ${}^2\mathbf{g}$ of the V_n is written:

$$ {}^2\mathbf{g} = \mathbf{a}\,\mathbf{a} = \mathbf{b}\,\mathbf{b} = \ldots, \qquad (72)$$

then the ideal factor $\mathbf{a}$ can be decomposed as follows:

$$\mathbf{a} = \mathbf{a}' + \mathbf{a}'', \qquad (73)$$

in which $\mathbf{a}'$ contains but $\mathbf{i}_k, \ldots, \mathbf{i}_n$, $\mathbf{a}''$ but $\mathbf{i}_1, \ldots, \mathbf{i}_{k-1}$.

${}^2\mathbf{g}' = \mathbf{a}'\,\mathbf{a}' = \mathbf{b}'\mathbf{b}' = \ldots.$ is the fundamental tensor of the $V_{n-k+1} \perp \mathbf{i}_1, \ldots, \mathbf{i}_{k-1}$ and the geodesic differentiation of a vector $\mathbf{v}$, which is wholly situated in this V_{n-k+1}, is determined by the equation:

$$\nabla' \mathbf{v} = {}^2\mathbf{g}' \overset{1}{\cdot} \nabla (\mathbf{a}' . \mathbf{v})\, \mathbf{a}' \qquad (74)$$

Hence for $\mathbf{i}_k$ we have:

$$\begin{aligned} \mathbf{i}_n \overset{1}{\cdot} \nabla \mathbf{i}_k &= \mathbf{i}_n \overset{1}{\cdot} \nabla (\mathbf{i}_k . \mathbf{a})\, \mathbf{a} = \mathbf{i}_n \overset{1}{\cdot} \nabla (\mathbf{i}_k . \mathbf{a}')\, \mathbf{a}' + \mathbf{i}_n \overset{1}{\cdot} \nabla (\mathbf{i}_k . \mathbf{a}')\, \mathbf{a}'' = \\ &= \mathbf{i}_n \overset{1}{\cdot} \nabla' \mathbf{i}_k + \mathbf{i}_n \overset{1}{\cdot} \nabla (\mathbf{i}_k . \mathbf{a}')\, \mathbf{a}''. \end{aligned} \qquad (75)$$

According to the supposition $\mathbf{i}_n$ is a congruence of lines of curvature for the V_{n-k} being $\perp \mathbf{i}_k$ in the considered V_{n-k+1}, so that according to (38):

$$\varkappa\, \mathbf{i}_n \overset{1}{.} \nabla' \mathbf{i}_k = \varrho_k \mathbf{i}_n\,, \quad \ldots \ldots \quad (76)$$

in which ϱ_k is a still unknown coefficient. Hence we conclude from (76):

$$\varkappa\, \mathbf{i}_n \overset{1}{.} \nabla \mathbf{i}_k = \varrho_k \mathbf{i}_n + \sum_j^{1,..,k-1} \mu_{kj} \mathbf{i}_j\,, \quad \ldots \ldots \quad (77)$$

in which μ_{kj} are still unknown coefficients. So it is supposed that it must be possible to arrange $\mathbf{i}_1, \ldots, \mathbf{i}_{n-1}$ in such a way that the equation (77) is satisfied in the same time for all values $k=1,\ldots,n-1$.

Since

$$\mathbf{i}_k \cdot \mathbf{i}_l = 0, \qquad k, l = 1, \ldots, n, \qquad k \neq l \quad \ldots \quad (78)$$

we find by application of $\mathbf{i}_n \cdot \nabla$:

$$\mathbf{i}_l\, \mathbf{i}_n \overset{2}{.} \nabla \mathbf{i}_k = -\, \mathbf{i}_k\, \mathbf{i}_n \overset{2}{.} \nabla \mathbf{i}_l \quad \ldots \ldots \quad (79)$$

For $k < l$ we have thus from (77), (78), and (79):

$$\mathbf{i}_k\, \mathbf{i}_n \overset{1}{.} \nabla \mathbf{i}_l = 0, \qquad l = 1, \ldots, n-1\,. \quad \ldots \quad (80)$$

hence:

$$\mu_{kj} = 0 \qquad \left.\begin{array}{l} k = 1, \ldots, n-1 \\ j = 1, \ldots, n-2 \end{array}\right\} \quad \ldots \ldots \quad (81)$$

By this the equations (77) pass into:

$$\varkappa\, \mathbf{i}_n \overset{1}{.} \nabla \mathbf{i}_k = \varrho_k \mathbf{i}_n\,, \qquad k = 1, \ldots, n-1 \quad \ldots \quad (82)$$

which can geometrically be interpreted in such a way that $\mathbf{i}_n$ is a congruence of lines of curvature in each of the $n-1$ given V_{n-1}-systems.

By application of $\mathbf{i}_k \cdot \nabla$ we conclude from (78):

$$\mathbf{i}_l\, \mathbf{i}_k \overset{2}{.} \nabla \mathbf{i}_n = -\, \mathbf{i}_n\, \mathbf{i}_k \overset{2}{.} \nabla \mathbf{i}_l, \quad k, l = 1, \ldots, n-1. \quad \ldots \quad (83)$$

Now $\mathbf{i}_l$ is V_{n-1}-normal, hence $\nabla \mathbf{i}_l$ is symmetrical in k and n, so that we have from (80) and (83):

$$\mathbf{i}_l\, \mathbf{i}_k \overset{2}{.} \nabla \mathbf{i}_n = 0, \qquad k, l = 1, \ldots, n-1\,, \quad \ldots \quad (84)$$

hence $\mathbf{i}_n$ is V_{n-1}-normal and $\mathbf{i}_1, \ldots, \mathbf{i}_{n-1}$ are the congruences of the lines of curvature of the $V_{n-1} \perp \mathbf{i}_n$.

