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Physics. - *"On the Brownian Motion"*. By Prof. L. S. ORNSTEIN. (Communicated by Prof. H. A. LORENTZ).

(Communicated in the meeting of December 29, 1917.)

VON SMOLUCHOWSKI [1]) observed that the function which gives the probability, that in the Brownian Motion a particle accomplishes a definite way in a given time is a solution of the equation of diffusion. For cases in which an exterior force also acts on the particles, he deduced a differential equation for the above-mentioned function of probability by a phenomenological method. Some time after Mr. H. C. BURGER [2]) deduced this differential equation following a method, which takes the essence of the function of probability more into consideration. Both deductions do not stand in direct connection with the mechanism of the Brownian motion; my object in this paper is to demonstrate, that starting from a relation which Mrs. DE HAAS—LORENTZ [3]) has used in her dissertation, to determine the average square of the distance accomplished, one is able to determine the function of probability of the Brownian motion. It is worth observing that the way in which different averages depend on the time may be calculated from the results obtained by Mrs. DE HAAS—LORENTZ by a slightly more careful transition of the limit than was necessary for the object she had put herself (viz. the determination of the stationary condition). First I want to determine these averages by a new method, which will offer the opportunity of demonstrating, that the opinion, from which Dr. A. SNETHLAGE [4]) starts in the theory of the Brownian motion that EINSTEIN's theory is in conflict with statistical mechanics, is incorrect.

Besides the function of probability for the distance I shall also deduce that for the velocity. The chain of thoughts which lead to

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1) Compare e.g. M. v. SMOLUCHOWSKI. Drei Vorträge über Diffusion, BROWN'sche Bewegung etc. Phys. Zeitschr. XVII p. 557 1916.

2) H. C. BURGER, Over de theorie der BROWN'sche beweging. Verslagen Kon. Ak. XXV p. 1482, 1917.

3) Mrs. Dr. G. L. DE HAAS—LORENTZ. Over de theorie der BROWN'sche beweging, Diss. Leiden 1912.

4) Miss Dr. A. SNETHLAGE, Moleculair-kinetische verschijnselen in gassen etc. Diss. Amst. 1917.

the results given below shows great similarity to the deductions which Lord RAYLEIGH[1]) gave utterance to already years ago. Kindred ways of regarding the stationary condition are also found in the work of Dr. FOKKER[2]) and M. PLANCK[3]).

§ 1. In the dissertation Mrs. DE HAAS—LORENTZ starts from the equation of motion for a emulsion particle, which she brings in the formula

$$m\frac{du}{dt} = -wu + mF \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Here u is the velocity of the particle, $w = 6\pi\mu a$ the resistance which according to STOKES' formula the spherical particle (radius a) would experience in a liquid with internal coefficient of friction μ. The force expended by the shocks of the molecules is divided into two parts, of which one is that according to STOKES, the second is quite irregular, so that $\overline{F} = 0$. The determination of the average is to be understood in this way that it is to be taken at a given moment for particles which all have had the same velocity u_0 a time before.

Now we are able to integrate the equation (1), if we introduce $\frac{w}{m} = \beta$, we have

$$u = u_0\, e^{-\beta t} + e^{-\beta t}\int_0^t e^{\beta t}\, F(t)\, dt \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where u_0 is the velocity at the time $t = 0$.

If then we determine the average of this equation in the way indicated, the result is

$$\overline{u} = u_0\, e^{-\beta t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

or expressed in words: when we start from a great number of particles of given velocity, the average velocity decreases in the same way as with large spheres; the damping coefficient also is deduced in the same way from radius and coefficient of friction of the fluid. Let us now calculate also the average of the square of the velocity. For this we find:

[1]) Lord RAYLEIGH, Phil. Mag. XXXII, p. 424. 1894. Papers III. Dynamical problems in illustration of the theory of gases.

[2]) Dr. A. FOKKER, Over de BROWN'sche beweging in het stralingsveld. Diss. Leiden, pg. 523, 1913.

