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Mathematics. — *"A Congruence of Orthogonal Hyperbolas".* By Prof. JAN DE VRIES.

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1. In any plane through the given point C lies one orthogonal hyperbola o^2, resting on the four crossing lines a_k. The congruence $[o^2]$ defined in this way will be examined here.

Any straight line k is a chord of one o^2. If however k passes through C, it is a chord of ∞^1 curves; it is in this case a *singular chord.*

Also the four lines a are *singular*; for the plane through C and a_1 contains a pencil (o^2), having for base-points the intersections of a_2, a_3, a_4 and the orthocentre of the triangle defined by them.

Finally also the two transversals b_{1234} of the lines a are *singular* chords, for in the plane Cb_{1234} any line cutting b_{1234} at right angles, forms with it a figure belonging to $[o^2]$.

2. To determine the order of the locus of the curves o^2 which have a straight line l through C as a chord, we first consider the surface formed by the orthogonal hyperbolas passing through two points P_1 and P_2 and resting on the lines a_1 and a_2.

The scroll which has a_1 and a_2 for directrices and a plane perpendicular to $l \equiv P_1P_2$ for director plane, contains two straight lines resting on l; for this reason l is a component of two figures o^2. From this it ensues, that the surface in question is a *dimonoid* O^4, with triple points P_1, P_2 and double torsal line l. Through P_1 and P_2 pass therefore *four* curves o^2 resting on a_1, a_2 and a_3.

Let us now consider the locus of the o^2 which have l as a chord, rest on a_1, a_2, a_3, and pass through P_1. There pass four curves o^2 through any other point of l; hence l is quadruple on the surface in question, which is for this reason a *monoid* O^6 with fivefold point P_1. From this appears, that the locus of the o^2 resting on a_1, a_2, a_3, a_4 and having a line l as a chord, is an *axial surface* O^8 with sixfold line l.

According to a wellknown property the axial surface O^8 contains *twenty* pairs of lines. To these belong the *eight* pairs each consisting of a transversal of l, a_k, a_l, a_m and the perpendicular to it intersecting a_n and l. Each of the other twelve pairs consists of a transversal of l, a_k, a_l and a transversal of l, a_m, a_n perpendicular to it.

3. Through any point P pass *six* curves of the congruence $[o^2]$. For the locus of the o^2 which have CP as a chord and which rest on the lines a, has CP as a sixfold straight line.

Any point A_k of the line a_k is *singular*. The curves o^2 through A_k form a *monoid* O^6 with vertex A_k and fourfold straight line A_kC. It contains *fourteen* pairs of lines arising in the following way. *Three* pairs consist each of a transversal through A_k to a_l, a_m and a straight line intersecting a_n and $l \equiv A_kC$. *Two* pairs consist each of a transversal of l, a_l, a_m, a_n and the perpendicular out of A_k to this transversal. In order to find the other pairs we consider the cone formed by the perpendiculars b_k out of A_k to the transversals of l, a_l, a_m. As two of these transversals are perpendicular to l, b_k coincides twice with l. The cone in question is therefore cubical and has l as a double generatrix. Consequently there are three orthogonal pairs of lines of which the line b_k passes through A_k. In this way the *nine* remaining pairs are found.

4. Also the point C is *singular*. The determination of the order of the surface Γ formed by the curves o^2 passing through C, comes to the determination of the number of orthogonal hyperbolas through C resting on five straight lines 1, 2, 3, 4, 5. Using the principle of the conservation of the number we can suppose the straight lines 1, 2, and 3 to lie in a plane φ. Through C and the point 12 pass *four* o^2, resting on 3, 4, and 5; analogously we find *four* of them through C and 23 and *four* through C and 13.

All the other figures satisfying the conditions are pairs of lines of which one line, s, lies in φ, while the other, t, passes through C. To these belongs in the first place the line s in φ intersecting 4 and 5, in combination with the perpendicular t out of C to s.

Let us now consider the plane pencil (s) in φ which has the intersection M of 4 for vertex. The perpendiculars out of C to the rays of (s) form a quadratic cone; the two generatrices t resting on 5, belong each to an orthogonal pair of lines (s, t). As we can interchange 4 and 5, the group considered contains *four* pairs (s, t).

Finally we find the figure formed by the transversal t through C to 4 and 5, combined with the line s in φ cutting it at right angles. In all we found $3 \times 4 + 1 + 2 \times 2 + 1 = 18$ figures o^2; the curves o^2 through C form consequently a surface Γ^{18}.

5. Any ray through C is a chord of *six* o^2, belonging to Γ; hence C is a *twelvefold* point.

The transversal b_{12} through C to a_1 and a_2 is cut at right angles by two transversals of a_3 and a_4; the *six* lines b_{kl} are accordingly *double lines* of Γ. To them 12 single lines are connected.

