Physics. — *"The Contributions from the Polarization and Magnetization Electrons to the Electric Current".* By Dr. A. D. FOKKER. (Communicated by Prof. H. A. LORENTZ).

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1. An important point in the theory of electrons is how to evaluate the electric current proceeding from the electrons which in their movements are bound to the atoms of matter. We require it for establishing the equations of the electromagnetic field in ponderable matter, and we know that it is responsible for the effects of polarization and magnetization.

Consider a stream of moving neutral atoms, and imagine them as consisting of a positive nucleus and one accompanying electron. The heavy nuclei will contain the centres of mass of the atoms their motion therefore will be identified with the motion of matter in bulk. The accompanying electrons will move round the nuclei or in their immediate neighbourhood. Now the stream of positive nuclei will form an electric current, and the stream of electrons of course will constitute another. For a great part these two currents will cancel one another, but not completely, as they would, if both motions were the same: the resulting current is clearly what arises from the intra-atomical motions of the bound electrons.

Obviously we shall know this current if, given the motion of a stream of particles, we can find the variation effected by displacing them slightly from their tracks, for it is by such small displacements that the motion of the electrons may be found from the motions of the nuclei. Our problem thus presents itself as a *variation problem*.

M. BORN has told us [1]) that the idea to put it thus is due to HERMANN MINKOWSKI. He has developed it after MINKOWSKI's death and compared his deductions with MINKOWSKI's posthumous notes. I venture to offer to the Academy a novel development of the same idea, which might claim a great simplicity and might be more exact in some points. In addition, a new second order contribution of the bound electrons is arrived at, which has been neglected until now, so far as I know (§§ 9 and 11).

[1]) H. MINKOWSKI—M. BORN, *Eine Ableitung der Grundgleichungen für die elektromagnetischen Vorgänge in bewegten Körpern vom Standpunkte der Elektronentheorie*, Math. Ann. 68, p. 526, 1910.

The Variational Displacements.

2. We consider a field of streaming discrete particles, the velocities being continuous functions of the coordinates and the time. We imagine a picture in a four-dimensional space-time-extension designing the motion-trails of the particles indicating their positions in successive instants. Now the displacements will consist of a shift in space and a shift in time, and we shall define these shifts with the aid of a field of a four-fold vector r^a, the components of which: $r^{(1)}, r^{(2)}, r^{(3)}$, being space-components and $r^{(4)}$ being the time-component, will be continuous functions of the coordinates and time x^a $(a = 1 \,.\,.\, 4)$.

Mathematically, we define the shifts as the one-membered infinitesimal transformation group determined by the functions r^a $(a = 1 \,.\,.\, 4)$, with parameter θ:

$$\Delta x^a = \theta r^a + \tfrac{1}{2}\, \theta^2 \sum_1^4 (c)\, r^c \frac{\partial r^a}{\partial x^c} + \ldots$$

This will be clearer if we explain the nature of the r^a. If the variational parameter increases by an amount $d\theta$, then the particles are supposed as suffering an additional shift given by

$$r^a\, d\theta \quad (a = 1, 2, 3, 4),$$

the values of r^a being taken such as they are in the momentary point-instant occupied by the particle. Leaving out second order terms with θ^2, we at once see that the first approximation of the total shift will be

$$\theta\, r^a, \quad (a = 1 \,.\,.\, 4),$$

and proceeding to second order terms we obviously get

$$\Delta x^a = \int_0^\theta \left\{ r^a + \Sigma\, (c) \frac{\partial r^a}{\partial x^c} r^c \vartheta \right\} d\vartheta,$$

$$\Delta x^a = \theta r^a + \tfrac{1}{2}\, \theta^2\, \Sigma\, (c)\, r^c \frac{\partial r^a}{\partial x^c},$$

where now the values of r^a and their derivatives have been taken in the point-instants of the particle's undisturbed motion.

The Variation of the Stream.

3. The following conception of the stream components will greatly facilitate our deductions.

Let N, a continuous function of space-time-coordinates, represent the density of the particles' distribution through space. At the instant

$x^{(4)}$, take an element of volume dV, situated at the point $x^{(1)}$, $x^{(2)}$, $x^{(3)}$. It will contain NdV particles. We assume that NdV is still a great number, notwithstanding dV being physically infinitesimal. Now, in our four-dimensional picture, consider the trails of these NdV particles, run through during an interval of time $dx^{(4)}$. These trails will cover an element of space-time-extension of magnitude $dVdx^{(4)}$. In the direction of the coordinate X^a the components will in the aggregate amount to

$$NdVdx^a.$$

It will readily be seen that the *streamcomponent in the direction of X^a is the aggregate of the X^a-components of the four-dimensional trails, run through by the particles per unit of volume per unit of time*:

$$\frac{N\,dV\,dx^a}{dV\,dx^{(4)}} = N\frac{dx^a}{dx^{(4)}} = Nw^a. \qquad (a = 1, 2, 3, 4).$$

We shall put $w^{(1)}$, $w^{(2)}$, $w^{(3)}$ for the components of the velocity: $dx^{(1)}/dx^{(4)}$, $dx^{(2)}/dx^{(4)}$, $dx^{(3)}/dx^{(4)}$. The fourth component equals unity: $w^{(4)} = dx^{(4)}/dx^{(4)}$, and accordingly the fourth streamcomponent $Nw^{(4)}$ is the number of particles per unit of volume.

It is obvious that the equation of continuity must be satisfied by these streamcomponents:

$$\Sigma\,(b)\,\frac{\partial Nw^b}{\partial x^b} = 0.$$

By the displacements the components will change to

$$Nw^a + \delta Nw^a + \tfrac{1}{2}\,\delta^2\,Nw^a,$$

where the first variation δNw^a is proportional to θ and the second variation $\delta^2 Nw^a$ will contain the second order terms with θ^2. It may be anticipated that the first variation will account for by far the greater part of the effects of polarization, whereas the second variation mainly gives the effects of magnetization.

4. We proceed to the evaluation of the *first variation*. Here we may consistently neglect θ^2.

The displacements will have changed the aggregate of the X^a-components of the trails under consideration: each dx^a passes into

$$dx^a + \Sigma\,(b)\,\frac{\partial\,\theta r^a}{\partial x^b}\,dx^b,$$

so that the aggregate becomes

$$N\,dV\left\{dx^a + \Sigma\,(b)\,\frac{\partial \theta r^a}{\partial x^b}\,dx^b\right\}.$$

On the other hand the four-dimensional extension covered by the trails has changed too: we find its magnitude by the aid of the JACOBIAN determinant:

$$(d\,V\,dx^{(4)})' = \begin{vmatrix} \dfrac{\partial\,(x^a + \Delta\,x^a)}{\partial x^a} & \dfrac{\partial\,(x^a + \Delta\,x^a)}{\partial x^b} & \cdot & \cdot \\ \dfrac{\partial\,(x^b + \Delta\,x^b)}{\partial x^a} & \dfrac{\partial\,(x^b + \Delta\,x^b)}{\partial x^b} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} d\,V\,dx^{(4)},$$

$$= \begin{vmatrix} 1 + \theta\dfrac{\partial r^a}{\partial x^a} & \theta\dfrac{\partial r^a}{\partial x^b} & \cdot & \cdot \\ \theta\dfrac{\partial r^b}{\partial x^a} & 1 + \theta\dfrac{\partial r^b}{\partial x^b} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} d\,V\,dx^{(4)} = \left[1 + \Sigma\,(b)\,\theta\frac{\partial r^b}{\partial x^b}\right] d\,V\,dx^{(4)}$$

We must divide by this, and so when we follow the displacement, we may state a change of the streamcomponent into

$$Nw^a + \Delta\,Nw^a = \left[Nw^a + \Sigma\,(b)\,Nw^b\,\theta\frac{\partial r^a}{\partial x^b}\right].\left[1 - \Sigma\,(b)\,\theta\frac{\partial r^b}{\partial x^b}\right].$$

But this is not the thing we want. This value is found in the point-instant $x^a + \Delta x^a$, after the displacement. We require the variation of the stream which we get if we stick to one and the same point-instant x^a both when the particles are shifted and when they are not. It is clear that the shifted particles which will by the displacement get to our point, had their starting-points elsewhere, in a point-instant which may be found if in the formula for Δx_a we change θ into $-\theta$.