Since $\mathbf{i}_1, \ldots, \mathbf{i}_{n-1}$ are V_{n-1}-normal and mutually perpendicular, we have also from (67):

$$\mathbf{i}_j\, \mathbf{i}_k \overset{2}{.} \nabla \mathbf{i}_l = 0, \qquad j, k, l = 1, \ldots, n-1\,. \quad \ldots \quad (85)$$

so that $\mathbf{i}_1, \ldots, \mathbf{i}_n$ are the congruences of the lines of curvature for each of the n systems $\perp \mathbf{i}_1, \ldots, \mathbf{i}_n$.

For a V_3' the proved theorem can be expressed in this way:

When two mutually orthogonal systems of surfaces intersect along a congruence of curves, which are the lines of curvature of one of *the two systems of surfaces, then there exists a system of surfaces orthogonal to the two given systems and the three systems intersect along their lines of curvature.*

For the R_3 this theorem has been first deduced by DARBOUX [1]).

8. LILIENTHAL'S *conditions.* We will now connect different shapes, in which the conditions occur in literature, for the case that $\mathbf{i}_n$ is V_{n-1}-normal, and inquire how far they remain valid, when more general manifolds are admitted.

In the same way as ${}^2\mathbf{h}$ the tensor ${}^2\mathbf{p}$ gets a simple significance when $\mathbf{i}_n$ is V_{n-1}-normal. Since on account of (19) and (42):

$$\nabla \smile \mathbf{i}_n = -\frac{\varkappa}{2} \sum_{\lambda\mu} \{(\mathbf{i}_n \cdot \nabla) g^{\lambda\mu}\} \mathbf{e}'_\lambda \mathbf{e}'_\mu + \mathbf{i}_n \smile \mathbf{u}_n, \quad . \quad . \quad . \quad (86)$$

the contravariant characteristic number of $\varkappa(\mathbf{i}_n \cdot \nabla) \nabla \smile \mathbf{i}_n$ is:

$$\left.\begin{aligned}
\varkappa\, \mathbf{e}_\beta \mathbf{e}_\alpha \overset{2}{\cdot} (\mathbf{i}_n \cdot \nabla)(\nabla \smile \mathbf{i}_n) &= -\tfrac{1}{2} \mathbf{e}_\beta \mathbf{e}_\alpha \overset{2}{\cdot} \sum_{\lambda\mu} [\mathbf{e}'_\lambda \mathbf{e}'_\mu (\mathbf{i}_n \cdot \nabla)^2 g^{\lambda\mu} + \\
&\quad + \{(\mathbf{i}_n \cdot \nabla) \mathbf{e}'_\lambda \mathbf{e}'_\mu\} (\mathbf{i}_n \cdot \nabla) g^{\lambda\mu}] + \mathbf{e}_\beta \mathbf{e}_\alpha \overset{2}{\cdot} \mathbf{u}_n \mathbf{u}_n = \\
&= -\tfrac{1}{2} (\mathbf{i}_n \cdot \nabla)^2 g^{\alpha\beta} + \mathbf{e}_\beta \mathbf{e}_\alpha \overset{2}{\cdot} \sum_{\lambda\mu} \{(\nabla \smile \mathbf{i}_n) \overset{2}{\cdot} \mathbf{e}_\lambda \mathbf{e}_\mu (\mathbf{i}_n \cdot \nabla) \mathbf{e}'_\lambda \mathbf{e}'_\mu\} + \mathbf{e}_\beta \mathbf{e}_\alpha \overset{2}{\cdot} \mathbf{u}_n \mathbf{u}_n = \\
&= -\tfrac{1}{2} (\mathbf{i}_n \cdot \nabla)^2 g^{\alpha\beta} - \sum_{\lambda\mu} (\nabla \smile \mathbf{i}_n) \overset{2}{\cdot} \mathbf{e}_\lambda \mathbf{e}_\mu \{\mathbf{e}'_\lambda \mathbf{e}'_\mu \overset{2}{\cdot} (\mathbf{i}_n \cdot \nabla) \mathbf{e}_\beta \mathbf{e}_\alpha\} + \mathbf{e}_\beta \mathbf{e}_\alpha \overset{2}{\cdot} \mathbf{u}_n \mathbf{u}_n = \\
&= -\tfrac{1}{2} (\mathbf{i}_n \cdot \nabla)^2 g^{\alpha\beta} + \sum_{\lambda\mu} (\nabla \smile \mathbf{i}_n) \overset{2}{\cdot} \mathbf{e}_\lambda \mathbf{e}_\mu \{\mathbf{e}'_\lambda \mathbf{e}'_\mu \overset{2}{\cdot} ((\nabla \mathbf{i}_n) \overset{1}{\cdot} \mathbf{e}_\beta \mathbf{e}_\alpha + \mathbf{e}_\beta (\nabla \mathbf{i}_n) \overset{1}{\cdot} \mathbf{e}_\alpha)\} + \\
&\quad + \mathbf{e}_\beta \mathbf{e}_\alpha \overset{2}{\cdot} \mathbf{u}_n \mathbf{u}_n = \\
&= -\tfrac{1}{2} (\mathbf{i}_n \cdot \nabla)^2 g^{\alpha\beta} + \sum_{\mu} \mathbf{e}_\alpha \overset{1}{\cdot} (\nabla \smile \mathbf{i}_n) \overset{1}{\cdot} \mathbf{e}_\mu \mathbf{e}'_\mu \overset{1}{\cdot} (\nabla \mathbf{i}_n) \overset{1}{\cdot} \mathbf{e}_\beta + \\
&\quad + \sum_{\lambda} \mathbf{e}_\beta \overset{1}{\cdot} (\nabla \smile \mathbf{i}_n) \overset{1}{\cdot} \mathbf{e}_\lambda \mathbf{e}'_\lambda \overset{1}{\cdot} (\nabla \mathbf{i}_n) \overset{1}{\cdot} \mathbf{e}_\alpha + \\
&\quad + \mathbf{e}_\alpha \overset{1}{\cdot} (\nabla \smile \mathbf{i}_n) \overset{1}{\cdot} \mathbf{i}_n \mathbf{i}_n \overset{1}{\cdot} (\nabla \mathbf{i}_n) \overset{1}{\cdot} \mathbf{e}_\beta + \mathbf{e}_\beta \overset{1}{\cdot} (\nabla \smile \mathbf{i}_n) \overset{1}{\cdot} \mathbf{i}_n \mathbf{i}_n \overset{1}{\cdot} (\nabla \mathbf{i}_n) \overset{1}{\cdot} \mathbf{e}_\alpha = \\
&= -\tfrac{1}{2} (\mathbf{i}_n \cdot \nabla)^2 g^{\alpha\beta} + 2\, \mathbf{e}_\alpha \mathbf{e}_\beta \overset{2}{\cdot} T (\nabla \smile \mathbf{i}_n) \overset{1}{\cdot} \nabla \mathbf{i}_n,
\end{aligned}\right\} \quad . (87)$$

