Physics. — *"On the Course of the Values of a and b for Hydrogen at Different Temperatures and Volumes".* IV. (Continued). By Dr. J. J. VAN LAAR. (Communicated by Prof. H. A. LORENTZ).

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§ XX. The value of a below the limiting temperature.

In this case the integrations need no longer take place in different stages, since a minimum distance r_m, which is dependent on θ, need no longer be reckoned with, so that first the integration with respect to θ can be carried out, and then with respect to r. All the entering molecules, from $\theta = 0$ to $\theta = 90°$, will now come in collision; for the limiting temperature T_0 the molecules that strike under an angle $\theta = 90°$ will just pass the rim of the molecule that is supposed not to move. We have, therefore, now to integrate (see § XVI):

$$a = {}^1/_2 \times (b_g)_\infty \alpha \times \frac{2a^4}{s(a^2-s^2)} \int_s^a \int_0^{1/_2\pi} \frac{dr \times \sin\theta\, d\theta}{r\sqrt{a^2\cos^2\theta + (a^2-r^2)(k^2\varphi-1)}},$$

in which $k^2\varphi$ is now always > 1, and in the limiting case $\varphi = \varphi_0 = 1 : k^2$ assumes the value 1. When we put $(a^2-r^2)(k^2\varphi-1) = q^2$, we get therefore:

$$a = {}^1/_2 \times (b_g)_\infty \alpha \times \frac{2a^3}{s(a^2-s^2)} \int_s^a \frac{dr}{r} \int_{1/_2\pi}^0 \frac{d(a\cos\theta)}{\sqrt{q^2+a^2\cos^2\theta}},$$

in which we may write for the second integral:

$$\log\left(a\cos\theta + \sqrt{q^2+a^2\cos^2\theta}\right)_{1/_2\pi}^0 = \log\frac{a+\sqrt{q^2+a^2}}{q},$$

so that we have still to integrate:

$$a = {}^1/_2 \times (b_g)_\infty \alpha \times \frac{2a^3}{s(a^2-s^2)} \int_s^a \frac{dr}{r} \log\left(\frac{a}{q} + \sqrt{1+\frac{a^2}{q^2}}\right) \quad . \quad (16)$$

If in the first place φ is near φ_0, then q approaches 0, and the integral approaches to

$$\int_s^a \frac{dr}{r} \log \frac{2a}{q} = \int_s^a \frac{dr}{r} \left[\log \frac{2}{\sqrt{k^2 \varphi - 1}} - \log \frac{\sqrt{a^2 - r^2}}{a} \right],$$

because q is $= \sqrt{k^2\varphi - 1} \times \sqrt{a^2 - r^2}$. Hence we have for the integral:

$$\log \frac{2}{\sqrt{k^2 \varphi - 1}} \log \frac{a}{s} - \int_s^a \log \frac{\sqrt{a^2 - r^2}}{a} \cdot \frac{dr}{r}.$$

We have for the last integral with $r : a = x$, $s : a = n$:

$$-\tfrac{1}{2} \int_n^1 \log (1 - x^2) \cdot \frac{dx}{x} = \tfrac{1}{4} \left(\frac{x^2}{1} + \frac{x^4}{4} + \frac{x^6}{9} + \ldots \right)_n^1 = \tfrac{1}{24} \pi^2 (1 - \varepsilon' n^2),$$

in which $\varepsilon' = 1$ for $n = 1$, and $6 : \pi^2 = 0{,}608$ for $n = 0$. For $\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \ldots = \frac{1}{6} \pi^2$, and $\frac{n^2}{1} + \frac{n^4}{4} + \frac{n^6}{9} + \ldots = n^2 \left(1 + \frac{n^2}{4} + \frac{n^4}{9} + \ldots \right)$ is $= \frac{1}{4} \pi^2$ for $n = 1$, and $= n^2$ for $n = 0$. (For $n = 0{,}6$ $\varepsilon' = 0{,}674$).