So we have to correct the above expression by accounting for this different starting-point: instead of Nw^a we are to take

$$Nw^a - \Sigma\,(b)\frac{\partial Nw^a}{\partial x^b}\theta r^b,$$

and we get

$$Nw^a + \delta Nw^a = Nw^a + \Sigma\,(b)\left[-\frac{\partial Nw^a}{\partial x^b}\theta r^b - Nw^a\,\theta\frac{\partial r^b}{\partial x^b} + Nw^b\,\theta\frac{\partial r^a}{\partial x^b}\right].$$

Availing ourselves of the equation of continuity, we may put our result in the symmetrical form:

$$\delta Nw^a = \Sigma\,(b)\frac{\partial}{\partial x^b}\{\theta r^a\,Nw^b - \theta r^b\,Nw^a\}.$$

This formula is also given by BORN. It may be found, without deduction however, in a paper by LORENTZ [1]).

5. *The second variation* is easily found, without calculation, by submitting the first variation in turn to the operation which we had to apply to Nw^a in order to find δNw^a. So we get without difficulty

$$\delta.\delta\, Nw^a = \Sigma\,(b)\frac{\partial}{\partial x^b}\{\theta r^a\,\delta Nw^b - \theta r^b\,\delta Nw^a\},$$

$$\delta^2\, Nw^a = \Sigma\,(bc)\frac{\partial}{\partial x}\left\{\theta r^a\frac{\partial}{\partial x^c}[\theta r^b\, Nw^c - \theta r^c Nw^b] - \theta r^b\frac{\partial}{\partial x^c}[\theta r^a Nw^c - \theta r^c Nw^a]\right\}.$$

It is, however, important to remark that this formula implies the accurate definitions of the displacements as given in § 2. This can be verified by a direct deduction, following throughout the same line of argument as in the case of the first variation. We may refrain from reproducing the calculus, but it will be good to point out, that one has to develop the Jacobian with the required exactness up to the terms with θ^2, and, above all, that at the last step to be taken one has to be careful to choose the right starting-point from where the displacements will carry the particles to the point under consideration, viz.,

$$x^a - \theta r^a + \tfrac{1}{2}\,\theta^2\,\Sigma\,(c)\,r^c\frac{\partial r^a}{\partial x^c},$$

and *not* $x^a - \Delta x^a$, as we might be tempted to take.

Next, we have to give an interpretation of our mathematical result in physical terms such as polarization and magnetization.

The Simultaneous Displacements.

6.1. Before turning to the physical interpretation we must look closer into the nature of our displacement vector r^a and the underlying assumptions.

We are to assume, that the trails of the electrons can be found from the trails of the nuclei with the aid of the vectors r^a, in the indicated manner. First of all, we have taken these to be continuous functions of the coordinates. This implies that neighbouring atoms are supposed as having their electrons at similar distances in similar directions, the positions of the electrons relative to the nuclei varying but extremely slowly from one atom to the next. Of course this will not

[1]) H. A LORENTZ, HAMILTON's *Principle in* EINSTEIN's *theory of Gravitation*, Proc. R. Ac. of Amsterdam, 19, p. 751, 1915.

exactly or even nearly exactly correspond to reality. But we can commit no essential error by assuming the atoms as behaving in such a continuous way.

Secondly, we must observe, that the only reality we are concerned with is the aggregate of trails of nuclei and electrons, and that the choice of the vectors r^a is entirely arbitrary provided they furnish us with the right motion of the electrons relative to the nuclei. Obviously the choice can be made in a great many different ways. Sometimes it will be suitable to choose the r^a such that the time-component $r^{(4)}$ vanishes in all points where matter is in a stationary state. We need not specify a particular choice.

6.2. As yet the displacements considered have been accompanied by a shift in time. In view of the physical interpretation of the formulae obtained, it will however be necessary to realize the simultaneous positions of the electrons relative to the nuclei.

Now, in a first approximation, we find the electron belonging to the nucleus, which at the instant $x^{(4)}$ is in the point $x^{(1)}$, $x^{(2)}$, $x^{(3)}$, shifted to the point

$$x^{(1)} + \theta r^{(1)},\ x^{(2)} + \theta r^{(2)},\ x^{(3)} + \theta r^{(3)}$$

at the instant

$$x^{(4)} + \theta r^{(4)}.$$

Thus we see that its position at the time $x^{(4)}$ will be given by

$$x^{(1)} + \rho^{(1)},\ x^{(2)} + \rho^{(2)},\ x^{(3)} + \rho^{(3)},$$

where

$$\rho^a = \theta r^a - w^a \theta r^{(4)}.$$

For an obvious reason $\rho^{(4)} = 0$.

Next, to obtain the second approximation, consider the nucleus at the instant

$$x^{(4)} - \theta r^{(4)} - \Sigma\,(c) \left\{ \tfrac{1}{2}\, \theta^2 r^c \frac{\partial r^{(4)}}{\partial x^c} - w^c \theta^2 r^{(4)} \frac{\partial r^{(4)}}{\partial x^c} \right\},$$

when its coordinates are

$$x^a - w^a \theta r^{(4)} - w^a\, \Sigma\,(c) \left\{ \tfrac{1}{2}\, \theta^2 r^c \frac{\partial r^{(4)}}{\partial x^c} - w^c \theta^2 r^{(4)} \frac{\partial r^{(4)}}{\partial x^c} \right\} + \tfrac{1}{2} \frac{dw^a}{dx^{(4)}} \theta^2 r^{(4)} r^{(4)}.$$

This line implies the preceding as a special case, for $a = 4$.

Then the displacements of the electron will be

$$\theta\, r^a + \tfrac{1}{2}\, \theta^2\, \Sigma\,(c)\, r^c \frac{\partial r^a}{\partial x^c} - \theta^2\, \Sigma\,(c)\, w^c r^{(4)} \frac{\partial r^a}{\partial x^c}.$$

so that its actual position will be given by

$$x^a + \theta r^a - w^a \theta r^{(4)} + \tfrac{1}{2} \frac{dw^a}{dx^{(4)}} \theta^2 r^{(4)} r^{(4)} + \Sigma (c) \left\{ \tfrac{1}{2} \theta^2 r^c \left(\frac{\partial r^a}{\partial x^c} - w^a \frac{\partial r^{(4)}}{\partial x^c} \right) - \right.$$
$$\left. - \theta^2 r^{(4)} w^c \left(\frac{\partial r^a}{\partial x^c} - w^a \frac{\partial r^{(4)}}{\partial x^c} \right) \right\}.$$

Taking $a = 4$, this formula yields the instant $x^{(4)}$, for $w^{(4)} = 1$, and all terms vanish except the first.

