**Physics.** — *"Calculation of some special cases, in* EINSTEIN'S *theory of gravitation"*. By Dr. GUNNAR NORDSTROM. (Communicated by Prof. H. A. LORENTZ).

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As an application of the theorems deduced in two preceding papers for EINSTEIN'S theory [1]) of gravitation, we shall now calculate the gravitation field and the stresses for some special *stationary systems with spherical symmetry.*

First the state at a surface of discontinuity will be investigated.

## § 1. *Introductory formulae.*

In a field with spherical symmetry a surface of discontinuity necessarily is a sphere. This surface will be considered as the limiting case of a layer of finite depth, and we shall only have to pay attention to such surfaces in which in the limit some component of the material stress-energy-tensor increases above every arbitrary limit so that the line-integral across the layer remains finite. In general at such a surface of discontinuity there evidently works a surface-tension $P$:

$$P = \lim_{r_2 - r_1 = 0} \int_{r_1}^{r_2} \mathfrak{T}_p^p \, dr, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $r_1$ denotes the inner radius of the layer, and $r_2$ the outer one. The radical component of the stress-tensor $\mathfrak{T}_r^r$ on the contrary we shall suppose never to pass every arbitrary limit; in other words we assume that:

$$\lim_{r_2 - r_1 = 0} \int_{r_1}^{r_2} \mathfrak{T}_r^r \, dr = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

First we shall consider a general surface of discontinuity and only afterwards we shall introduce special assumptions. We start from the first and third formulae (38) I and from (39) I. (From these three formulae the second formula (38) I may also be derived, but

[1]) G. NORDSTRÖM, On the mass of a material system according to the gravitation theory of EINSTEIN These Proceedings XX, 1917, p. 1076 (cited further on as I) and: On the energy of the gravitation field in EINSTEIN'S theory. These Proceedings XX, 1918 p. 1238 (cited further on as II).

we do not need this). The system of coordinates will be fixed by the conditions:

$$v = r, \quad \text{viz.} \quad p = r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

Putting further:

$$\mathfrak{T} = \sqrt{-g}\, T = u\, w\, T, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

and applying a simple transformation, we can write for the mentioned starting formulae:

$$-\frac{1}{u^2}\left(1 + 2r\frac{w'}{w}\right) + 1 = r^2 \varkappa\, T_r^r, \quad . \quad . \quad . \quad . \qquad (5)$$

$$\frac{d}{dr}\left\{r\left(1 - \frac{1}{u^2}\right)\right\} = r^2 \varkappa\, T_4^4, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

$$\frac{2}{r}(T_p^p - T_r^r) + \frac{w'}{w}(T_4^4 - T_r^r) = \frac{dT_r^r}{dr}. \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

These formulae hold for each stationary gravitation field with spherical symmetry; the system of coordinates only is determined by the condition (3). The quantities $u$ and $w$ determine (when $p = 1$) all components $g_{\mu\nu}$ of the fundamental tensor according to the formulae (25) I.

When $T_4^4$ is given, the equation (6) determines $u$ as a function of $r$. By integration across a layer which afterwards by a passage to the limit is changed into a surface of discontinuity with radius $r_1 = r_2 = R$ and after division by $R$ we obtain

$$\frac{1}{u_1^2} - \frac{1}{u_2^2} = R\varkappa \lim_{r_2 - r_1 = 0} \int_{r_1}^{r_2} T_4^4\, dr. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

This formula shows that *u changes discontinuously at a surface of discontinuity where*

$$\lim_{r_2 - r_1 = 0} \int_{r_1}^{r_2} T_r^r\, dr$$

*differs from zero.* Such a surface which moreover satisfies the condition (2) will be called a *material surface.* The system of coördinates might be chosen in such a way that at the surface $u$ changes continuously, but then $p$ would change discontinuously. In general at least one of the space-components of the fundamental tensor changes discontinuously at a material surface. With the aid of formula (5) we shall now prove, that $w$ on the contrary changes continuously at our material surface, when only the condition (2) is satisfied. Equation (5) gives

$$2\frac{w'}{w} = \frac{u^2}{r}\left(1 - r^2 \varkappa\, T_r^r\right) - \frac{1}{r}, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

and by integration across the layer we obtain

$$\log w_2{}^2 - \log w_1{}^2 = \int_{r_1}^{r_2} \left\{ \frac{u^2}{r} (1 - r^2 \varkappa T_r^r) - \frac{1}{r} \right\} dr \quad . \quad . \quad . \quad (10)$$

We shall only consider gravitation fields in which $u$ is every where finite and when in the limit we pass to an infinitely thin layer the limiting value of the integral on the right-hand side becomes zero according to the assumption (2).