Hence we get finally:

$$\underline{(\varphi \gtreqless \varphi_0)} \quad a = \frac{1}{2n(1 - n^2)} (b_q)_\infty \, a \left[\frac{1}{12} \pi^2 (1 - \varepsilon' n^2) + \log \frac{1}{n^2} \log \frac{2}{\sqrt{k^2 \varphi - 1}} \right] . \, (17)$$

When we compare this with (14^b), where we found for values of φ in the neighbourhood of φ_0 (but $< \varphi_0$, while φ remains $> \varphi_0$ in (16)):

$$(\varphi \lesseqgtr \varphi_0) \quad a = \frac{1}{2n(1 - n^2)} (b_q)_\infty \, a \left[\frac{1}{4} \pi^2 (1 - \varepsilon n) + \log^2 \frac{1}{n} + \log \frac{1}{n^2} \log \frac{2}{\sqrt{1 - k^2 \varphi}} \right],$$

we observe with regard to the member that is independent of T, a discontinuity appearing at $\varphi = \varphi_0$. [We have added, for a comparison, to the first (finite) term the term $\log^2 \frac{1}{n}$, which was cancelled in § 18 in form. ($13b$) by the side of the infinitely large logarithmic term].

For $n = 1$ we find (with the factor $\frac{1}{1 - n^2}$ from the factor before the sign of integration) in the first case $\frac{1}{12} \pi^2 \frac{1 - n^2}{1 - n^2} = \frac{1}{12} \pi^2$, in the second case $\frac{1}{4} \pi^2 \frac{1 - n}{1 - n^2} + \log^2 \frac{1}{n} = \frac{1}{8} \pi^2$. And for $n = 0$ we find $\frac{1}{12} \pi^2$, resp. $\frac{1}{4} \pi^2 + \log^2 \frac{1}{n} = \frac{1}{4} \pi^2 + \infty^2$. This difference can be partly accounted for by the sudden disappearance at $\varphi = \varphi_0$ of the terms which refer to the *passing* molecules, and which, therefore, do not occur any more in (17). But in any case the difference is of no importance, as these terms, which are independent of φ, remain finite with respect to the term that depends on φ, and logarithmically approaches infinity. (In the case $n=0$, where — for infinitely large spheres of attraction — the entire quantity a would become infinite, and accordingly our derivation is no longer valid, the fact that $log^2 \frac{1}{n}$ becomes infinite, is of no importance at all).

We will still point out that for $\varphi=\varphi_0$ a does not only become logarithmically infinite with the form of $f(r)$ assumed by us, but with any arbitrary assumption about this. Compare for this Appendix C.

We suppose in the second place in (16) φ near ∞ (i.e. T near 0).

For the integral in (16) we may then write, as q becomes very large:

$$\int_s^a \frac{dr}{r} log\left(\frac{a}{q}+1+\tfrac{1}{2}\frac{a^2}{q^2}\right)=\int_s^a \frac{dr}{r}\times\frac{a}{q}=\frac{a}{\sqrt{k^2\varphi-1}}\int_s^a \frac{dr}{r\sqrt{a^2-r^2}},$$

i.e.

$$\frac{1}{\sqrt{k^2\varphi-1}}\left(log\frac{a-\sqrt{a^2-r^2}}{r}\right)_s^a=\frac{1}{\sqrt{k^2\varphi-1}}\left(log\, 1-log\frac{a-\sqrt{a^2-s^2}}{s}\right)=$$

$$=\frac{1}{\sqrt{k^2\varphi-1}}\, log\frac{a+\sqrt{a^2-s^2}}{s}.$$

When the factor before the sign of integration is taken into account we get therefore:

$$\left(\begin{matrix}\varphi\to\infty\\ T\to 0\end{matrix}\right)\qquad \mathrm{a}=\frac{1}{n\,(1-n^2)}\times(b_g)_\infty a\times\frac{1}{\sqrt{k^2\varphi-1}}\, log\frac{1+\sqrt{1-n^2}}{n}\ . \quad (18)$$

This approaches 0 therefore, when φ approaches ∞ (T approaches 0). We may write for $k^2\varphi-1$, after substitution of the value for φ, the expression $\frac{n^2}{1-n^2}\frac{{}^1/_3\, a}{RT}-1=\frac{n^2}{1-n^2}\frac{{}^1/_3\, a}{RT}$, when T is near 0.

Hence after the *maximum* for a at $\varphi=\varphi_0$ the attraction steadily decreases, and *disappears* at 0° abs. This result was to be foreseen. In the original integral of the virial of attraction the radical quantity in the denominator becomes namely $=\infty$ at 0° abs., when φ becomes $=\infty$. This radical quantity expresses the relative increase of velocity in the sphere of attraction, and as this increase remains *finite* with respect to $u_0=0$, the *relative* increase will become infinitely great.