So we see that for $a = 1, 2, 3$ it gives the simultaneous displacements. We can simplify considerably. Writing

$$\Sigma (c)\, w^c \frac{\partial}{\partial x^c} = \frac{d}{dx^{(4)}},$$

$$\Sigma (c)\, \theta r^c \frac{\partial}{\partial x^c} = \Sigma (c)\, \varrho^c \frac{\partial}{\partial x^c} + \theta r^{(4)} \frac{d}{dx^{(4)}},$$

we get for the simultaneous displacements:

$$s^a = \varrho^a - \tfrac{1}{2} \theta r^{(4)} \left[\frac{d\varrho^a}{dx^{(4)}} - \Sigma (c)\, \varrho^c \frac{\partial w^a}{\partial x^c} \right] + \tfrac{1}{2} \Sigma (c)\, \varrho^c \frac{\partial \varrho^a}{\partial x^c}. \qquad (6.2)$$

For $a = 4$ we have $s^{(4)} \equiv 0$.

6.3. Let us inquire what will be the polarization of matter, viz. the electrical moment per unit of volume. The electrical moment of one atom being es^a, where e is the charge of an electron, the answer is, in a first approximation, that the polarization has components

$$N e s^a, \qquad (a = 1, 2, 3).$$

Proceeding more carefully, we must take some closed surface, a sphere, say, sum up the electrical moments of the atoms within and divide by the volume. But what about the border atoms, which are intersected by the sphere? Must we leave them out, or must we reckon them as lying within the sphere? The difference will be of second order only, but it does make a difference.

A similar question has been raised by LORENTZ in his Theory of Electrons (note 53). LORENTZ decides himself to leave out the intersected atoms, and this is certainly right when we restrict ourselves to the first order terms, neglecting θ^2. But here we retain θ^2. Fortunately, our calculus leads us to the answer: it will show a correction to be made to the same effect as establishing the rule: the atoms are to be reckoned as lying within the surface, whenever more than half of the line joining nucleus and electron lies within the surface. This is a quite satisfactory rule.

Thus the polarization is:

$$N e s^a - \tfrac{1}{2} \Sigma (c) \frac{\partial N e s^a s^c}{\partial x^c}. \qquad (6.3)$$

6.4. The magnetic momentum of an atom has the components

$$\frac{1}{2c} e \left(s^a \frac{ds^b}{dx^{(4)}} - s^b \frac{ds^a}{dx^{(4)}} \right),$$

Hence the components of the magnetization are

$$cm^{ab} = \tfrac{1}{2} N e \left(s^a \frac{ds^b}{dx^{(4)}} - s^b \frac{ds^a}{dx^{(4)}} \right). \tag{6.4}$$

It is possible this ought to be corrected in the same way as shown for the polarization. The correction, however would be of the third order and contain θ^3; and this we drop throughout our investigation.

For this same reason we are justified in replacing s^a by ϱ^a in the expression for the magnetization.

Interpretation of the Variation of the Stream.

7. If e be the charge carried by an electron, then the current carried by the electrons is

$$e N w^a + e \delta N w^a + \tfrac{1}{2} e \delta^2 N w^a, \qquad (a = 1, 2, 3, 4).$$

Adding the current carried by the nuclei, viz. $-eNw^a$, we get for the resulting current:

$$e \delta N w^a + \tfrac{1}{2} e \delta^2 N w^a.$$

Our results indicate that this can be written as a divergency of a skew-symmetrical tensor T^{ab}:

$$e \delta N w^a + \tfrac{1}{2} e \delta^2 N w^a = \Sigma\, (b) \frac{\partial T^{ab}}{\partial x^b},$$

where T^{ab} is given by

$$T^{ab} = e\theta (r^a N w^b - r^b N w^a) + \tfrac{1}{2} e\theta^2 \left\{ r^a \Sigma\, (c) \frac{\partial}{\partial x^c} (r^b N w^c - r^c N w^b) - \right.$$

$$\left. - r^b \Sigma \frac{\partial}{\partial x^c} (r^a N w^c - r^c N w^a) \right\},$$

and

$$T^{ab} = - T^{ba}.$$

We shall see what this tensor contains. First writing

$$\frac{T^{ab}}{e} = (\varrho^a N w^b - \varrho^b N w^a) +$$

$$+ \tfrac{1}{2} \theta r^{(4)} w^a \Sigma \frac{\partial}{\partial x^c} (\varrho^b N w^c - \varrho^c N w^b) - \tfrac{1}{2} \theta r^{(4)} w^b \Sigma \frac{\partial}{\partial x^c} (\varrho^a N w^c - \varrho^c N w^a) +$$

$$+ \tfrac{1}{2} \varrho^a \Sigma \frac{\partial}{\partial x^c} (\varrho^b N w^c - \varrho^c N w^b) - \tfrac{1}{2} \varrho^b \Sigma \frac{\partial}{\partial x^c} (\varrho^a N w^c - \varrho^c N w^a),$$

we can arrange terms in such a way as to get

$$T^{ab} = w^b Ne\left[\varrho^a - \tfrac{1}{2}\theta r^{(4)} \Sigma \left\{ w^c \frac{\partial \varrho^a}{\partial x^c} - \varrho^c \frac{\partial w^a}{\partial x^c} \right\} + \tfrac{1}{2}\Sigma \varrho^c \frac{\partial \varrho^a}{\partial x^c}\right] - \tfrac{1}{2} w^b \Sigma \frac{\partial Ne\varrho^a\varrho^c}{\partial x^c} -$$

$$- w^a Ne\left[\varrho^b - \tfrac{1}{2}\theta r^{(4)} \Sigma \left\{ w^c \frac{\partial \varrho^b}{\partial x^c} - \varrho^c \frac{\partial w^b}{\partial x^c} \right\} + \tfrac{1}{2}\Sigma \varrho^c \frac{\partial \varrho^b}{\partial x^c}\right] + \tfrac{1}{2} w^a \Sigma \frac{\partial Ne\varrho^b\varrho^c}{\partial x^c} -$$

$$- \tfrac{1}{2} Ne\varrho^a \Sigma \varrho^c \frac{\partial w^b}{\partial x^c} + \tfrac{1}{2} Ne\varrho^b \Sigma \varrho^c \frac{\partial w^a}{\partial x^c} + \tfrac{1}{2} Ne\left[\varrho^a \Sigma w^c \frac{\partial \varrho^b}{\partial x^c} - \varrho^b \Sigma w^c \frac{\partial \varrho^a}{\partial x^c}\right].$$

We recognize the simultaneous displacements (6.2), and find

$$T^{ab} = w^b \left\{ N e s^a - \tfrac{1}{2} \Sigma \frac{\partial N e s^a s^c}{\partial x^c} \right\} - w^a \left\{ N e s^b - \tfrac{1}{2} \Sigma \frac{\partial N e s^b s^c}{\partial x^c} \right\} -$$

$$- \tfrac{1}{2} N e s^a \Sigma s^c \frac{\partial w^b}{\partial x^c} + \tfrac{1}{2} N e s^b \Sigma s^c \frac{\partial w^a}{\partial x^c} + \tfrac{1}{2} Ne\left(s^a \frac{ds^b}{dx^{(4)}} - s^b \frac{ds^a}{dx^{(4)}} \right).$$

8. Taking $b = 4$, some terms vanish, and we get

$$T^{a4} = N e s^a - \tfrac{1}{2} \Sigma\,(c) \frac{\partial N e s^a s^c}{\partial x^c}.$$

Remembering what has been found about the polarization in (6.3), we at once see that T^{a4} $(a = 1, 2, 3)$ are the *components of the polarization.* Thus the polarization is no 4-dimensional vector: its components are the space-time-components of a tensor.