Now we shall apply formula (7) and substitute in it the expression (9) for $\frac{w'}{w}$ and the expression (6) for $T_4^4$. Multiplying further by $\frac{urdr}{2}$, we find

$$u\,(T_p^p - T_r^r)dr + \tfrac{1}{4}\{u^3(1 - r^2 \varkappa T_r^r) - u\} \left\{ \frac{1}{\varkappa r^2} \frac{d}{dr} r\left(1 - \frac{1}{u^2}\right) - T_r^r \right\} dr = \frac{ur}{2}\frac{dT_r^r}{dr}\,dr. \quad (11)$$

This equation must be integrated over a layer and afterwards we must pass to the case of an infinitesimal depth. In order to obtain as a first term on the left-hand side the surface tension $P$ as defined by equation (1) we must moreover multiply by $w$. We shall however not continue our general investigation, but rather consider two more special cases.

## § 2. *Investigation of the state at a material surface.*

First we investigate the case that at the limit $T_4^4$ surpasses any value, so that the right-hand side does not become zero, but that $\frac{dT_r^r}{dr}$ remains finite, so that on both sides of the surface of discontinuity $T_r^r$ has the same value.

In (11) we first consider the part of the left-hand side which after integration gives

$$I = \tfrac{1}{4} \int_{r_1}^{r_2} \{u^3 (1 - r^2 \varkappa T_r^r) - u\} \frac{1}{\varkappa r^2} \frac{d}{dr} r\left(1 - \frac{1}{u^2}\right) dr =$$

$$= \frac{1}{4\varkappa} \int \frac{1}{r^2} \{u^3 (1 - r^2 \varkappa T_r^r) - u\}\, d \left\{ r\left(1 - \frac{1}{u^2}\right) \right\}.$$

We have to calculate the value of this expression for the limit $r_2 - r_1 = 0$. In this limiting case $r$ constant $= r_1 = r_2 = R$, so that we have

$$d \left\{ r \left(1 - \frac{1}{u^2}\right) \right\} = 2\,R\,\frac{du}{u^3}.$$

We thus obtain

$$\lim_{r_2 - r_1 = 0} I = \frac{1 - R^2 \varkappa T^r_r}{2\varkappa R} \int_{u_1}^{u_2} du - \frac{1}{2\varkappa R} \int_{u_1}^{u_2} \frac{du}{u^2} = \frac{u_2 - u_1}{2\varkappa R} \left(1 - \frac{1}{u_1 u_2} - R^2 \varkappa T^r_r\right).$$

Now we have treated one part of the left-hand side of (11) by integration and by passage to the limit. Of the remaining parts of this left-hand side those containing $T^r_r$ remain zero at the passage to the limit according to our assumption (2), $u$ remaining moreover finite. The part containing $T^p_p$ on the contrary does not become zero. The right-hand side has the value zero at the limit, as we have assumed $T^r_r$ to change continuously at the surface of discontinuity. Multiplying our equation still by $w$, which quantity we have proved to change continuously at the surface, so that at the limit it may be considered as constant, we obtain:

$$P = -\frac{w}{2\varkappa R}(u_2 - u_1)\left(1 - \frac{1}{u_1 u_2} - R^2 \varkappa T^r_r\right) \quad . \quad . \quad . \quad (12)$$

Together with (8) this formula expresses the laws for a surface of discontinuity of the kind we now consider. These formulae will be applied to the special case that all matter that is present is situated in the material surface. $T^r_r$ being continuous, we have in this case $T^r_r = 0$. Further we have according to (6) both inside and outside the surface

$$r\left(1 - \frac{1}{u^2}\right) = \text{const.} \qquad (r \neq R) \quad . \quad . \quad . \quad . \quad (13)$$

When $r = 0$, $u$ cannot be zero, so that the value of the constant within the surface must be zero. We thus find for $r < R$, $u = 1$ and therefore also

$$u_1 = 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Within the spherical material surface we thus have a euclidic space. (This is of course true for every hollow sphere; the distribution of mass and stress on the outside only has spherical symmetry). Outside the material surface the constant in equation (13) has not the value zero, but a value, proportional to the mass of the system which is given by formula (15) II:

$$m = \frac{4\pi\alpha}{\varkappa}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15a)$$