To each t_{123} of a_1, a_2, a_3 we draw the perpendicular b out of C and we consider the cone which has the straight lines b as generatrices. Let γ be a plane through C and a straight line c of the scroll to which a_1, a_2, a_3 belong. Through the intersection D of t_{123} we draw in γ the straight line d perpendicular to c. As c is cut at right angles by two lines t_{123}, d coincides twice with c, envelops consequently a curve of the third class with double tangent c. The three lines d meeting in C are generatrices of the cone (b); this is consequently cubical and there are three pairs of lines (b, t_{123}). In all we find *twelve* pairs of lines o^2 of which one of the lines rests on three straight lines a.

Finally there lie on Γ the two transversals b_{1234} each connected to a straight line through C.

Each of the four o^2 which have a line a as a chord, is a *double curve* of Γ.

6. To find the order of the surface Δ formed by the o^2 resting on a straight line l, we try to find the number of curves o^2, in planes through C, which rest on six straight lines 1, 2, 3, 4, 5, 6, and again suppose 1, 2, 3 to lie in a plane φ.

Through the point 12 pass *six* o^2 resting on 3, 4, 5, 6, while their planes pass through C. Analogously *six* pass through 23 and *six* through 13. All the other figures degenerate into a straight line s of φ and a line t cutting it at right angles.

The plane through C and the intersections of 4 and 5 with φ contains a figure (s, t) of which the line t rests on 6. We obtain here a group of three pairs (s, t).

If s is to pass through the point $D \equiv (4, \varphi)$, t must rest on 5, 6 and CD_4. The orthogonal projections t' of the straight lines of the scroll $(t)^2$ envelop a conic. Let the perpendicular r out of D_4 to t' be associated to the ray s joining D_4 with the intersection T of a line t; r being perpendicular to two lines t', hence associated to two rays s, there are three coincidences $r \equiv s$. We find therefore *three* pairs of lines (s, t) satisfying the given conditions; in all a group of 3×3 figures o^2.

From this it ensues at the same time, that the straight line r cutting the ray t in T at right angles, envelops a curve of the fourth

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class, for through D_4 passes also the line r at D_4 perpendicular to the ray t of which D_4 is the intersection.

Finally we have to consider the case that the line t rests on 4, 5 and 6. If we now also project the scroll $(t)^2$ orthogonally on φ and draw through the intersection T of t and φ the line r perpendicular to t, r envelops, as appeared above, a curve of the fourth class. From this follows, that also the plane (rt) envelops a curve of the fourth class, so that through C there pass *four* planes in each of which a transversal of 4, 5, 6 is cut perpendicularly by a transversal of 1, 2, 3.

In all we found $3 \times 6 + 3 + 3 \times 3 + 4 = 34$ figures o^2; the locus of the o^2 resting on a straight line l, is consequently a surface A^{34}.

The curve o^2 in the plane (Cl) is apparently a *double curve.* The four lines a are *sixfold* on A; for the curves o^2 through a point of a form a surface O^6.

7. The planes Ca_k may be called singular because they contain ∞^1 orthogonal hyperbolas. This will also be the case when a plane through C cuts the lines a_k in an orthocentrical group. Now the orthocentres of the triangles $A_1 A_2 A_3$ of which the planes pass through C, form a surface; there must therefore be a finite number of singular planes of the kind in question.

In order to determine this number, we first consider the locus of the orthocentre H of a triangle CA_1A_2, when A_1 lies on a_1, A_2 on a_2. The plane through a point A_1 perpendicular to the ray A_1C contains one point A_2, hence one triangle A_1A_2C of which H lies in A_1. Consequently the surface in question contains the straight lines a_1 and a_2.

In the plane Ca_1 lie ∞^1 triangles A_1A_2C; their orthocentres lie in a conic H^2 through C and the intersection D_2 of a_2. The intersection of the surface with Ca_1 consists of a_1 and H^2; we have therefore a surface H^3. Three times H lies on a_3, or, in other words, through C pass three planes in which the orthocentre of $A_1A_2A_3$ lies in C.

We consider now the surface formed by the orthocentres of the triangles $A_1A_2A_3$ of which the planes pass through C.

If H is to get on a_1, A_1A_2 must be perpendicular to A_1A_3. In each plane through a fixed straight line A_1C we draw through A_1 the line l perpendicular to A_1A_2. If this plane is perpendicular to A_1C, l coincides with A_1C; hence l describes a quadratic cone. Two of the generatrices intersect a_3; through A_1C pass consequently two planes in which H coincides with A_1. But then a_1 is a *double straight line* of the surface in question.

A line A_1C is cut at right angles by two lines A_2A_3; it contains therefore two points H, which as a rule lie neither on a_1 nor in C. It has appeared above, that there are three rays A_1C on each of which one of the points H lies in C; the pairs of points H form consequently a curve H_5 with triple point C.

Finally the plane Ca_1 contains a conic which is the locus of the orthocentre of a triangle $A_1D_2D_3$ (where D_3 is intersection of a_2 and Ca_1).

We may conclude, that the orthocentres of the *triangles* $A_1A_2A_3$ lie on a *surface* H^9 with *double lines* a_1, a_2, a_3 and *triple point* C.

From this it ensues, that there are *nine singular planes* in which the four points A_1, A_2, A_3, A_4 form an *orthocentrical group*.

Any straight line of such a plane is apparently singular.