When neither a nor b have the value 4, then the part of T^{ab} containing the polarization:

$$w^b \left\{ N e s^a - \tfrac{1}{2} \Sigma \frac{\partial N e s^a s^c}{\partial x^c} \right\} - w^a \left\{ N e s^b - \tfrac{1}{2} \Sigma \frac{\partial N e s^b s^c}{\partial x^c} \right\}$$

is nothing else but a component of the well known RONTGEN-*vector*, which in three-dimensional analysis is written [**p** . **w**], where **p** and **w** are the three-dimensional polarization and velocity vectors. We see that in our tensor the *components of polarization are always accompanied by the components of the corresponding* RÖNTGEN-*vector.*

9. In another part of T^{ab} $(a \neq 4,\ b \neq 4)$, viz.

$$cm^{ab} = \tfrac{1}{2} Ne\left(s^a \frac{ds^b}{dx^{(4)}} - s^b \frac{ds^a}{dx^{(4)}} \right),$$

we recognize the *components of magnetization.*

The remaining part however:

$$- \tfrac{1}{2} Nes^a \Sigma\,(c)\, s^c \frac{\partial w^b}{\partial x^c} + \tfrac{1}{2} Nes^b \Sigma\,(c)\, s^c \frac{\partial w^a}{\partial x^c},$$

indicates the existence of *a new effect.* It is of the second order and therefore has been neglected by LORENTZ[1]) and by CUNNINGHAM[2]). BORN does not separate it from the magnetization. But we can imagine an experiment (see below) where this effect will manifest itself apart from magnetism. So we shall keep these terms apart.

Here the quadratic electric moments of the atoms appear:

$$es^a\, s^b,$$

the same quantities which occur in recent papers of DEBIJE and HOLTSMARK on the broadening of spectral lines from luminous gases under increased pressures.[3]) Half the sum of these quantities per unit of volume we shall call the *electrical extension* of matter, unless a better name be proposed. If an atom contains more than one electron, then we can have an electrical extension without polarization. We denote it by

$$K^{ab} = \tfrac{1}{2}\, Nes^a\, s^b.$$

and the corresponding part of the tensor can be written

$$k^{ab} = -\Sigma\,(c) \left\{ K^{ac} \frac{\partial w^b}{\partial x^c} - K^{bc} \frac{\partial w^a}{\partial x^c} \right\}.$$

10. In order to review the results reached thus far, let us gather them in a scheme, and let us for convenience' sake use rectangular coordinates x, y, z; t for the time, and three-dimensional notations for the (three-dimensional) vectors of polarization, magnetization, and velocity: $\mathbf{p}, \mathbf{m}\ (\mathbf{m}_x = m^{23}$, etc.) and $\mathbf{w}$. In addition, write ${}^2\mathbf{K}$ for the three-dimensional extension tensor, and for the new vector $\mathbf{k}$:

$$\mathbf{k} = -\,[({}^2\mathbf{K} \cdot \nabla) \cdot \mathbf{w}],$$

where $({}^2\mathbf{K}.\nabla)$ is an operator having vector properties. Thus $\mathbf{k}_x = k^{23}$, etc. Then the contents of the tensor T^{ab} are:

$$T^{ab}: \quad \begin{array}{c} \xrightarrow{b} \\ \downarrow a \end{array} \quad \begin{array}{cccc} & c\mathbf{m}_z + \mathbf{k}_z + [\mathbf{p.w}]_z & -c\mathbf{m}_y - \mathbf{k}_y - [\mathbf{p.w}]_y & \mathbf{p}_x \\ -c\mathbf{m}_z - \mathbf{k}_z - [\mathbf{p.w}]_z & & c\mathbf{m}_x + \mathbf{k}_x + [\mathbf{p.w}]_x & \mathbf{p}_y \\ c\mathbf{m}_y + \mathbf{k}_y + [\mathbf{p.w}]_y & -c\mathbf{m}_x - \mathbf{k}_x - [\mathbf{p.w}]_x & & \mathbf{p}_z \\ -\mathbf{p}_x & -\mathbf{p}_y & -\mathbf{p}_z & \end{array} .$$

Applying the formula for the current from the bound electrons:

1) Encyclopaedie der Mathem. Wissenschaften.

2) The Principle of Relativity, Camb. Univ. Press.

3) P. DEBYE, *Das molekulare elektrische Feld in Gasen*, Phys. Ztschr. **20**, p. 160, 1919.

J. HOLTSMARK, *Ueber die Verbreiterung von Spektrallinien*, ib. p. 162.

See also P. DEBIJE, *Die* VAN DER WAALS*schen Kohasionskrafte*, Phys. Zschr. **21**, p. 178, 1920.

$$\Sigma\,(b)\,\frac{\partial T^{ab}}{\partial x^c},$$

and putting it in the right hand members of the equations of the field, we get for the fundamental equations for moving non-conducting media, in three-dimensional vector notation:

$$rot\,\mathbf{B} - \frac{1}{c}\dot{\mathbf{E}} = rot\,\mathbf{m} + \frac{1}{c}\,rot\,\mathbf{k} + \frac{1}{c}\,rot\,[\mathbf{p}.\mathbf{w}] + \frac{1}{c}\dot{\mathbf{p}},$$

and

$$div\,\mathbf{E} = -\,div\,\mathbf{p}.$$

These are LORENTZ' equations with the addition of $rot\,\mathbf{k}$ to the current. We see a polarization current $\dot{\mathbf{p}}$, a RÖNTGENcurrent $rot\,[\mathbf{p}.\mathbf{w}]$ and the current of magnetization $rot\;c\mathbf{m}$.

A proposed Experiment.

11. Let us inquire further into the nature of the second order current

$$rot\,\mathbf{k}.$$

Referring to the definition:

$$\mathbf{k} = -\,[({}^{2}\mathbf{K}\,.\,\nabla)\,.\,\mathbf{w}],$$

we see that it is an effect due to the non-uniformity of motion in matter where the atomical charges lie outside one another. If these charges had fixed positions, i.e. if the electrons were rigidly fixed between the nuclei and if they therefore could be said to have exactly the motion of matter in bulk (i.e. of the nuclei, or rather motions interpolated between the nuclei) then our calculus indicates, that there would be no current resulting from the charges: the streams of positive and negative particles cancelling each other.

But in this case, the motion of matter being non-uniform, the electrons clearly would turn round the nuclei in an absolute sense, and the atoms would have a magnetic momentum. It is the part of $\mathbf{k}$ to counterbalance this slight magnetization, it then equals $c\mathbf{m}$ with opposite sign.

On the other hand, in case the electrons, instead of being rigidly fixed in the frame of the nuclei, always kept the same distance and in the same direction from the nuclei, not turning round in the rotating motion of matter, then $\mathbf{k}$ comes into play, not being balanced by a slight magnetization; so an induction field will be produced.

It should be possible to keep the electrons in the same direction from the nuclei by applying an electric field and maintaining a constant polarization. A rotatory motion then should produce an induction. We must be careful, however, to separate this from the

RÖNTGEN-effect, by eliminating the latter. This might be done in the following way:

Take a sphere of insulating material, which is mounted to perform rotatory oscillations round a vertical axis. Surround its equator by a circuit fixed in space. Apply an electric field of constant horizontal direction, and the oscillations of the sphere must induce an oscillating current in the circuit.

The effect will be small, but it should be detectable with the aid of the modern detectors of radiotelegraphy. It will be proportional to the square of the electric field applied.

It might be pointed out that a comparison of the effect with the produced polarization, would provide us with means to determine the number of electrons per atom, which are involved in the polarization, because, for a given polarization, the displacement **s** of the electrons is inversely proportional to the number n of displaced electrons per atom, and so the effect of **k** per electron is inversely to n^2. Materials with the same di-electric constant should show the effect to a degree inversely proportional to the number of polarizing electrons per atom.