from which in connection with (59) we conclude:

$${}^2\mathbf{p} = -\frac{\varkappa}{2} \mathbf{e}'_\alpha \mathbf{e}'_\beta (\mathbf{i}_n \cdot \nabla)^2 g^{\alpha\beta} = \mathbf{e}'_\alpha \mathbf{e}'_\beta (\mathbf{i}_n \cdot \nabla) h^{\alpha\beta} \quad . \quad . \quad (88)$$

Hence the condition that ${}^2\mathbf{h}$ and ${}^2\mathbf{p}$ have the same principal directions, for the case $n = 3$, can be written in coordinates:

$$\boxed{\begin{vmatrix} g^{aa} & g^{ab} & g^{bb} \\ (\mathbf{i}_n \cdot \nabla) g^{aa} & (\mathbf{i}_n \cdot \nabla) g^{ab} & (\mathbf{i}_n \cdot \nabla) g^{bb} \\ (\mathbf{i}_n \cdot \nabla)^2 g^{aa} & (\mathbf{i}_n \cdot \nabla)^2 g^{ab} & (\mathbf{i}_n \cdot \nabla)^2 g^{bb} \end{vmatrix} = 0,} \qquad (C_2)$$

1) G. DARBOUX, Sur les surfaces orthogonales. Annales sc. de l'Ecole Normale 3 (66) 97—141, p. 110.

and this is exactly the equation given for the first time for R_3 by LILIENTHAL [1]), and to which lately, also for R_3, WIERINGA [2]) has again drawn the attention. So this condition is a special case from RICCI's first. It remains also valid for an arbitrary linear element, and also for $n > 3$, then however it is no longer the only condition.

9. RICCI's *conditions.* Be $\mathbf{i}_n$ again V_{n-1}-normal. Then we can choose an original variable y^n and vectors $\mathbf{s}_n$ and $\mathbf{s}'_n$, so that:

$$\mathbf{i}_n = \sigma_n \mathbf{s}_n = \frac{1}{\sigma_n} \mathbf{s}_n' \quad . \; . \; . \; . \; . \; . \; . \; . \quad (89)$$

By means of this equation we can eliminate $\mathbf{i}_n$ from (C) and (D) and substitute $\mathbf{s}_n$ for it.

Since:

$$(\mathbf{i}_n \,.\, \nabla)(\nabla \smile \mathbf{i}_n) = (\mathbf{i}_n \,.\, \nabla)\{\sigma_n \nabla \mathbf{s}_n + \tfrac{1}{2}(\nabla \sigma_n)\mathbf{s}_n + \tfrac{1}{2}\mathbf{s}_n \nabla \sigma_n\}, \quad (90)$$

we have:

$$\left.\begin{aligned} \overset{4}{\mathbf{g}}_n \overset{2}{.} (\mathbf{i}_n \,.\, \nabla)(\nabla \smile \mathbf{i}_n) = \overset{4}{\mathbf{g}}_n \overset{2}{.} \{(\mathbf{i}_n \,.\nabla \sigma_n)\nabla \mathbf{s}_n + \sigma_n \mathbf{i}_n \overset{1}{.} \nabla\nabla \mathbf{s}_n + \\ + \tfrac{1}{2}(\nabla \sigma_n)\mathbf{i}_n \overset{1}{.} \nabla \mathbf{s}_n + \tfrac{1}{2}(\mathbf{i}_n \overset{1}{.} \nabla \mathbf{s}_n)\nabla \sigma_n\}, \end{aligned}\right\} \quad (91)$$

or, since:

$$\nabla \sigma_n = \nabla (\varkappa \mathbf{s}_n \,.\, \mathbf{s}_n)^{-\frac{1}{2}} = -\varkappa \sigma_n^3 (\nabla \mathbf{s}_n) \overset{1}{.} \mathbf{s}_n = -\sigma_n \mathbf{u}_n + \varkappa \sigma_n \mathbf{s}_n \overset{1}{} (\nabla \sigma_n)\mathbf{s}_n, \quad (92)$$

also:

$$\overset{4}{\mathbf{g}}_n \overset{2}{.} (\mathbf{i}_n \,.\, \nabla)(\nabla \smile \mathbf{i}_n) = \overset{4}{\mathbf{g}}_n \overset{2}{.} \{(\mathbf{i}_n \,.\, \nabla \sigma_n)\nabla \mathbf{s}_n + \sigma_n^2 \mathbf{s}_n \overset{1}{.} \nabla\nabla \mathbf{s}_n - \varkappa \mathbf{u}_n \mathbf{u}_n\}. \quad (93)$$

Since on account of (31) and (69):

$$\mathbf{i}_j \mathbf{i}_k \overset{2}{.} \{2\varkappa T (\nabla \smile \mathbf{i}_n) \overset{1}{.} \nabla \mathbf{i}_n\} = \mathbf{i}_j \mathbf{i}_k \overset{2}{} \mathbf{u}_n \mathbf{u}_n, \quad . \; . \; . \quad (94)$$

the condition (C^1) gets the shape:

$$\boxed{\mathbf{i}_j \mathbf{i}_k \overset{2}{.} \{\varkappa \sigma_n^2 \mathbf{s}_n \overset{1}{} \nabla\nabla \mathbf{s}_n - 2\,\mathbf{u}_n \mathbf{u}_n\} = 0}\;^{3)}, \quad \begin{matrix} j \neq k \\ j, k = 1, 2, \ldots, n-1. \end{matrix} \quad (C_3)$$