We thus have for $u_2$:

$$u_2 = \frac{1}{\sqrt{1 - \frac{\alpha}{R}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

For $w$ we have at our surface:

$$w = c\sqrt{1 - \frac{\alpha}{R}} = \frac{c}{u_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

This may be proved e.g. by putting $\varepsilon = 0$ in formula (12) II which holds outside our surface. Also by putting $r = R$ we obtain the value (16) at the surface, and formula (9) shows afterwards (as within the surface $u = 1$ and $T'_r{}^r = 0$), that this constant value of $w$ holds also everywhere inside the material surface.

*Introducing the expressions* $u_1$, $u_2$, and $w$, we find for the surface tension $P$

$$P = -\frac{c}{2\varkappa R}\left(1 - \sqrt{1 - \frac{\alpha}{R}}\right)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

This formula expresses the relation between the surface-tension, the mass and the radius. Expressed in the usual units the surface-tension is $cP$ (comp. I p. 1079). The constant of mass $\alpha$ is also connected with the right-hand side of equation (8). After introduction of the values of $u_1$ and $u_2$ this equation gives

$$\alpha = \varkappa R^2 \lim_{r_2 - r_1 = 0} \int_{r_1}^{r_2} T_4^4 \, dr. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

In the euclidic space inside the material surface we have not the same velocity of light as at an infinite distance from our system, but a smaller velocity

$$c\sqrt{1 - \frac{\alpha}{R}}.$$

We thus have a representation of EINSTEIN's idea on the influence of distant masses on the velocity of light in our part of the world.

Expanding the expression (17) for $P$ in powers of $\alpha/R$ we obtain:

$$P = -\frac{c}{2\varkappa R}\left(\tfrac{1}{4}\frac{\alpha^2}{R^2} + \tfrac{1}{8}\frac{\alpha^3}{R^3} + \ldots\right) \quad . \quad . \quad . \quad . \quad (17a)$$

NEWTON's theory gives for $cP$:

$$cP = -\frac{km^2}{16\,\pi\,R^3}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17b)$$

where $k$ is the NEWTONIAN gravitation constant:

$$k = \frac{c^2\varkappa}{8\pi}.$$

Introducing in (17*b*) the expressions for $k$ and $m$, we find for $P$ an expression, corresponding to the first term of (17*a*). As to the terms of lower order the theory of EINSTEIN agrees therefore with that of NEWTON.

### §. 3. *Second example of a surface of discontinuity.*

Now we shall consider another kind of surface of discontinuity viz. one in which

$$\lim_{r_2-r_1=0}\int_{r_1}^{r_2} T_4^4\, dr = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

but where $T_r^r$ changes discontinuously. Such a surface of discontinuity we have e. g. when an electric charge is spread over the surface.

Formula (8) shows that in the case in question $u$ changes continuously at the surface:

$$u_2 = u_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Above we showed already by formula (10) that $w$ changes continuously.

This time too we must multiply formula (11) by $w$, integrate a layer and pass to the limit of an infinitesimal thickness. As in the last part of the left-hand side all quantities remain finite at the limit, this part gives the limiting value zero. As further $u$ and $w$ change continuously, we obtain

$$P = \frac{R}{2}\, u\, w\, (T_{r_2}^r - T_{r_1}^r),$$

or, introducing the components of the volume-tensor $\mathfrak{T}$

$$P = \frac{R}{2}\, (\mathfrak{T}_{r_2}^r - \mathfrak{T}_{r_1}^r). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

The meaning of this equation is trivial. It expresses the equilibrium between the surface-tension $P$ at the spherical surface and the normal force perpendicular to that surface, the magnitude of which is $\mathfrak{T}_{r_2}^r - \mathfrak{T}_{r_1}^r$, per unit of surface. The gravitation has evidently no influence.