Spontaneous Electric Polarization of Moving Magnets.

12. Though we have used in the title of this paper the denominations "Polarization and Magnetization Electrons", yet it is well known that it is impossible to make a rigorous distinction between the two. For even though there may be in some cases electrons which only produce polarization and no magnetization, there can be no electron which gives rise to a magnetization and never produces polarization.

In fact, whenever magnetized matter moves in a direction perpendicular to the magnetization, then it shows a polarization at right angles both to magnetization and motion.

The explanation runs as follows. A magnetic atom contains electrons sweeping round the nucleus, in circles, say, with uniform velocity, under the actions of electromagnetic forces. When the atom acquires a motion in the plane of the circling electrons, then the forces are modified in a way given by the theory of electrons and of relativity. The effect of this alteration of the forces will be that the orbit is no longer a circle, and becomes an ellipse, and that the velocity changes in such a way that the electrons during a longer time stay in one part of the ellipse near an end of the long axis than in the other. This clearly results into a polarization.

We shall call this the *polarization of moving magnetism*. It explains why no current is set up in a moving magnet on account of a motion perpendicular to its own internal induction field, so that with sliding contacts no current can be taken off. Thus, e. g., if we take a circular spring, the two ends pressing together, we can put a long magnet into it. Suppose that we can draw the magnet across the ring, the ends of the spring giving way and making a sliding contact: there will arise no current in the ring if we do it.

Again, this polarization is responsible for the electric force set up in a homogeneous magnetic field if the magnets producing the latter acquire a uniform motion at right angles to the field. The magnetic field may remain stationary and homogeneous: nevertheless an electric force will be induced by the motion of the magnets.

Afterwards these problems will be treated more adequately when we shall have explained the character of our deductions from the relativity point of view (see below § 20).

Then we shall also define a distinction between the di-electric polarization which is independent in itself, and the polarization of moving magnetism.

The Invariancy of the Results.

13. Thus far we did not want to refer to a single theorem of the theory of relativity to deduce our results. Nevertheless they possess the property of complete invariancy, not only in EINSTEIN-MINKOWSKI's theory of restricted relativity, but also in EINSTEIN's theory of general relativity. We proceed to show this.

This theory ascribes to a four-dimensional track the length ds:

$$ds^2 = \Sigma\,(ab)\,g_{ab}\,dx^a\,dx^b,$$

if dx^a $(a = 1\,..\,4)$ define the increments of the coordinates and time. The determinant of the g_{ab} is called g, and its minors divided by g are denoted g^{ab}.

What is the character of Nw^a? Remembering the definition (§ 3:)

$$Nw^a = \frac{\sqrt{g}\,N\,dV\,dx^a}{\sqrt{g}\,dV\,dx^{(4)}},$$

we notice that $N\,dV$ is a number, dx^a is a contravariant vector and $\sqrt{g}\,dV\,dx^{(4)}$ constitutes a scalar. Thus Nw^a is a contravariant vector multiplied by $\sqrt{g}$.

θr^a is a contravariant vector too, and so

$$\theta r^a\,Nw^b - \theta r^b\,Nw^a$$

is an skew-symmetrical contravariant tensor, multiplied by $\sqrt{g}$. (This is sometimes called a volume-tensor or a tensor-density, after WEYL). Then we know that

$$\delta\, Nw^a = \Sigma\,(b)\frac{\partial}{\partial x^b}\{\theta r^a\, Nw^b - \theta r^b\, \delta\, Nw^a\}$$

is the contravariant vector-divergency of this tensor, multiplied by $\sqrt{g}$, and thus of the same nature as Nw^a itself.

In like manner the second variation

$$\delta^2\, Nw = \Sigma\,(b)\,\frac{\partial}{\partial x^b}\{\theta r^a\, \delta Nw^b - \theta r^b\, \delta Nw^a\}$$

is a contravariant vector multiplied by $\sqrt{g}$ again.

It follows that our results are in complete accordance with relativity theory in the most general sense, and we are justified in applying any theorem of that theory.

Having thus recognized the true character of our tensor, we shall henceforth write $\sqrt{g}\; T^{ab}$ instead of T^{ab}

$$\sqrt{g}\; T^{ab} = e\theta r^a\, Nw^b - e\theta r^b\, Nw^a + \tfrac{1}{2}\, e\{\theta r^a\, \delta\, Nw^b - \theta r^b\, \delta\, Nw^a\}.$$

This will cause no confusion.

We must further keep in mind that w^a is no four-dimensional vector, but $w^a\, dx^{(4)}/ds$ is. We shall not introduce a new notation for this velocity vector.

The General Covariant Equations for the Field.

14. The covariant tensor of the field can be written as the rotation of the potential vector φ_a:

$$f_{ab} = \frac{\partial \varphi_b}{\partial x^a} - \frac{\partial \varphi_a}{\partial x^b}, \qquad (a = 1, 2, 3, 4;\quad b = 1 \,..\, 4). \tag{14.1}$$

From these we get the contravariant components:

$$f^{ab} = \Sigma\,(cd)\, g^{ac}\, g^{bd}\, f_{cd},$$

and the fundamental equations of the theory of electrons are

$$\Sigma\, \frac{\partial}{\partial x^b}\,(\sqrt{g}\, f^{ab}) = \varrho v^a. \tag{14.2}$$

where ϱ is the density of the electric charges, and ϱv^a is a contravariant vector multiplied by $\sqrt{g}$.

From the relations (14.1) arises another equation. Multiply by the contravariant fourth rank tensor $\frac{1}{2}\delta^{abcd}/\sqrt{g}$, and contract twice. Here δ^{abcd} is 1 whenever the figures $abcd$ constitute an even permutation of 1234, and in other cases vanishes. Then we get the conjugate tensor f_*^{ab}[1]):

$$f_*^{ab} = \Sigma\,(cd)\,\frac{1}{2\sqrt{g}}\,\delta^{abcd} f_{cd}.$$

[1]) In order to get the covariant conjugate tensor components f^*_{ab} from the contravariant f^{cd}, multiply in the same way by the covariant tensor

$$\tfrac{1}{2}\sqrt{g}\,\delta_{abcd}\cdot(\delta_{abcd} = \delta^{abcd}).$$

If now we write

$$\Sigma\,(b)\frac{\partial}{\partial x^b}(\sqrt{g} f_*^{ab}) = 0, \tag{14.3}$$

this must be an identity in virtue of (14.1).

The MINKOWSKIAN force acting on a moving charge e has the covariant components:

$$f_a = e\,\Sigma\,(b)\,\frac{dx^{(4)}}{ds}\,w^b f_{ab}.$$

These equations are supposed to hold within the finest structure of matter.

To obtain the equations of matter in bulk, we take the mean over a small region, containing a great many atoms. We define

$$F_{ab} = \frac{\int f_{ab}\sqrt{g}\,dx^{(1)}\ldots dx^{(4)}}{\int \sqrt{g}\,dx^{(1)}\ldots dx^{(4)}}, \qquad F^{ab} = \frac{\int f^{ab}\sqrt{g}\,dx^{(1)}\ldots dx^{(4)}}{\int \sqrt{g}\,dx^{(1)}\ldots dx^{(4)}};$$

It is readily seen that still $F^{ab} = \Sigma(cd)\,g^{ac}g^{bd}F_{cd}$.