Since:

$$\nabla \mathbf{s}_n = \frac{1}{\sigma_n} \nabla \mathbf{i}_n + \left(\nabla \frac{1}{\sigma_n}\right)\mathbf{i}_n, \quad . \; . \; . \; . \; . \; . \quad (95)$$

we further have, in connection with (30) and (33):

$$\mathbf{i}_j \mathbf{i}_k \overset{2}{.} \nabla \mathbf{s}_n = 0, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (96)$$

from which by application of $(\mathbf{i}_k \,.\nabla)$ may be concluded:

$$\frac{1}{\sigma_n}(\mathbf{i}_l \overset{1}{.} \nabla \mathbf{i}_j) \overset{1}{.} (\nabla \mathbf{i}_n) \overset{1}{.} \mathbf{i}_k + \frac{1}{\sigma_n}(\mathbf{i}_l \overset{1}{.} \nabla \mathbf{i}_k) \overset{1}{.} (\nabla \mathbf{i}_n) \overset{1}{.} \mathbf{i}_j + \mathbf{i}_j \mathbf{i}_k \mathbf{i}_l \overset{3}{.} \nabla\nabla \mathbf{s}_n = 0. \quad (97)$$

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[1]) R. v. LILIENTHAL, Ueber die Bedingung, unter der eine Flächenschar einem dreifach orthogonalen Flächensystem angehört. Math. Annalen 44 (94), 449—457.

[2]) W. G. L. WIERINGA, Over drievoudig orthogonale oppervlakkensystemen. Diss. Groningen, (18) 59 pp., see p. 13.

[3]) See note [1]) of next page.

$(\nabla \mathbf{i}_n)^1_{\cdot} \, \mathbf{i}_k$ containing but $\mathbf{i}_k$ and $\mathbf{i}_n$ on account of (38), we find in connection with (67):

$$\boxed{\mathbf{i}_j \, \mathbf{i}_k \, \mathbf{i}_l \, {}^3 \, \nabla\nabla \, \mathrm{s}_n = 0}, \quad j \neq k, \; j \neq l, \; k \neq l, \; j, k, l = 1, 2, \ldots, n-1.\text{[1]} \qquad (D_1)$$

This equation (D_1) can be decomposed into:

$$\boxed{\mathbf{i}_j \, \mathbf{i}_k \, \mathbf{i}_l \, {}^3_{\cdot} \, (\nabla \smile \nabla) \, \mathrm{s}_n = 0}, \; \ldots \ldots \qquad (D'_1)$$

or:

$$\mathbf{i}_j \, \mathbf{i}_k \, \mathbf{i}_l \, {}^3_{\cdot} \, (\nabla \smile \nabla) \, \nabla y^n = 0, \; \ldots \ldots \qquad (98)$$

and:

$$\mathbf{i}_j \, \mathbf{i}_k \, \mathbf{i}_l \, {}^3_{\cdot} \, (\nabla \frown \nabla) \, \mathrm{s}_n = 0. \; \ldots \ldots \qquad (99)$$

When $\overset{4}{\mathbf{K}}$ is the RIEMANN-CHRISTOFFEL-affinor of V_n, (99) can be written:[2]

$$\mathbf{i}_j \, \mathbf{i}_k \, \mathbf{i}_l \, {}^3_{\cdot} \, \overset{4}{\mathbf{K}} \, {}^1_{\cdot} \, \nabla \, y^n = 0, \; \ldots \ldots \qquad (100)$$

or

$$\boxed{\mathbf{i}_j \, \mathbf{i}_k \, \mathbf{i}_l \, {}^3_{\cdot} \, \overset{4}{\mathbf{K}} \, {}^1_{\cdot} \, \mathbf{i}_n = 0.} \; \ldots \ldots \qquad (D_1'')$$

The equations (C_3), (D_1), (D'_1) and (100) are deduced by RICCI.[3] The number of the equations (D'_1) is $\frac{(n-1)\,(n-2)\,(n-3)}{1\,.\,2\,.\,3}$, the number of the equations (D'') is $\frac{(n-1)\,(n-2)\,(n-3)}{3}$, because we may permutate not only j and k, but also k and l[4]. (D_1') contains third, (D_1'') only first differential quotients of y^n.

The conditions (D_1'') vanish identically, when the characteristic numbers $l\,k\,j\,n$ of $\overset{4}{\mathbf{K}}$ vanish. Since in a space of constant RIEMANN-curvature K_0:

$$\overset{4}{\mathbf{K}} = 2\,K_0 \, (\mathrm{a} \frown \mathrm{b}) \, (\mathrm{a} \frown \mathrm{b}) \text{ [5]} . \; \ldots \ldots \qquad (101)$$

the equation holds:

$$\mathbf{i}_n \, \mathbf{i}_l \, \mathbf{i}_k \, \mathbf{i}_j \, {}^4_{\cdot} \, \overset{4}{\mathbf{K}} = 0, \; \ldots \ldots \qquad (102)$$

so that the condition (D_1'') is an identity in such a space, and hence also in a euclidean space. Thus (D_1) reduces in this case to (D'_1). For

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1) (C_3) can also be deduced from (84) in an analogous way as (D_1).

2) Comp. A. R. page 59.

3) G. RICCI, Dei sistemi etc., p. 314. Here the equations (C_3) and (D_1) are lettered (A_1) and (B_1). G. RICCI, Sui sistemi, p. 151.

4) Compare the observations of RICCI on occasion of a paper of DRACH, Comptes Rendus 125 (97) 598—601 and 810—811.

5) Compare Cf. BIANCHI-LUKAT, 1st. german edition, p. 574.

euclidean space the condition (D'_1) has been given by DARBOUX [1]) [2]). The characteristic numbers $(lkjn)$ of $\overset{4}{\mathbf{K}}$ vanish too, when the $V_{n-1} \perp \mathbf{i}_n$ are geodesic.