When on the surface we have an electric charge $e$ and inside the surface no matter, we find (II, note p. 1240)

$$\mathfrak{T}_{r_1}^r = 0, \qquad \mathfrak{T}_{r_2}^r = \frac{u\, w\, e^2}{8\pi R^4}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

Now we shall assume that neither outside the surface there is any matter except the electric field, and we shall calculate the mass of the electric sphere. As was proved in II § 1 we have outside the sphere

$$u\,w = c \qquad (r > R), \quad \dots \dots \quad (23)$$

$$T_r^r = T_4^4 = \frac{e^2}{8\pi\, r^4} \qquad (r > R) \quad \dots \dots \quad (24)$$

As inside the sphere and at its surface $u = 1$, we find from (6) by integration up to an upper limit $r > R$

$$r\left(1 - \frac{1}{u^2}\right) = \varkappa \int_R^r r^2\, T_4^4\, dr = -\frac{\varkappa e^2}{8\pi\, r} + \frac{\varkappa e^2}{8\pi\, R} \qquad (r > R),$$

$$\frac{1}{u^2} = 1 - \frac{\varkappa e^2}{8\pi \cdot Rr} + \frac{\varkappa e^2}{8\pi\, r^2} \quad \dots \dots \quad (25)$$

A comparison with equation (11) II shows, that we must have:

$$a = \frac{\varkappa e^2}{8\pi\, R},$$

and (15*a*) gives for the mass $m$

$$m = \frac{e^2}{2R} \quad \dots \dots \quad (26)$$

The charge $e$ being expressed in electro-magnetic units (see II p. 1202) this expression for $m$ is equal to the electro-static energy divided by $c^2$. Besides the electro-static energy no energy occurs in our system. That outside the electric body no gravitation energy, is present has been proved already in II § 2. The last result says therefore that neither in the electric surface any gravitation energy is accumulated.

### § 4. *A sphere of an incompressible fluid.*

This problem has been treated already by SCHWARZSCHILD, [1] but as the formulae (5), (6), and (7) lead us by another way quickly to the same result, it may be allowed to develop these calculations as shortly as possible.

That the medium is incompressible means that when at rest

$$T_4^4 = \varrho \quad \dots \dots \quad (27)$$

is a constant characteristic for the medium. The fluid character of the medium demands further that no tangential stresses can occur, so that we have

$$T_r^r = T_p^p = -p, \quad \dots \dots \quad (28)$$

[1]) K. SCHWARZSCHILD, Ueber das Gravitationsfeld einer Kugel aus inkompressibeler Flüssigkeit nach der Einsteinschen Theorie Berl. Ber 1916 p. 424.

where the pressure scalar $p$ [1]) meanwhile is a function of the place viz. of $r$. The radius $R$ of the sphere and the mass $m$ and $\varrho$ are related by an equation which is found by integrating (6) from $r=0$ to $r=R$. As for $r=0$ $u$ is not zero, while for $r=R$ it has the value $\dfrac{1}{\sqrt{1-\dfrac{\alpha}{R}}}$ (see II equation (11)), we find

$$\alpha = \frac{\varkappa\varrho}{3}R^3$$

and therefore

$$m = \varrho \tfrac{4}{3}\pi R^3 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \qquad (29)$$

This shows that $\varrho$ plays the part of density.

Integrated from $r=0$ to an arbitrary upper limit $r<R$ (6) gives further $u$ as a function of $r$. We obtain:

$$r\left(1-\frac{1}{u^2}\right)=\frac{\varkappa\varrho}{3}r^3,$$

$$u^2 = \frac{1}{1-\dfrac{\varkappa\varrho}{3}r^2} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \qquad (30)$$

Now $w$ and $p$ have still to be determined as functions of $r$. The quantities $w$ and $p$ are connected by equation (7). This gives

$$\frac{w'}{w}(\varrho+p) = -\frac{dp}{dr}, \quad . \; . \; . \; . \; . \; . \; . \; . \qquad (31)$$

so that

$$\frac{dw}{w}(\varrho+p) = -\,dp.$$

This must be integrated. The integration constant is determined by the fact that at the spherical surface $p=0$ and $w = c\sqrt{1-\dfrac{\alpha}{R}} = c\sqrt{1-\dfrac{\varkappa\varrho}{3}R^2}$ (see II equation (12)). We thus obtain the asked connection between $w$ and $p$:

$$w(\varrho+p) = \varrho c\sqrt{1-\frac{\varkappa\varrho}{3}R^2} \quad . \; . \; . \; . \; . \qquad (32)$$

Now $p$ will be calculated as a function of $r$. Introducing in (5) the expression (30) for $u$ and simplifying the equation we obtain

$$2\frac{w'}{w}\left(1-\frac{\varkappa\varrho}{3}r^2\right) = \frac{\varkappa}{3}r(\varrho+3p) \quad . \; . \; . \; . \; . \; . \qquad (33)$$

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[1]) We need not be afraid that this $p$ will be confused with the quantity $p$ which in § 1 has been put equal to 1.