The mean of the convection current ϱv^a, as produced by the bound electrons, we have just found, and so the equations for non-conducting matter are:

$$\Sigma\,(b)\frac{\partial}{\partial x^b}(\sqrt{g}\,F^{ab}) = \Sigma\,(b)\frac{\partial}{\partial x^b}(\sqrt{g}\,T^{ab}). \tag{14.41}$$

In conducting matter, the current from the conduction electrons $\sqrt{g}I^a$ must be added in the right hand member.

The other equations become

$$\Sigma\,(b)\frac{\partial}{\partial x^b}(\sqrt{g}\,F_*^{ab}) = 0 \tag{14.42}$$

Now, we could try a solution $F^{ab} = T^{ab}$, and add a solution E^{ab} of the equations

$$\Sigma\,(b)\frac{\partial}{\partial x^b}(\sqrt{g}\,E^{ab}) = 0 \tag{14.51}$$

and

$$\Sigma\,(b)\frac{\partial}{\partial x^b}(\sqrt{g}\,E_*^{ab}) = -\,\Sigma\,(b)\frac{\partial}{\partial x^b}(\sqrt{g}\,T_*^{ab}), \tag{14.52}$$

E_*^{ab} and T_*^{ab} being the conjugate tensors of E^{ab} and T^{ab}. Then

$$F^{ab} = T^{ab} + E^{ab}$$

is a solution of equations (14.41) and (14.42). We shall call T^{ab} the *internal*, and E^{ab} the *external* field.

Separation of the Polarization and the Magnetization Tensor.

15.1. It has been remarked, that in our tensor $\sqrt{g}\,T^{ab}$ the polarization and magnetization are intimately interwoven. Indeed, it is not always easy to separate them.

We shall assume that our system of reference is such that g_{a4} vanishes for $a = 1, 2, 3$. This means, that in our system of reference the velocity of beams of light in opposite directions is the same. This also implies that whenever the first three contravariant components of velocity vanish, also the first three covariant components vanish.

Then we may in stationary points of matter (or in an arbitrary point after changing variables in a way that renders it stationary) separate T^{ab} into two tensors, one consisting of the $(a4)$-components, the other components vanishing, and vice versa. These tensors we may call the *polarization and magnetization tensors.* It should be understood, that a change of variables, or a motion of matter, cannot leave half of the components zero, as they are in stationary points: the magnetization tensor, e.g., completes itself with polarization components.

Write ${}_0T^{ab}$ for T^{ab} in a stationary point, and separate:

$${}_0T^{ab} = {}_0M^{ab} + {}_0P^{ab},$$

so that

$${}_0M^{ab} \,(=)\quad \begin{matrix} & {}_0T^{12} & {}_0T^{13} & 0 \\ {}_0T^{21} & & {}_0T^{23} & 0 \\ {}_0T^{31} & {}_0T^{32} & & 0 \\ 0 & 0 & 0 & \end{matrix}\,,$$

and

$${}_0P^{ab} \,(=)\quad \begin{matrix} & 0 & 0 & {}_0T^{14} \\ 0 & & 0 & {}_0T^{24} \\ 0 & 0 & & {}_0T^{34} \\ {}_0T^{41} & {}_0T^{42} & {}_0T^{43} & \end{matrix}\,.$$

We could have taken the covariant components of the tensor T_{ab} and the separation would have had the same effect. This is due to the fact that g_{a4} vanishes for $a = 1, 2, 3$. If this is not the case, then we have first to change variables to make them vanish and afterwards make the separation.

15.2. We shall now briefly indicate what becomes of the constitutive relations between polarization and electric force, magnetization and magnetic force, and between the conduction current and electric force. We proceed in a quite formal way

First, to find the generalization of the equation $\mathbf{P} = (\varepsilon - 1)\,\mathbf{E}$, we form from the field-tensor a force-vector F^a :

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$$F^a = \Sigma\,(b)\,\frac{dx^{(4)}}{ds}\,w_b\,F^{ab},$$

and from the polarization tensor we form a vector P^a:

$$P^a = \Sigma\,(b)\,\frac{dx^{(4)}}{ds}\,w_b\,P^{ab}.$$

and the required generalization will be

$$P^a = -(\varepsilon - 1)\,F^a.$$

Secondly, to generalize the relation $\mathbf{B} = \mu\mathbf{H}$, or rather

$$\mathbf{M} = \frac{\mu - 1}{\mu}\,\mathbf{B},$$

we proceed in a similar manner. From the conjugate field tensor we form a vector G_a:

$$G_a = \Sigma\,(b)\,\frac{dx^{(4)}}{ds}\,w^{(b)}\,F_{*ab},$$

and from the conjugate magnetization tensor a vector Q_a:

$$Q_a = \Sigma\,(b)\,\frac{dx^{(4)}}{ds}\,w^b\,M_{*ab}.$$

The generalized relation is

$$Q_a = -\frac{\mu - 1}{\mu}\,G_a.$$

The current of the free electrons is partly a convection current, partly a conduction current. The latter will be the component of the four-dimensional vector-density $\sqrt{g}I^a$ in a direction perpendicular to the four-dimensional velocity vector. The conduction vector thus is:

$$J^a = I^a - w^a \left\{\frac{dx^{(4)}}{ds}\right\}^2 \Sigma\,(b)\,w_b\,I^b.$$

This can be put otherwise, if we first form a skew-symmetrical tensor

$$I^{ab} = \frac{dx^{(4)}}{ds}\{I^a w^b - I^b w^a\}$$

and afterwards from this tensor form a vector again:

$$J^a = \Sigma\,(b)\,\frac{dx^{(4)}}{ds}\,w_b\,I^{ab}.$$

The equation for the conduction current must be

$$J^a = -\lambda\,F^a.$$

We notice that in the common equation $\mathbf{J} = \sigma\mathbf{E}$, $\sigma = \lambda\sqrt{g}$.

16.1. Now take the contravariant tensor P^{ab} and form its conjugate:

$$P_{*ab} = \Sigma\,(cd)\,\tfrac{1}{2}\,\sqrt{g}\,\delta_{abcd}\,P^{cd}.$$

then we get the conjugate tensor with covariant components

$$
P_{*ab} \,(=)\, \begin{array}{cccc}
 & \sqrt{g}P^{34} & -\sqrt{g}P^{24} & \sqrt{g}P^{23} \\
-\sqrt{g}P^{34} & & \sqrt{g}P^{14} & \sqrt{g}P^{31} \\
\sqrt{g}P^{24} & -\sqrt{g}P^{14} & & \sqrt{g}P^{12} \\
-\sqrt{g}P^{23} & -\sqrt{g}P^{31} & -\sqrt{g}P^{12} &
\end{array}
$$

By multiplying by the velocity vector and contracting,

$$\Sigma\,(b)\,\frac{dx^{(4)}}{ds}\,w^b P_{*ab},$$

we get a vector. This vector clearly vanishes in a stationary point, because $w^{(1)}$, $w^{(2)}$, $w^{(3)}$, and ${}_0P_{*a4}$ vanish, and it therefore *always* vanishes. Thus we conclude that we shall always have

$$0 = w^{(2)}\sqrt{g}P^{34} - w^{(3)}\sqrt{g}P^{24} + \sqrt{g}P^{23}, \tag{16.1}$$

and similar relations for cyclic permutations of the figures 123. It is thus confirmed that where $\sqrt{g}P^{a4}$ $(a = 1, 2, 3)$ are polarization components, the other components of this tensor consist of components of the corresponding RÖNTGEN-vector.