10. LÉVY's, CAYLEY's *and* DARBOUX's *conditions.* Differentiating the relation:

$$\mathbf{i}_n = \sigma_n \mathbf{s}_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (103)$$

we get

$$\nabla \mathbf{i}_n = (\nabla \sigma_n) \mathbf{s}_n + \sigma_n \nabla \mathbf{s}_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (104)$$

Differentiating again, we get:

$$\nabla\nabla \mathbf{i}_n = (\nabla\nabla \sigma_n) \mathbf{s}_n + (\nabla \mathbf{s}_n) \overset{1}{} \mathbf{a} (\nabla \sigma_n) \mathbf{a} + (\nabla \sigma_n) \nabla \mathbf{s}_n + \sigma_n \nabla\nabla \mathbf{s}_n, \quad (105)$$

and from this and (104) we have for $\nabla\nabla \sigma_n$:

$$\varkappa \nabla\nabla \sigma_n = \sigma_n (\nabla\nabla \mathbf{i}_n) \overset{1}{\cdot} \mathbf{i}_n - \sigma_n^2 (\nabla\nabla \mathbf{s}_n) \overset{1}{\cdot} \mathbf{i}_n + \frac{2\varkappa}{\sigma_n} (\nabla \sigma_n)(\nabla \sigma_n). \quad (106)$$

Since:

$$(\nabla\nabla \mathbf{i}_n) \overset{1}{\cdot} \mathbf{i}_n = \nabla \{(\nabla \mathbf{i}_n) \overset{1}{\cdot} \mathbf{i}_n\} - (\nabla \mathbf{i}_n) \overset{1}{\cdot} \mathbf{a} (\nabla \mathbf{i}_n) \overset{1}{\cdot} \mathbf{a} = -{}^2\mathbf{h} \overset{1}{\cdot} {}^2\mathbf{h} - (\mathbf{u}_n . \mathbf{u}_n) \mathbf{i}_n \mathbf{i}_n, \quad (107)$$

we get, in connection with (92):

$$\overset{4}{\mathbf{g}}_n \overset{2}{\cdot} \nabla\nabla \sigma_n = -\varkappa\sigma_n {}^2\mathbf{h}^1 {}^2\mathbf{h} - \varkappa\sigma_n^3 \overset{4}{\mathbf{g}}_n \overset{2}{\cdot} (\nabla\nabla \mathbf{s}_n) \overset{1}{\cdot} \mathbf{s}_n + 2\,\sigma_n \mathbf{u}_n \mathbf{u}_n. \quad (108)$$

In connection with (C_3) this equation gives a new shape to the first condition:

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[1]) G. DARBOUX, Leçons sur les systèmes orthogonaux et les coòrdonnées curvilignes I (98), p. 130, form. (35).

[2]) As a simple example for the application of (C_2) and (D'_1) for euclidean space, we can take the system $u = Y_1(y^1) + \ldots + Y_n(y^n)$, in which $y^1, \ldots, y^n$ are Cartesian coordinates. To calculate $g^{a_1 a_2}$ etc., it is necessary to find a system of $n-1$ V_{n-1} which determines in the V_{n-1} $u = \text{const.}$ a system of coordinates $\mathbf{e}_{a_1} \ldots$ Then $\varkappa\mathbf{e}_{a_1} . \mathbf{e}_{a_1} = g_{a_1 a_1}$, etc. For this purpose we must try to find $n-1$ independent solutions of the differential equations

$$\overset{1,\ldots,n}{\underset{i}{\Sigma}} \frac{\partial u}{\partial y^i} \frac{\partial \psi}{\partial y^i} = 0 \quad \text{or} \quad \overset{1,\ldots,n}{\underset{i}{\Sigma}} Y_i (y^i) \frac{\partial \psi}{\partial y^i} = 0$$

For the calculation compare e.g. WIERINGA, Diss. p. 21 and seq. Then we can see that the condition (D'_1) is identically satisfied, so that only LILIENTHAL's condition (C_2) remains, which can be written in this case:

$$\begin{vmatrix} \overset{1}{Y_i''} & \overset{1}{Y_l''} & \overset{1}{Y_k''} \\ Y_i' Y_i''' - 2\, Y_i''^2 & Y_l' Y_l''' - 2\, Y_l''^2 & Y_k' Y_k''' - 2\, Y_k''^2 \end{vmatrix} = 0,$$

or $Y_i' Y_i''' - 2\, Y_i''^2 = A Y_i'' + B$, in which A and B are constants.

This result has been deduced for $n = 3$ by SERRET, and for a general n by DARBOUX in another way as has been done here. Comp. DARBOUX, Leçons sur les systèmes orthogonaux etc., p. 140 and 141.

$$\mathrm{i}_j\,\mathrm{i}_k \overset{2}{\cdot} \nabla\nabla\,\sigma_n = 2\,\varkappa\sigma_n^{\,2}\,\mathrm{i}_j\,\mathrm{i}_k \overset{2}{\cdot} \{\mathrm{s}_n \,{}^1\, (\nabla \frown \nabla)\,\mathrm{s}_n\} = \varkappa\sigma_n\,\mathrm{i}_j\,\mathrm{i}_k \,{}^2\, (\mathrm{i}_n \,{}^1\, \overset{4}{\mathrm{K}} \,{}^1\, \mathrm{i}_n) \quad (109)$$

or

$$\boxed{\mathrm{i}_j\,\mathrm{i}_k \overset{2}{\cdot} \nabla\nabla\,\sigma_n = \varkappa\sigma_n\,\mathrm{i}_n\,\mathrm{i}_j\,\mathrm{i}_k\,\mathrm{i}_n \overset{4}{\cdot} \overset{4}{\mathrm{K}}.} \quad (C_4)$$

Thus for a V_n, for which the characteristic numbers $(n\,k\,j\,n)$ of $\overset{4}{\mathbf{K}}$ vanish, this first condition can be written:

$$\boxed{\mathrm{i}_j\,\mathrm{i}_k \overset{2}{\cdot} \nabla\nabla\,\sigma_n = 0.} \quad (C'_4)$$

This equation expresses that the tensor $\nabla\nabla\sigma_n$ *has the same principal directions as* ${}^2\mathbf{h}$. The geometrical signification of σ_n is that this quantity is proportional to the infinitesimal distance between succeeding $V_{n-1} \perp \mathrm{i}_n$ measured along i_n.