We eliminate $\frac{w'}{w}$ between this equation and (31). In this way we find

$$\frac{2\,dp}{(\varrho+3p)(\varrho+p)}+\frac{\varkappa}{3}\frac{rdr}{1-\frac{\varkappa\varrho}{3}r^2}=0 \quad . \quad . \quad . \quad . \quad (34)$$

The integration gives

$$log\,\frac{\varrho+3p}{\varrho+p}-log\sqrt{1-\frac{\varkappa\varrho}{3}r^2}=const.$$

The integration constant has to be determined with the aid of the condition that for $r=R$ $p=0$. We therefore find

$$\frac{\varrho+3p}{\varrho+p}=\sqrt{\frac{1-\frac{\varkappa\varrho}{3}r^2}{1-\frac{\varkappa\varrho}{3}R^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

Thus the pressure-scalar $p$ is determined as a function of $r$.

Eliminating $p$ between this equation and (32) we obtain for $w$ as a function of $r$ the expression:

$$w=\frac{c}{2}\left(3\sqrt{1-\frac{\varkappa\varrho}{3}R^2}-\sqrt{1-\frac{\varkappa\varrho}{3}r^2}\right). \quad . \quad . \quad (36)$$

In this way we have perfectly determined the gravitation field and the pressure distribution inside our sphere. The formulae we obtained become identical with those of SCHWARZSCHILD when for $r$ we substitute

$$r=\sqrt{\frac{3}{\varkappa\varrho}}\sin\chi.$$

## § 5. *On the gravitation field as it may be imagined to exist in the inside of an atom.*

In the theory of atomic structure of RUTHERFORD-BOHR we meet with difficulties arising from the assumption that in an atomic nucleus of very small dimensions there exist units of charge which — at least when they are liberated in the form of electrons — have a greater diameter than the atomic nucleus. As now EINSTEIN's gravitation theory states that the space in a gravitation field when expressed in natural units is non-euclidic, the question arises whether this theory leaves the possibility of the assumption that the atomic nucleus fills a greater space with a narrow neck or perhaps a space which crosses itself at a certain point. This question will be investigated here.

We consider again a stationary system with spherical symmetry. In the same way as above we may define the distance $r$ from the centre of symmetry by putting $p = 1$ viz. by demanding that the periphery of a circle with its centre at the centre of symmetry is $2\pi r$, when expressed in natural units. If we do so in the case in question, the state in the field is not a single-valued but within a certain interval at least a more-valued function of $r$. It is therefore useful to introduce a new radial space-coordinate of which the quantities in the field are single-valued functions. As such a coordinate the distance $s$ from the centre of symmetry expressed in natural units suggests itself. In order to specialize our discussion we can prescribe a relation between the radius defined by the condition $p = 1$ and $s$ and investigate afterwards whether this is in agreement with a possible distribution of the components $T_\mu^\nu$ of the stress-energy-tensor.

As a trial we put

$$r = \pm s\left(\frac{s^2}{3a^2} - 1\right), \quad \ldots \ldots \quad (37)$$

where $a$ is a constant, and we choose the sign thus that a positive value of $r$ corresponds to a positive value of $s$. For small values of $s$ $r$ and $s$ are proportional and the three-dimensional space is dilated when we come farther away from the centre (viz. from the point $s = 0$). For $s = a$ $r$ reaches however a maximum and when $s$ increases still further the space is contracted and crosses itself at a point in the neighbourhood of $s = \sqrt{3}\,a$. For still higher values of $s$ the space is again dilated.

Before proceeding we still remark that in fact the sign of $r$ does not play a role. Inversing the sign of $r$ in our fundamental formulae (5), (6) and (7) and interchanging also the signs of $dr$ and $w'$ we find from the formulae the same values as above for all remaining quantities. For this reason we take in (37) everywhere the $+$ sign, so that $r$ is taken negative in the interval $0 < s < \sqrt{3}\,a$.