16.2. Apply a similar reasoning to the magnetization tensor. Multiply by the velocity vector and contract:

$$\Sigma\,(bc)\,g_{bc}\frac{dx^{(4)}}{ds}\,w^c\,M^{ab} = \Sigma(b)\frac{dx^{(4)}}{ds}\,w_b\,M^{ab},$$

This will be a vector vanishing in stationary points, since w_1, w_2, w_3, and ${}_0M^{a4}$ vanish. Therefore it will always vanish, and we shall have

$$0 = w_2\,M^{12} + w_3\,M^{13} + w_4\,M^{14}. \quad (cycl.\ 123). \tag{16.2}$$

Here we meet the polarization of moving magnetism, $\sqrt{g}M^{a4}$, in terms of M^{ab}. We know from §§ 8, 9 that $\sqrt{g}M^{ab}$ must contain, besides the components of the magnetization and of k^{ab}, the components of the RÖNTGEN-vector corresponding to the polarization of moving magnetism also.

This will afford us means completely to express the polarization of moving magnetism in terms of the magnetization and **k** of moving matter (§ 19).

Comparison with Other Theories.

17. In constructing the polarization tensor EINSTEIN, following MINKOWSKI, starts from the vector P^a defined in § 15.2, and he puts for his tensor [1])

[1]) *Die formale Grundlage der allgemeinen Relativitatstheorie*, Berl. Sitz, 41, p. 1065, 1914.

$$\frac{dx^{(4)}}{ds}\{P^a w^b - P^b w^a\}.$$

In order to show that this is the same as our tensor P^{ab}, take a special case, $a=1$, $b=2$ e.g., and write in full

$$\frac{dx^{(4)}}{ds}\{P^a w^b - P^b w^a\} = \left\{\frac{dx^{(4)}}{ds}\right\}^2 \Big\{ w^{(2)}(w_2 P^{12} + w_3 P^{13} + w_4 P^{14}) -$$
$$- w^{(1)}(w_1 P^{21} + w_3 P^{23} + w_4 P^{24})\Big\}.$$

We can rearrange:

$$= \left\{\frac{dx^{(4)}}{ds}\right\}^2 \Big\{ P^{12}(w^{(1)} w_1 + w^{(2)} w_2 + w^{(3)} w_3 + w^{(4)} w_4) +$$
$$+ w_3 (w^{(1)} P^{32} + w^{(2)} P^{13} + w^{(3)} P^{21}) + w_4 (w^{(1)} P^{42} + w^{(2)} P^{14} + w^{(4)} P^{21})\Big\}.$$

and now we remark that the latter two bracket forms vanish in virtue of (16.1), for

$$\frac{dx^{(4)}}{ds}(w^{(1)} P^{32} + w^{(2)} P^{13} + w^{(3)} P^{21}) =$$
$$= \frac{1}{\sqrt{g}}\frac{dx^{(4)}}{ds}(w^{(1)} P_{*41} + w^{(2)} P_{*42} + w^{(3)} P_{*43}) = 0.$$

As

$$\left\{\frac{dx^{(4)}}{ds}\right\}^2 \Sigma w^b w_b = 1.$$

the required identity is shown to exist.

In the same way it can be shown that the magnetization tensor or rather its conjugate in the form

$$\frac{dx^{(4)}}{ds}\{Q_a w_b - Q_b w_a\}$$

agrees with our M_{*ab}.

18. Let us make the simplifying assumption of the absence of gravitation. Then the g_{ab} and g^{ab} have the values:

$$g_{ab}\,(=)\;\begin{matrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & c^2 \end{matrix}\;,\quad g^{ab}\,(=)\;\begin{matrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1/c^2 \end{matrix}\;,\quad g = -c^2 \;^{1)}$$

If $\mathbf{A}$ and φ denote the common vector and scalar potentials, then the components φ_a are $\mathbf{A}_x, \mathbf{A}_y, \mathbf{A}_z$ and $-c\varphi$. The components of the field are

[1]) In order to avoid imaginaries, we shall everywhere in $\sqrt{g}$ take $|g|$.

$$F_{ab} \quad (=) \quad \begin{array}{cccc} & \mathbf{B}_z & -\mathbf{B}_y & c\mathbf{E}_x \\ -\mathbf{B}_z & & \mathbf{B}_x & c\mathbf{E}_y \\ \mathbf{B}_y & -\mathbf{B}_x & & c\mathbf{E}_z \\ -c\mathbf{E}_x & -c\mathbf{E}_y & -c\mathbf{E}_z & \end{array},$$

$$F^{ab} \quad (=) \quad \begin{array}{cccc} & \mathbf{B}_z & -\mathbf{B}_y & -\mathbf{E}_x/c \\ -\mathbf{B}_z & & \mathbf{B}_x & -\mathbf{E}_y/c \\ \mathbf{B}_y & -\mathbf{B} & & -\mathbf{E}_z/c \\ \mathbf{E}_x/c & \mathbf{E}_y/c & \mathbf{E}_z/c & \end{array}.$$

The equations for the field are (**14.41**)

$$\Sigma(b)\frac{\partial}{\partial x^b}(\sqrt{g}\,F^{ab}) = \Sigma(b)\frac{\partial}{\partial x^b}(\sqrt{g}\,P^{ab} + \sqrt{g}\,M^{ab}),$$

and we have, if $\mathbf{P}$ is the principal di-electric polarization:

$$\sqrt{g}\,P^{ab} \quad (=) \quad \begin{array}{cccc} & [\mathbf{Pw}]_z & -[\mathbf{Pw}]_y & \mathbf{P}_x \\ -[\mathbf{Pw}]_z & & [\mathbf{Pw}]_x & \mathbf{P}_y \\ [\mathbf{Pw}]_y & -[\mathbf{Pw}]_x & & \mathbf{P}_z \\ -\mathbf{P}_x & -\mathbf{P}_y & -\mathbf{P}_z & \end{array}, \qquad (18.1)$$

and

$$\sqrt{g}M^{ab}(=) \begin{array}{cccc} & c\mathfrak{m}_z+\mathfrak{k}_z+[\mathfrak{n}.\mathbf{w}]_z & -c\mathfrak{m}_y-\mathfrak{k}_y-[\mathfrak{n}.\mathbf{w}]_y & \mathfrak{n}_x \\ -c\mathfrak{m}_z-\mathfrak{k}_z-[\mathfrak{n}.\mathbf{w}]_z & & c\mathfrak{m}_x+\mathfrak{k}_x+[\mathfrak{n}.\mathbf{w}]_x & \mathfrak{n}_y \\ c\mathfrak{m}_y+\mathfrak{k}_y+[\mathfrak{n}.\mathbf{w}]_y & -c\mathfrak{m}_x-\mathfrak{k}_x-[\mathfrak{n}.\mathbf{w}]_x & & \mathfrak{n}_z \\ -\mathfrak{n}_x & -\mathfrak{n}_y & -\mathfrak{n}_z & \end{array} \qquad (18.2)$$

where $\mathfrak{n}$ denotes the (electric) polarization of moving magnetism.