In space of constant RIEMANN-curvature K_0 we have, in connection with (101):

$$\mathrm{i}_j\,\mathrm{i}_k \,{}^2\, \{\mathrm{i}_n \,{}^1\, \overset{4}{\mathrm{K}} \overset{1}{\cdot} \mathrm{i}_n\} = -K_0\,\mathrm{i}_j\,\mathrm{i}_k \overset{2}{\cdot} (\varkappa^2\mathrm{g} - \mathrm{i}_n\,\mathrm{i}_n) = 0, \quad . \quad . \quad . \quad (110)$$

from which we conclude that in this manifold the first condition has the shape (C_4'), hence also in euclidean space. In this latter case the condition is deduced for $n = 3$ by LEVY [1]), CAYLEY [2]), DARBOUX [3]), and for general values of n by DARBOUX [4]). Thus the necessary and sufficient conditions for manifolds of constant RIEMANN-curvature are (C_4') and (D_1').

11. WEINGARTEN's condition. We will try to find a shape of the conditions that only depends on i_n and no more on $\mathrm{i}_j, j=1, 2, \ldots, n-1$. When a tensor, whose principal directions do not coincide with those of ${}^2\mathbf{h}$, be transvected once with ${}^2\mathbf{h}$, an affinor arises whose alternating part is certainly not annihilated. Thus the condition that the principal directions coincide, is that the alternating part of the first transvection with ${}^2\mathbf{h}$ vanishes. Hence (109) is equivalent to:

$$\overset{4}{B}\,\mathrm{g} \overset{2}{\cdot} \{(\nabla\,\mathrm{i}_n)\,{}^1\,(\nabla\nabla\,\sigma_n) - \sigma_n\,(\nabla\,\mathrm{i}_n)\,\mathrm{i}_n \,{}^2\, \overset{4}{\mathrm{K}} \overset{1}{\cdot} \mathrm{i}_n\} = 0, \quad . \quad . \quad . \quad (111)$$

in which B, may indicate that the bivector-part has to be taken.

1) M. LÉVY, Mémoire etc., p. 170.

2) A. CAYLEY, Sur la condition pour qu'une famille de surfaces fasse partie d'un système orthogonal, Comptes Rendus 75 (72), a series of articles.

3) G. DARBOUX, Sur l'équation du troisième ordre dont dépend le problème des surfaces orthogonales. Comptes Rendus 76 (73) 41—45, 83—86. See also e. g. BIANCHI-LUKAT 1st german edition.

4) G. DARBOUX, Leçons sur les systèmes orthogonaux etc., p. 128. His formula (32) is our formula (C'_4).

Since:

$$\nabla \{ (\nabla \mathbf{i}_n) \overset{1}{.} (\nabla \sigma_n) \} = (\nabla\nabla \mathbf{i}_n)^1 \nabla \sigma_n + \mathbf{a} (\nabla \mathbf{i}_n) \overset{1}{.} (\mathbf{a} . \nabla) \nabla \sigma_n, \quad (112)$$

we have:

$$\nabla \frown \{(\nabla \mathbf{i}_n) \overset{1}{.} \nabla \sigma_n\} = B \nabla \{(\nabla \mathbf{i}_n)^1 \nabla \sigma_n \} = {}^1/_2 \overset{4}{\mathbf{K}} \overset{2}{.} \mathbf{i}_n \nabla \sigma_n - B (\nabla \mathbf{i}_n)^1 \nabla\nabla \sigma_n, (113)$$

so that (111) is equivalent to:

$$\overset{4}{\mathbf{g}} \overset{2}{.} \left[- \nabla \frown \{(\nabla \mathbf{i}_n) \overset{1}{.} \nabla \sigma_n\} + \frac{1}{2} \overset{4}{\mathbf{K}} \overset{2}{.} \mathbf{i}_n \nabla \sigma_n - \sigma_n B (\nabla \mathbf{i}_n) \mathbf{i}_n \overset{2}{.} \overset{4}{\mathbf{K}}{}^1 \mathbf{i}_n \right] = 0. \quad (114)$$

Since in a space of constant RIEMANN-curvature on account of (92) and (101):

$$\overset{4}{\mathbf{g}} \overset{2}{.} \overset{4}{\mathbf{K}} \overset{2}{.} \mathbf{i}_n \nabla \sigma_n = - \sigma_n \overset{4}{\mathbf{g}} \overset{2}{.} \overset{4}{\mathbf{K}} \overset{2}{.} \mathbf{i}_n \mathbf{u}_n = - 2 \sigma_n K_0 \overset{4}{\mathbf{g}}{}^2 \mathbf{i}_n \frown \mathbf{u}_n = 0, \quad (115)$$

the condition for such a manifold is, on account of (110), that the component of $\nabla \frown \{(\nabla \mathbf{i}_n) \overset{1}{.} \nabla \sigma_n\}$ in the region $\perp \mathbf{i}_n$ vanishes. On account however of STOKES' law [1]), we have for each vector $\mathbf{v}$:

$$\int_s \mathbf{v} . d\mathbf{x} = - 2 \varkappa \int_\sigma {}_2\mathbf{f} \overset{2}{.} (\nabla \frown \mathbf{v}) \, d\sigma, \quad . \quad . \quad . \quad . \quad (116)$$

in which s is a closed curve and ${}^2\mathbf{f} d\sigma$ the bivector of the surface-element of any surface σ bounded by this curve. From this we conclude *that in a space of constant* RIEMANN-*curvature we can also give as first condition that the linear integral of the vector* $(\nabla \mathbf{i}_n) \overset{1}{.} \nabla \sigma_n$ *along each closed curve in a* $V_{n-1} \perp \mathbf{i}_n$ *vanishes.* This condition is the only one for V_3. For an R_3 it has been first indicated by WEINGARTEN [2]) and RICCI [3]) has observed on occasion of WEINGARTEN's paper that the condition holds also for a V_3 of constant RIEMANN-curvature. From the above-mentioned we see that the condition, but no more as the only one, holds also for manifolds of constant RIEMANN-curvature, for which $n > 3$.