And this relative increase of velocity entirely determines the densi in the sphere of attraction, which is in inverse ratio to it.

We observe once more here, that the earlier BOLTZMANN theoı would give an *exponentially infinite* value for a at 0° abs., whereı in ıeality it is $=0$.

For $n=1$ ($a=s$) the limiting value of $\frac{1}{1-n^2}$ will be $= log \frac{1+\sqrt{1-n^2}}{n}$ $= \frac{1}{\sqrt{1-n^2}}$. With $\sqrt{\frac{n^2}{1-n^2}}$ in $\sqrt{k^2\varphi-1}$ (see above) this becom $1:n$, so that then a will approach $(b_g)_\infty\, \alpha \times \sqrt{\frac{RT}{{}^1/_8\alpha}}$.

For $n=0$ (a great with respect to s) the absolute zero coincid with the limiting temperature, given by $\varphi_0 = 1:k^2 = (1-n^2):n$ For then $\varphi_0=\infty$ ($T_0=0$). In (18) $Lim \frac{1}{n}\, log$ becomes further $= \frac{1}{n} log \frac{2}{n}$ so that then a will approach $(b_g)_\infty\, \alpha \times \frac{1}{n}\, log \frac{2}{n} \times \frac{1}{\sqrt{n^2 \frac{{}^1/_8\alpha}{RT}-1}}$, whic again becomes $=0$ for $T=0$, so long as n is not absolutely $=$ (which of course would be practically impossible.

Summarising we can therefore state, in agreement with the abov developed exact theory concerning the quantity a for *very larg volume*, that a, from a limiting value at $T=\infty$, steadily increase to a *maximum value* at $T=T_0$, after which it decreases again till a has become $=0$ at the absolute zero. The mentioned limitin temperature T_0 is then determined by $RT_0 = {}^1/_8\, \alpha : \varphi_0$, in whicl $\varphi_0 = (1-n^2):n^2$. ($n = s:a$, in which s represents the diameter o a molecule, and a the radius of the sphere of attraction). For H T_0 is about $= \frac{1}{2} T_k$, the ratio of the values of a_∞, a_k, and a_0 bein $1 : 1^1/_8 : 2$.

In the next paper we shall briefly discuss the influence of MAXWELL's distribution of velocity, and then treat the course of the quantity b from $T=\infty$ to $T=0$, likewise at large volume. Then the value of a and b for *small* volumes will be considered, so as to make a complete theoretical insight possible concerning the *whole* course oı a and b along the boundary line, both along the vapour branch and along the liquid branch.

Fontanivent, January 1918. (*To be continued*).

APPENDIX.

A. The integral $k\int_0^{\frac{1}{2}\pi}\frac{\sin\psi}{\sqrt{1+k^2\sin\psi}}\psi d\psi$. (addition to § **XVII**).

When we expand this into a series through repeated partial integration, we get:

$$\int\frac{\sin\psi}{\sqrt{1+k^2\sin^2\psi}}\psi d\psi=\int P.\psi d\psi=\frac{\psi^2}{2}P-\frac{\psi^3}{6}\frac{dP}{d\psi}+\frac{\psi^4}{24}\frac{d^2\psi}{d\psi^2}-\frac{\psi^5}{120}\frac{d^3P}{d\psi^3}+\ldots,$$

in which (through ψ) all the terms at the lower limit 0 disappear. And for the upper limit all the *odd* differential quotients of P will disappear, because in this $\cos\psi$ occurs as factor. Indeed, when we put $1+k^2\sin\psi=\omega$, so that $\frac{d\omega}{d\psi}$ becomes $=2k^2\sin\psi\cos\psi$, we have:

$$\frac{dP}{d\psi}=-\tfrac{1}{2}\frac{\sin\psi}{\omega^{3/2}}(2k^2\sin\psi\cos\psi)+\frac{\cos\psi}{\omega^{1/2}}=\cos\psi\left(\frac{-k^2\sin^2\psi}{\omega^{3/2}}+\frac{1}{\omega^{1/2}}\right)=$$

$$=\cos\psi\left(\frac{1-\omega}{\omega^{3/2}}+\frac{1}{\omega^{1/2}}\right)=\frac{\cos\psi}{\omega^{3/2}}.$$