For the conjugate tensor of the field we have

$$F_*{}^{ab} \quad (=) \quad \begin{array}{cccc} & \mathbf{E}_z & -\mathbf{E}_y & \mathbf{B}_x/c \\ -\mathbf{E}_z & & \mathbf{E}_x & \mathbf{B}_y/c \\ \mathbf{E}_y & -\mathbf{E}_x & & \mathbf{B}_z/c \\ -\mathbf{B}_x/c & -\mathbf{B}_y/c & -\mathbf{B}_z/c & \end{array}$$

We see that the equations (**14.42**) amount to

$$c\,rot\,\mathbf{E} + \dot{\mathbf{B}} = 0, \qquad (18.31)$$

$$div\,\mathbf{B} = 0, \qquad (18.32)$$

From the equations of the field we see that

$$div\,\mathbf{E} = -\,div\,(\mathbf{P} + \mathfrak{n}), \qquad (18.41)$$

and

$$c\,rot\,\mathbf{B} - \dot{\mathbf{E}} = rot\,(c\mathfrak{m} + \mathfrak{k} + [\mathfrak{n}.\mathbf{w}] + [\mathbf{P}.\mathbf{w}]) + [\dot{\mathbf{P}} + \dot{\mathfrak{n}}]. \qquad (18.42)$$

These are the equations we have met in § 10. Only we had not yet separated $\mathfrak{p} = \mathbf{P} + \mathfrak{n}$ there.

19. Let us solve $\mathfrak{n}$ in terms of $\mathfrak{m}$ and $\mathfrak{k}$. Referring to the equation of § 16 2 we must notice that

$$w_1 = -\mathfrak{w}_x, \qquad w_2 = -\mathfrak{w}_y, \qquad w_3 = -\mathfrak{w}_z, \qquad w_4 = g_{44} w^{(4)} = c^2,$$

and we get

$$c^2 \mathfrak{n}_x = \mathfrak{w}_y (c\mathfrak{m}_z + \mathfrak{k}_z + [\mathfrak{n} \cdot \mathfrak{w}]_z) - \mathfrak{w}_z (c\mathfrak{m}_y + \mathfrak{k}_y + [\mathfrak{n} \cdot \mathfrak{w}]_y)$$

or

$$c^2 \mathfrak{n} = [\mathfrak{w} \cdot (c\mathfrak{m} + \mathfrak{k} + [\mathfrak{n} \cdot \mathfrak{w}])]. \tag{19.1}$$

From this it is easily seen that

$$(\mathfrak{n} \cdot \mathfrak{w}) = 0,$$

and as

$$[\mathfrak{w} \cdot [\mathfrak{n} \cdot \mathfrak{w}]] = w^2 \mathfrak{n} - \mathfrak{w} (\mathfrak{n} \cdot \mathfrak{w}),$$

we get

$$\mathfrak{n}_x = \sqrt{g}\, M^{14} = \frac{[\mathfrak{w} \cdot (c\mathfrak{m} + \mathfrak{k})]_x}{c^2 \left(1 - \frac{w^2}{c^2}\right)}, \text{ a.so.} \tag{19.2}$$

and

$$\sqrt{g}\, M^{12} = \frac{c\mathfrak{m}_z + \mathfrak{k}_z}{1 - \frac{w^2}{c^2}} - \frac{\mathfrak{w}_z (\mathfrak{w} \cdot (c\mathfrak{m} + \mathfrak{k}))}{c^2 \left(1 - \frac{w^2}{c^2}\right)}, \text{ a.s.o.} \tag{19.3}$$

In this form our result for the magnetization tensor can be readily compared with the corresponding formulae of BORN [1]). He also points out the existence of the vector $\mathfrak{n}$ and states that it is the magnetic analogon to the RÖNTGEN-vector. We see that the factor $1/(1-w^2/c^2)$ disturbs the analogy. The difference in the appreciation of the result is this that BORN (apart from not separating $\mathfrak{k}$) takes the whole of the components $\sqrt{g}M^{23}$, $\sqrt{g}M^{31}$ and $\sqrt{g}M^{12}$ to be the components of magnetization and seems not to have become aware of the fact that they contain the RÖNTGEN-vector components belonging to $\mathfrak{n}$ as well as the magnetization components proper.

BORN emphasizes the complete symmetry of his electric and magnetic equations and certainly one can enjoy the mathematical beauty of the formulae thus written. It would, however, be erroneous to believe that the difference from LORENTZ' equations is more than a difference in form. Our investigation shows that the *physical contents* of BORN's equations is no other than what has been expressed by LORENTZ.

Action of Polarization of Moving Magnetism.

20. Let us illustrate some effects of $\mathfrak{n}$ by considering a long

[1]) l.c. form. 39 and 39', pp. 546 and 547.

magnet moving at right angles to its magnetization. We shall follow the distinction of "internal" and "external" field at the end of § 14. The effect of this electric polarization **n**, called into existence by the motion of magnetized matter, is to produce an internal electric field (18.41):

$$\mathrm{E} = -\,\mathrm{n}.$$

This could be expected to act on free electrons, present in the magnet, and cause a conduction current. But these electrons are carried along with matter and therefore are moving with velocity **w** through the internal magnetic field where the induction vector is (see § 18.42):

$$c\mathbf{B} = c\mathrm{m} + \mathrm{k} + [\mathrm{n}\,.\,\mathrm{w}]$$

and, where the external field may be neglected [1]), they consequently are subjected to the NEWTONian force

$$e\Big(\mathrm{E} + \frac{1}{c}\,[\mathrm{w}\,.\,\mathrm{B}]\Big).$$

This expression vanishes according to the formulae of §§ 16.2 and 19, so that the free electrons moving along with the magnet are not driven sideways.

Therefore it is impossible with sliding contacts at the magnet's sides to get a current from it, and the experiment with the long magnet drawn across a circular spring is explained. (§ 12).

On the other hand, if we cut the magnet at right angles to the magnetization, and take out an infinitely thin lamella, so that a thin wire might be kept in the same place while the magnet is drawn across, then the "external" field in this split will simply be the continuation of the internal field. and the free electrons in the wire, not sharing the motion of the magnet, will be subjected to the electric force **E** only, so that an induction current will be set up in the wire.

Thus we see that it is *the polarization of moving magnetism that accounts for the inductive force,* when a magnetic pole moves across a wire, *in a case where the magnetic-field is homogeneous and stationary.*

Conclusive Remarks.

21. In conclusion we may remark that the result of the first variation is wholly incorporated in the polarization tensor. The

[1]) Suppose the magnetization as being homogeneous, and the free poles of the magnet as being at infinite distance.

greater part of the result of the second variation is represented in the magnetization tensor.

Consider once more the complete polarization (6.3 and 6.2):

$$N e\left[\varrho^a - \tfrac{1}{2}\,\theta r^{(4)}\left\{\frac{d\varrho^a}{dx^{(4)}} - \Sigma\,\varrho^c\frac{\partial w^a}{\partial x^c}\right\} + \tfrac{1}{2}\,\Sigma\,\varrho^c\frac{\partial\varrho^a}{\partial x^c}\right] - \tfrac{1}{2}\,\Sigma\frac{\partial N e\varrho^a\varrho^c}{\partial x^c}.$$

Here $Ne\varrho^a$ is the term, by far the most important, which results from the first variation. It is difficult to tell in a few words, which part from the second order terms is exactly the polarization of moving magnetism. If the r^a are so chosen that $r^{(4)}$ vanishes in stationary points, then we can say that the greater part of

$$\tfrac{1}{2}\,Ne\,\Sigma\,\varrho^c\frac{\partial\varrho^a}{\partial x^c} - \tfrac{1}{2}\,\Sigma\frac{\partial N e\varrho^a\varrho^c}{\partial x^c} = \tfrac{1}{2}\,\delta N\,.\,e\varrho^a.$$

figures in the polarization tensor. A small fraction of it (in as much as δN is no scalar) appears, however, in the magnetization tensor, together with

$$-\tfrac{1}{2}\,Ne\,\theta r^{(4)}\left\{\frac{d\varrho^a}{dx^{(4)}} - \Sigma\,\varrho^c\frac{\partial w^a}{\partial x_c}\right\}$$

as the polarization of moving magnetism. But we refrain from entering into detail here.