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1) Comp. A. R., page 37 and 61.

2) WEINGARTEN, Ueber die Bedingung, unter welcher eine Flachenfamilie einem orthogonalen Flächensystem angehört. Crelle 83 (77), 1—12.

3) G. RICCI, Della equazione di condizione dei parametri dei sistemi di superficie, che appartengono ad un sistema triplo ortogonale. Rendiconti Acc Lincei Ser. V, III_2 (94) 93—96.

RICCI observes for the case $n = 3$ that WEINGARTEN's theorem remains also valid, when $\overset{4}{\mathbf{K}}$ has the shape:

$$\overset{4}{\mathbf{K}} = \mu (\mathbf{a} \frown \mathbf{b}) (\mathbf{a} \frown \mathbf{b}) + \nu (\mathbf{i}_1 \frown \mathbf{i}_2) (\mathbf{i}_1 \frown \mathbf{i}_2)$$

when ν is an arbitrary coefficient. This however holds also for general values of n.

12. *Mutually orthogonal V_{n-1}-systems through a given congruence, the canonical congruences being not singly determined.*

When the roots of (24) are not all different, these roots determine in general q mutually perpendicular regions $R_{p_1}, \ldots R_{p_q}$. Within the region R_{p_α} every set of p_α mutually perpendicular directions satisfies the canonical conditions. The equations (47—51) teach us that it must be possible to choose the canonical directions in each of the regions R_{p_α} in such a way that they are V_{n-1}-normal, when through $\mathbf{i}_n$ there shall pass $n-1$ mutually orthogonal V_{n-1}-systems. Thus the conditions (C') and (D), depending on (55) resp. (67), i.e. of the being V_{n-1}-normal of *all* canonical congruences, will no more remain valid without any restriction.

When ${}_{p_1}\mathbf{i}, \ldots {}_{p_q}\mathbf{i}$ are the unit-p-vectors belonging to the regions $R_{p_1}, \ldots, R_{p_q}$, the equations:

$$\mathbf{i}_n \overset{1}{.} \nabla y^{\varkappa} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (117)$$

$$_{p_\alpha}\mathbf{i} \overset{1}{.} \nabla y^{\varkappa} = 0 \quad \alpha = 1, .. , \beta-1, \beta+1, ..., q$$

must be satisfied by p_β independent solutions. On account of (B) we thus have:

$$(\mathbf{i}_n \, {}_{p_1}\mathbf{i} \cdots {}_{p_{\beta-1}}\widehat{\mathbf{i}} \, {}_{p_{\beta+1}}\mathbf{i} \cdots {}_{p_q}\mathbf{i})^2 \;\; \nabla \frown {}_{p_\beta}\mathbf{i} = 0. \quad . \quad . \quad . \quad (118)$$

and from this we conclude:

$$\mathbf{i}_k \frown \mathbf{i}_n \overset{2}{.} \nabla \mathbf{i}_j = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (119)$$

$$\mathbf{i}_k \frown \mathbf{i}_l \overset{2}{.} \nabla \mathbf{i}_j = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (120)$$

in which $\mathbf{i}_j$ belongs to another region than $\mathbf{i}_k$ and $\mathbf{i}_l$, and for the rest the choice is arbitrary, provided $k \neq l$.

(119) has entirely the same form as (55) and from (120) follows for the special case that $\mathbf{i}_j$, $\mathbf{i}_k$, $\mathbf{i}_l$ each belong to different regions:

$$\mathbf{i}_k \mathbf{i}_l \overset{2}{.} \nabla \mathbf{i}_j = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (121)$$

an equation of the same form, and deduced in the same way as (67).

The equations (C') and (D) only remain valid under the abovementioned restricting conditions. They are besides no longer sufficient. A supplementary condition will be found in the following way:

The equation (65) shows:

$$\left.\begin{aligned}(\lambda_k - \lambda_j)\, \mathbf{i}_k \mathbf{i}_l \overset{2}{.} \nabla \mathbf{i}_j + \varkappa \mathbf{i}_j \mathbf{i}_k \mathbf{i}_l \overset{3}{.} \nabla {}^2\mathbf{h} = 0,\\ (\lambda_l - \lambda_j)\, \mathbf{i}_l \mathbf{i}_k \overset{2}{.} \nabla \mathbf{i}_j + \varkappa \mathbf{i}_j \mathbf{i}_l \mathbf{i}_k \overset{3}{.} \nabla {}^2\mathbf{h} = 0.\end{aligned}\right\} \quad . \quad . \quad . \quad (122)$$

valid for the case that $\mathbf{i}_k$ and $\mathbf{i}_l$ belong to the same region and $\mathbf{i}_j$ to another one. Then, subtracting the equations (122) one from the other we conclude, in connection with (121):

$$\boxed{\mathbf{i}_j (\mathbf{i}_k \frown \mathbf{i}_l)^2 \nabla {}^2\mathbf{h} = 0 \text{ [1]} \quad j \neq k,\ j \neq l,\ k \neq l,\ j,\ k = 1, 2, \ldots,\ n-1.} \quad (E)$$

Under the mentioned conditions the equations (C'), (D) and (E) are not only necessary, but also sufficient. In fact, from (E) may be concluded, in connection with (122), since $\lambda_k = \lambda_l$, that $\nabla \mathbf{i}_j$ is symmetrical in l and k, when l and k belong to the same region, but j and l do not. From (D) we conclude, in analogical way as we have explained in the first part of this paper, that $\nabla \mathbf{i}_j$ is symmetrical in l and k, when l and k belong to different regions, different from j. (C') tells that $\nabla \mathbf{i}_j$ is symmetrical in n and k, when k differs from j. Hence these three conditions are sufficient to show that $\nabla \mathbf{i}_j$ is symmetrical in the region $\perp \mathbf{i}_j$, and thus that $\mathbf{i}_j$ is V_{n-1}-normal.