$$\frac{d^2P}{d\psi^2}=-\tfrac{3}{2}\frac{\cos\psi}{\omega^{5/2}}(2k^2\sin\psi\cos\psi)-\frac{\sin\psi}{\omega^{3/2}}=-\sin\psi\left(\frac{3k^2\cos^2\psi}{\omega^{5/2}}+\frac{1}{\omega^{3/2}}\right)=$$

$$=-\sin\psi\left(\frac{3(1+k^2)}{\omega^{5/2}}-\frac{2}{\omega^{3/2}}\right),$$

because $k^2\cos^2\psi=k^2-k^2\sin^2\psi=k^2-(\omega-1)=(1+k^2)-\omega$. We have further:

$$\frac{d^3P}{d\psi^3}=-\sin\psi\left(\frac{15(1+k^2)}{\omega^{7/2}}-\frac{6}{\omega^{5/2}}\right)(-k^2\sin\psi\cos\psi)-\cos\psi\left(\frac{3(1+k^2)}{\omega^{5/2}}-\frac{2}{\omega^{3/2}}\right)$$

$$=-\cos\psi\left[\left(\frac{15(1+k^2)}{\omega^{7/2}}-\frac{6}{\omega^{5/2}}\right)(1-\omega)+\left(\frac{3(1+k^2)}{\omega^{5/2}}-\frac{2}{\omega^{3/2}}\right)\right]$$

$$=-\cos\psi\left(\frac{15(1+k^2)}{\omega^{7/2}}-\frac{12(1+k^2)+6}{\omega^{5/2}}+\frac{4}{\omega^{3/2}}\right).$$

And also:

$$\frac{d^4P}{d\psi^4}=-\cos\psi\left(\frac{105(1+k^2)}{\omega^{9/2}}-\frac{60(1+k^2)+30}{\omega^{7/2}}+\frac{12}{\omega^{5/2}}\right)(-k^2\sin\psi\cos\psi)+$$

$$+\sin\psi\left(\frac{15(1+k^2)}{\omega^{7/2}}-\frac{12(1+k^2)+6}{\omega^{5/2}}+\frac{4}{\omega^{3/2}}\right)$$

$$=\sin\psi\left[\left(\frac{105\,(1+k^2)}{\omega^{9/2}}-\frac{60\,(1+k^2)+30}{\omega^{7/2}}+\frac{12}{\omega^{5/2}}\right)((1+k^2)-\omega)+\right.$$
$$\left.+\left(\frac{15\,(1+k^2)}{\omega^{7/2}}-\frac{12\,(1+k^2)+6}{\omega^{5/2}}+\frac{4}{\omega^{3/2}}\right)\right]$$
$$=\sin\psi\left[\frac{105(1+k^2)^2}{\omega^{9/2}}-\frac{60(1+k^2)^2+120(1+k^2)}{\omega^{7/2}}+\frac{60(1+k^2)+24}{\omega^{5/2}}-\frac{8}{\omega^{3/2}}\right].$$

Etc. Etc. As has been said, all the odd differential quotients disappear for $\psi=\frac{1}{2}\pi$, and as ω becomes $=1+k^2$ for $\psi=\frac{1}{2}\pi$, we keep:

$$\left(\frac{d^2P}{d\psi^2}\right)_{1/2\pi}=-\frac{1}{\omega^{3/2}}=-\frac{1}{(1+k^2)^{3/2}};$$
$$\left(\frac{d^4P}{d\psi^4}\right)_{1/2\pi}=\frac{15}{\omega^{5/2}}-\frac{12}{\omega^{3/2}}-\frac{6}{\omega^{5/2}}+\frac{4}{\omega^{3/2}}=\frac{9}{(1+k^2)^{5/2}}-\frac{8}{(1+k^2)^{3/2}}=\frac{1-8\,k^2}{(1+k^2)^{5/2}}.$$

For the sake of brevity we have only taken the part with $\sin\psi$ into account in the last calculation of the two differential quotients: that with $\cos\psi$ is namely $=0$. I. e. of $\frac{d^2P}{d\psi^2}$ only the part $-\frac{\sin\psi}{\omega^{3/2}}$, and of $\frac{d^4P}{d\psi^4}$ only the part with $\sin\psi$ in the first of the three lines belonging to this. The other parts have every time been necessary for the determination of the next higher differential quotient. Proceeding, we should have found:

$$\frac{d^6P}{d\psi^6}=-\left(\frac{225}{(1+k^2)^{7/2}}-\frac{360}{(1+k^2)^{5/2}}+\frac{136}{(1+k^2)^{3/2}}\right)=-\frac{1-88k^2+136k^4}{(1+k^2)^{7/2}}.$$