While the following discussions will be based on the fundamental equations (5), (6), (7), we suppose $u, w, r, T_1^1, T_p^p, T_4^4$ to be functions of $s$. As $s$ is the distance from the centre of symmetry expressed in natural units we obtain, attending to the meaning of the quantity $u$ (see I § 3)

$$ds = u\,dr, \quad \ldots \ldots \quad (38)$$

As (37) gives by differentiation

$$dr = \left(\frac{s^2}{a^2} - 1\right) ds, \quad \ldots \ldots \quad (39)$$

we find for $u$

$$u = \frac{1}{\frac{s^2}{a^2} - 1} \quad \ldots \ldots \ldots \ldots \quad (40)$$

That $u$ is negative for $s < a$, does not cause any trouble, as the fundamental tensor depends on $u^2$ only.

Now we must introduce in equation (6) the expressions (37) and (40) for $r$ and $u$. Introducing to begin with the expressions on the left hand-side only we obtain

$$-\frac{s^2}{a^2}\left(\frac{7}{3}\frac{s^2}{a^2} - 6\right) = r^2 \varkappa T_4^4 \quad \ldots \ldots \quad (41a)$$

Introducing the expressions on the right-hand side too we find for $T_4^4$ as a function of $s$

$$\varkappa T_4^4 = \frac{6 - \frac{7}{3}\frac{s^2}{a^2}}{a^2\left(\frac{s^2}{3a^2} - 1\right)^2} \quad \ldots \ldots \ldots \quad (41)$$

The formulae derived here hold evidently only inside the material system of which the outer boundary may be indicated by $s = S$. In order that the space occupied by the system may cross itself at any point we must have because of (37).

$$S > \sqrt{3}\, a.$$

In the limiting surface $s = S$ we have according to (40) $u < 1$. In order that in that surface $u$ may pass continuously into the value it has in the field on the outside, $u$ must also in the outer field be smaller than 1 for $s = S$. This follows also from formula (11) II, when the system has only a sufficient great electric charge. Further it does not matter that $u$ would change discontinuously at the boundary, if only this is a material plane as considered in § 2.

Formula (41) shows that in the interval $\sqrt{\frac{18}{7}}\, a < s < S$ $T_4^4$ is negative, which though somewhat startling is not at all absurd. Further formula (41) indicates that $T_4^4$ becomes infinite for $s = \sqrt{3}\, a$. Within a finite extension there is however only a finite mass of matter, which follows from the fact that $r^2 T_4^4$ is everywhere finite according to (41).

The equations (40) and (41) for $u$ and $T_4^4$ involve together with (37) that the fundamental equation (6) is satisfied. Now we must still determine $w$, $T_r^r$ and $T_p^p$ as functions of $s$, so that also the equations (5) and (7) are satisfied. As (5), (6) and (7) form the complete set of field equations for a stationary gravitation field, we may choose for one of the quantities $w$, $T^r_r$ and $T^p_p$ an arbitrary single-valued function of $s$. When also the expressions for $r$, $u$ and $T^4_4$ are introduced, the equations (5) and (7) determine now the two quantities. All these possible material systems give — if only the distribution of $T^4_4$ is the same — a three-dimensional space of the same curvature, because the formulae $(25)I$ perfectly determine the space-components of the fundamental tensor (when $p = 1$). The curvature of the four-dimensional space-time continuum on the contrary depends also on the distribution of $T^r_r$ which quantity according to (5) influences $w$. The following simple assumptions might e. g. be made to obtain a definite system : $w =$ constant, $T^r_r = 0$ or $T^r_r = T^p_p$ (normal pressure in all directions). Performing the integration of (15) and (7), we might choose the integration constants in such a way, that at the boundary $s = S$ $w$ takes the value that holds there for the outer field, for, as was proved in § 1, $w$ changes continuously into a surface of discontinuity.

The purpose of our investigation being reached no further calculations will be added. We have shown that EINSTEIN's theory of gravitation really admits such a distribution of the stress-energy-tensor $T$, that the (three-dimensional) space crosses itself at a certain point. We can also prove without difficulty, that systems can exist in which the space filled with the matter runs out into a narrow neck.

It is still of some importance to investigate the action of the electric forces within the space which is just dilated and afterwards again contracted. We might e. g. investigate the state, when, with constant $T^4_4$, $T^r_r$, $T^p_p$ for the non electro-magnetic matter, a point charge was placed at the centre of symmetry. The gravitation field will evidently change. We have not only to calculate this field, but also the laws of the equilibrium and the motion of other electric (point)-charges in the new electric field. Here we must treat the matter as perfectly permeable. These indications may however suffice, which show already that EINSTEIN's theory opens wide possibilities to explain the state in the inside of an atom.