When we call [2]) $p_1, p_2 \ldots p_q$ the multiplicity of the roots of the algebraic characteristic equation (24), the number of equations (C') is the sum of the two-factorial products of the numbers $p_1, p_2, \ldots p_q$, and the number of the equations (D) is thrice the sum of the three-factorial products of these numbers. The number of the equations (E) is equal to the sum of the products of the form $p_k p_h \left(\frac{p_h + p_k}{2} - 1\right)$.

13. *Simplifications for the case that the given congruence is V_{n-1}-normal.*

When $\mathbf{i}_n$ is V_{n-1}-normal, (C) passes into (C_1) or (C_2), being valid for the case that $\mathbf{i}_j$ and $\mathbf{i}_k$ belong to different regions. (D) can also be brought into the form (D_1) and is then valid for the case that $\mathbf{i}_j$, $\mathbf{i}_k$ and $\mathbf{i}_l$ belong to different regions.

From (97) follows for the case that $\mathbf{i}_k$ and $\mathbf{i}_l$ belong to the same region and $\mathbf{i}_j$ to another:

$$\mathbf{i}_j (\mathbf{i}_k \frown \mathbf{i}_l)^3 \nabla\nabla \mathrm{s}_n = 0 \quad \ldots\ldots\ldots \quad (123)$$

This equation can also be written in the form:

$$\boxed{\mathbf{i}_j \mathbf{i}_k \mathbf{i}_l \overset{3}{.} \overset{4}{\mathrm{K}} {}^1 \mathbf{i}_n = 0 \text{ [3]}} \quad \ldots\ldots \quad (E_1)$$

which has a formal analogy to (D_1''), but which is valid under different conditions. But the increment of the vector $\mathbf{i}_n$, when

[1]) (E) is the equation (C) of Ricci, Dei sistemi, page 312, but deduced from ${}^2\mathbf{h}$, and not from $\nabla \smile \mathbf{i}_n$.

[2]) Compare Ricci, Dei sistemi, p. 312.

[3]) (E_1) is (C_1) of Ricci, Dei sistemi, p. 314.

geodesically moved along the boundary of the surface-element $d\sigma$, is: [1])

$$D_{kl}\, \mathbf{i}_n = d\sigma\, \mathbf{i}_k\, \mathbf{i}_l\, {}^2\, \overset{4}{\mathbf{K}}\, {}^1_{\cdot}\, \mathbf{i}_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (124)$$

So (E_1) demands this increment to remain in the region formed by $\mathbf{i}_k$ and $\mathbf{i}_l$. [2])

Thus we have obtained the following theorem:

III. *A system of* ∞^1 V_{n-1} *in a* V_n, *whose second fundamental tensor, apart from determined* V_r, $r < n$, *has* q *singly determined principal regions* $R_{p_1}, \ldots, R_{p_q}$, *but within the regions of more than one dimension no singly determined principal directions, belongs then and only then to an n-uple orthogonal system, when by moving perpendicular to m of the principal regions of* ${}^2\mathbf{h}$, *the component of the geodesic differential of* ${}^2\mathbf{h}$, *in the manifold determined by these m regions, has principal regions that coincide with the m mentioned principal regions of* ${}^2\mathbf{h}$, *and when besides the increment of* $\mathbf{i}_n$, *when* $\mathbf{i}_n$ *is geodesically moved along the boundary of a surface-element in any principal region, remains entirely in this same principal region.*

14. *Necessary and sufficient conditions that a* V_n *may admit n-uple orthogonal* V_{n-1}*-systems.*

The condition (D_1'') is a condition for the V_n in which the n-uple orthogonal system exists. If we wish every system of n mutually perpendicular $(n-1)$-directions in each point of the V_n to belong to an n-uple orthogonal V_n-system, then (D_1'') must be valid for every set of four mutually perpendicular unit-vectors. It can be proved that $\overset{4}{\mathbf{K}}$ can then be written in the form:

$$\boxed{\overset{4}{\mathbf{K}} = (\mathbf{a} \frown \mathbf{z})\,(\mathbf{a} \frown \mathbf{z})} \qquad (F)$$

in which $\mathbf{z}^2$ is an arbitrary tensor. For $n = 3$ $\overset{4}{\mathbf{K}}$ can *always* get this shape and, as has been proved by COTTON [3]), every set of three mutually perpendicular directions in any point of a V_3 can belong to a triple orthogonal system. It can be proved that (F) is sufficient for $n > 3$ too.

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1) A. R. p. 64.

2) An analogous geometrical interpretation can also be given to condition (D_1'').

3) E. COTTON Sur une généralisation du problème de la représentation conforme aux variétes à trois dimensions, Comptes Rendus 125 (97) 225—228, compare also E. COTTON, Annales de Toulouse 1 (99) 385—438, Chap. III.

15. *Addendum.*

In this paper the product $\mathbf{i}.\mathbf{i} = \varkappa$ of the system R_n^0 [1]) is used. $\varkappa$ can be found from the dualities existing in the orthogonal group, on which the identifications used in the system R_n^0 are founded. Now in investigations on differential geometry these identifications (e.g. of $\mathbf{i}_1$ and $\mathbf{i}_2 \frown \ldots \frown \mathbf{i}_n$) are practically not used. In this case it is convenient to substitute $\varkappa$ by $+1$, then $\varkappa$ vanishes in all formulae, and the calculation grows much easier. It has however to be noted, that taking $+1$ for $\varkappa$ it is no longer permitted to make use of the identifications founded on the dualities of the orthogonal group.

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[1]) J A. SCHOUTEN, On the direct analyses of the linear quantities etc., These Proceedings 21 (17) 327–341; Die Zahlensysteme der geometrischen Groszen, Nieuw Archief (20) 141—156.