The coefficients of the highest powers of $1+k^2$ are in all these results resp. $=1^2$, $(1\times 3)^2$, $(1\times 3\times 5)^2$, etc. The sum of the coefficients is always $=1$. $(9-8=1;\ 225-360+136=1)$. Hence we get now, taking into consideration that $k:\sqrt{(1+k^2)}=$ $=\frac{s}{\sqrt{a^2-s^2}}:\frac{a}{\sqrt{a^2-s^2}}=\frac{s}{a}=n$, and $(P)_{1/2\pi}=1:\sqrt{1+k^2}$:

$$k\int\limits_0^{1/2\pi}\frac{\sin\psi}{\sqrt{1+k^2\sin^2\psi}}\,\psi\,d\psi=n\left[\frac{(\frac{1}{2}\pi)^2}{2}-\frac{1}{1+k^2}\,\frac{(\frac{1}{2}\pi)^4}{24}+\right.$$
$$\left.+\frac{1-8k^2}{(1+k^2)^2}\,\frac{(\frac{1}{2}\pi)^6}{720}-\frac{1-88k^2+136k^4}{(1+k^2)^3}\,\frac{(\frac{1}{2}\pi)^8}{40320}+\text{etc.}\right],$$

in which we may also write $1-n^2$ for $1:(1+k^2)=(a^2-s^2):a^2$. The above series is convergent, as is easily seen from the structure of the factors $(1-8k^2):(1+k^2)^{5/2}=9:(1+k^2)^{5/2}=8\,(1+k^2)^{5/2}$, etc.

For *large* values of k ($a = s$, i. e. $n = 1$) it converges very greatly, and rapidly approaches the first term, i. e. $n \times {}^1/_8\pi^2$.

For *small* values of k (near 0, i. e. a large with respect to s, $n = 0$) the series becomes:

$$n\left[\tfrac{1}{2}\left(\tfrac{1}{2}\right)\pi^2 - \tfrac{1}{24}\left({}^1/_2\pi\right)^4 + \tfrac{1}{720}\left({}^1/_2\right)\pi^6 - \text{etc.}\right] = n\left(1 - \cos\tfrac{1}{2}\pi\right) = n.$$

For the two limiting cases $n = 1$ and $n = 0$ we, therefore, find back the same values as we had already found by direct integration in the text of § 17.

When $n = 0{,}6$, we get $1 - 8k^2 = 1 - 4{,}5 = -3{,}5$, $1 - 88k^2 + 136k^4 = 1 - 49{,}5 + 43{,}0 = -5{,}5$, $1:(1+k^2) = 0{,}64$, so that with ${}^1/_4\pi^2 = 2{,}4674$, the integral being put $= \varepsilon n \times {}^1/_8\pi$ (cf. the text of § 17), we find from

$$\varepsilon = 1 - \frac{1}{1+k^2}\frac{({}^1/_2\partial)^2}{12} + \frac{1-8k^2}{(1+k^2)^2}\frac{({}^1/_2\pi)^4}{360} - \frac{1-88k^2+136k^4}{(1+k^2)^3}\frac{({}^1/_2\pi)^6}{20160} + \text{etc.}$$

for ε the value

$$1 - 0{,}1316 - 0{,}02425 + 0{,}00107 \ldots = 0{,}8452 \ldots = 0{,}845.$$

B. The integral $k\int_{tg\theta_0 k}^{1}\frac{\cosh\psi}{\sqrt{1+k^2\cos^2 h\,\psi}}\,\psi d\psi$ (addition to § **XVIII**).

In entirely the same way as for the above treated integral we find through repeated partial integration:

$$k\int_{tg\theta_0 k}^{1}\frac{\cos h\psi}{\sqrt{1+k^2\cos^2 h\,\psi}}\,\psi\,d\psi = -\left[\frac{tg\theta_0}{\sec\theta_0}\frac{\log^2}{2} - \frac{k\sqrt{\frac{tg^2\theta_0}{k^2}-1}}{\sec^3\theta_0}\frac{\log^3}{6} + \right.$$

$$+ tg\,\theta_0\left(\frac{3(1+k^2)}{\sec^5\theta_0} - \frac{2}{\sec^3\theta_0}\right)\frac{\log^4}{24} -$$

$$\left. - k\sqrt{\frac{tg^2\theta_0}{k^2}-1}\left(\frac{15(1+k^2)}{\sec^7\theta_0} - \frac{12(1+k^2)+6}{\sec^5\theta_0} + \frac{4}{\sec^3\theta_0}\right)\frac{\log^5}{120} + \text{etc.}\right],$$

in which *log* represents $\log\left(\frac{tg\theta_0}{k} + \sqrt{\frac{tg^2\theta_0}{k^2} - 1}\right)$.

In this it has been taken into account that $d\cosh\psi = \sinh\psi$ and $d\sinh\psi = \cosh\psi$, and that further $-k^2\cos^2 h\,\psi$ can again be replaced by $1 - \omega$ (when namely $1 + k^2\cos^2 h\,\psi$ is put $= \omega$) and $-k^2\sin^2 h\,\psi$ by $-k^2\cos^2 h\,\psi + k^2 = (1+k^2) - \omega$. Now the terms with odd powers of ψ do *not* disappear, because at the lower limit the factor $\sinh\psi$, which occurs for these powers, does not disappear (as for the above treated integral $cos\,\psi$ at the upper limit), but becomes $=\sqrt{\frac{tg^2\theta_0}{k^2}-1}$, because $cosh\,\psi$ then is $=tg\theta_0:k$. At the upper limit everything disappears, because then $\psi=0$. (Besides, the terms with odd powers of ψ still contain the factor $sinh\,\psi$, which now likewise becomes $=0$, because $cosh\,\psi$ becomes $=1$ at the upper limit. (Cf. further the text of § 18)). We may, therefore, write:

$$k\int=-\left[sin\,\theta_0\left\{\frac{log^2}{2}+\left(\frac{3(1+k^2)}{(1+tg^2\theta_0)^2}-\frac{2}{1+tg^2\theta_0}\right)\frac{log^4}{4}+\text{etc.}\right\}-\right.$$
$$-\sqrt{\frac{tg^2\theta_0-k^2}{1+tg^2\theta_0}}\left\{\frac{1}{1+tg^2\theta_0}\frac{log^3}{6}+\left(\frac{15(1+k^2)}{(1+tg^2\theta_0)^3}-\frac{12(1+k^2)+6}{(1+tg^2\theta_0)^2}+\right.\right.$$
$$\left.\left.\left.+\frac{4}{1+tg^2\theta_0}\right)\frac{log^5}{120}+\text{etc.}\right\}\right].$$

Let us now introduce the quantity φ, determined by equation (6) of the last paper but one, viz.

$$\frac{a^2}{s^2}sin^2\theta_0=1+\frac{M}{{}^1/_2\mu u_0{}^2}=1+\varphi,$$

in which, therefore, φ depends on the *temperature* (determined by ${}^1/_2\mu u_0{}^2$). For $1+tg^2\theta_0$ we may write $\frac{1-k^2\varphi}{1+k^2}$, because $k^2(1+\varphi):$ $:(1-k^2\varphi)$ may be substituted for $tg^2\theta_0=\frac{s^2}{a^2}(1+\varphi):\left(1-\frac{s^2}{a^2}(1+\varphi)\right)$ with $\frac{s^2}{a^2-s^2}=k^2$. For $tg^2\theta_0-k^2$ we find $k^2(1+k^2)\varphi:(1-k^2\varphi)$, so that we get:

$$k\int=-\left[\frac{k}{\sqrt{1+k^2}}\sqrt{1+\varphi}\left\{\frac{log^2}{2}+\frac{(1-k^2\varphi)(1-3k^2\varphi)}{1+k^2}\frac{log^4}{24}+\text{etc.}\right\}-\right.$$
$$-k\sqrt{\varphi}\left\{\frac{1-k^2\varphi}{1+k^2}\frac{log^3}{6}+\left(\frac{(1-k^2\varphi)^2(9-15k^2\varphi)}{(1+k^2)^2}-\right.\right.$$
$$\left.\left.\left.-\frac{(1-k^2\varphi)(8-12k^2\varphi)}{1+k^2}\right)\frac{log^5}{120}+\text{etc.}\right\}\right]$$

in which

$$log=log\left(\frac{tg\theta_0}{k}+\sqrt{\frac{tg^2\theta_0}{k^2}-1}\right)=log\frac{\sqrt{1+\varphi}+\sqrt{(1+k^2)\varphi}}{\sqrt{1-k^2\varphi}}.$$

Let us now examine, what are the limiting values to which the found integral approaches at high temperatures, and at low temperatures (φ near $\varphi_0=1\cdot k^2$).

At *high* temperatures ($\varphi=0$) log draws near to $log\,1=0$, so that all the terms with high powers of log are cancelled by the side of the first term, and besides the whole part with $k\sqrt{\varphi}$ disappears. That in this case only the first term with log^2 remains, follows also from this that $tg\,\theta_0 = k^2(1+\varphi):(1-k^2\varphi)$ approaches k for $\varphi = 0$, so that in case of equality of the limits of the original integral the factor $k\cos h\psi : \sqrt{1+k^2\cos^2 h\,\psi} = k : \sqrt{1+k^2}$ does not change between them (with respect to the log that becomes 0 at both the limits), and can accordingly be brought outside the integral sign.

At *low* temperatures (but higher than the limiting temperature T_0, determined by $\varphi_0 = 1 : k^2$) the whole second part of $k\int$ will again disappear in consequence of the factor $1-k^2\varphi$, which approaches 0, whereas of the first part again only the first term with log^2 remains. In this case $\cos h\,\psi = tg\,\theta_0 : k = \infty$ at the lower limit, and the factor of $\psi d\psi$ in the integral can again be placed outside the integral sign at this limit, which now prevails since the log becomes infinite there. At the other limit the log is namely $= 0$.

With close approximation we may, therefore, write (n has been written for $k:\sqrt{1+k^2} = s:a$):

$$k\int = -\tfrac{1}{2}\, n\sqrt{1+\varphi}\; log^2 \frac{\sqrt{1+\varphi}+\sqrt{(1+k^2)\varphi}}{\sqrt{1-k^2\varphi}},$$

with neglect of all the terms with higher powers of log. Only at intermediary temperatures the omitted part can have any influence — but the difference brought about by this might possibly be made to disappear entirely on a somewhat modified assumption concerning $f(r)$ between a and s (see § XVI).

C. The quantity a for $\varphi = \varphi_0 = 1 : k^2$. (addition to § XX).

The original integral was (cf § 16):

$$a = \tfrac{1}{2} \times (b_g)_\infty\, a \times \frac{2a^4}{s(a^2-s^2)} \int_s^a \int_0^{\frac{1}{2}\pi} \frac{r(-f'(r))dr \times \sin\theta\, d\theta}{\sqrt{1-\frac{a^2}{r^2}\sin\theta + \varphi f(r)}}.$$

We may also write for the integral:

$$\int_s^a \int_{\frac{1}{2}\pi}^0 \frac{r^2(-f'(r))\,dr\;\; d\,(a\cos\theta)}{\sqrt{r^2\varphi f(r)-(a^2-r^2)+a^2\cos^2\theta}} = \frac{1}{a}\int_s^a r^2(-f'(r))\,dr\,log\left(\frac{a}{q}+\sqrt{1+\frac{a^2}{q^2}}\right),$$

when $r^2\varphi f(r) - (a^2-r^2) = q^2$ is put. When $f(r)$ is generally $= \frac{s^t}{r^t}$, so that this duly becomes $= 1$ for $r = s$, then $-f'(r) = \frac{ts^t}{r^{t+1}}$ and $q^2 = \frac{\varphi s^t}{r^{t-2}} - (a^2-r^2)$. Hence we now have:

$$a = (b_g)_\infty \alpha \times \frac{ta^3 s^{t-1}}{a^2 - s^2} \int_s^a \frac{dr}{r^{t-1}} log\left(\frac{a}{q} + \sqrt{1 + \frac{a^2}{q^2}}\right),$$

in which the quantity q for the lower limit passes into $\varphi s^2 - (a^2 - s^2)$, which becomes $= 0$ for $\varphi = \frac{a^2 - s^2}{s^2} = \frac{1 - n^2}{n^2} = \frac{1}{k^2}$ as before. The value of a will, therefore, again approach to *logarithmically infinite* for $\varphi = \varphi_0 = 1 : k^2$. This is, accordingly, entirely independent of the exponent t in the assumed law of force $f(r) \therefore r^{-t}$.