[3]) M. PLANCK, Ueber einen Satz der Statistischen Dynamik u.s.w. Berl. Ber. p. 324. 1917.

$$\overline{u^2} = u_0^2\, e^{2\beta t} + e^{-2\beta t} \overline{\left\{ \int_0^t e^{\beta t}\, F(t)\, dt \right\}^2} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In order to determine the integral in the second member we proceed in the following way. We write for it

$$\int_0^t \int_0^t \overline{F(\xi)\, F(\eta)}\, e^{\beta(\xi+\eta)}\, d\eta\, d\xi.$$

Now $\overline{F(\xi)\, F(\eta)}$ is only differing from zero if η and ξ differ very slightly, i.e. there is for short periods a correlation of the forces F. If we introduce $\eta = \xi + \psi$, it is allowed to replace η in the exponent by ξ and to split up the integral into a product of integrals according to ξ and ψ where we may integrate from $-\infty$ to $+\infty$. If then we assume

$$\int_{-\infty}^{+\infty} \overline{F(\xi)\, F(\xi + \psi)}\, d\psi = \vartheta \quad . \quad . \quad . \quad . \quad . \quad (4a)$$

which is a constant characteristic of the problem and if we perform the integration towards ξ, then (4) is transformed after substitution into

$$\overline{u^2} = u_0^2\; e^{-2\beta t} + \frac{(1 - e^{-2\beta t})\, \vartheta}{2\beta} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

When applying this equation for $t = \infty$, $\overline{u^2} = \frac{kT}{m}$ and thus we get

$$\vartheta = \frac{kT}{m}\, 2\beta$$

In the same way we are able to determine the average square of the distance accomplished. From (2) or by direct integration from (1) we get namely

$$u - u_0 = -\beta s + \int_0^t F\, dt$$

$$\beta^2\, \overline{s^2} = \overline{(u - u_0)^2} + \overline{\left\{ \int_0^t F\, dt \right\}^2} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

For the last integral we find in a quite analogous way

$$\vartheta\, t$$

If we calculate the first average with the help of (3) and (5) we obtain

$$\beta^2 \overline{s^2} = u_0^2 (1 - 2e^{-\beta t} + e^{-2\beta t}) + \frac{\vartheta}{2\beta}(-3 - e^{-2\beta t} + 4e^{-\beta t}) + \vartheta t \quad . \quad (7)$$

Consequently for very long periods we find

$$\overline{s^2} = \frac{2kT\vartheta}{\beta} = \frac{kT}{3\,a\mu}\,t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

which is the well-known formula for the average distance in the Brownian motion. If we determine the average of (7) with reference to all possible initial velocities and if we consider that $\overline{u_0^2} = \frac{\vartheta}{2\beta}$, we find for the average square of the distance accomplished as an arbitrary initial velocity:

$$\beta^2 \overline{s^2} = \frac{\vartheta}{\beta}(t\beta - 1 + e^{-\beta t}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7a)$$

As long as βt is large in relation to $1 - e^{-\beta t}$ the formula of EINSTEIN is thus the right one. For the cases, considered in experiments, the lowest limit for t to be obtained in this way is of the order of 0.01 second.

§ 2. On the basis of statistical mechanics objections have been raised by Prof. J. D. v. D. WAALS JR. and Miss A. SNETHLAGE [1]) to the application of the division which has been applied to this case upon the example of EINSTEIN and HOPF in their treatment of another problem.

Starting from the supposition than in an "ensemble".

$$\overline{Ku} = 0$$

where K is the active force, they work out another fundamental formula viz. (with a slight variation in notation)

$$\frac{d^2u}{dt^2} = -\varrho^2 u + w \quad . \quad . \quad . \quad . \quad . \quad . \quad - \quad . \qquad (9)$$

where $\overline{w}$ has to been taken zero. We can again integrate this equation and obtain then

$$u = u_0 \cos \varrho t + \frac{\dot{u}_0}{\varrho} \sin \varrho t + \frac{1}{\varrho}\int_0^t w(\xi) \sin \varrho (t - \xi)\, d\xi \quad . \quad . \qquad (10)$$

If taking the average we get:

$$\overline{u} = u_0 \cos \varrho t + \frac{\dot{u}_0}{\varrho} \sin \varrho t$$

The average velocity would in this way possess a definite period. If however we work out $\overline{u^2}$ we arrive at an incompatibility.

[1]) Cf. Versl. Kon. Ak. v. Wet. XXIV. 1916. p. 1272.

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Because for $\overline{u^2}$ we get

$$\overline{u^2} = \left(u_0 \cos \varrho t + \frac{\dot{u}^0}{\varrho} \sin \varrho t\right)^2 + \frac{1}{\varrho^2} \overline{\left\{\int_0^t w(\xi) \sin \varrho (t-\xi) d\xi\right\}^2} \quad . \quad (11)$$

For the integral, if again we make a double integral of it and if we introduce the constant θ

$$\theta = \int_{-\infty}^{+\infty} w(\xi) w(\xi + \psi) d\psi \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

we can write

$$\frac{\theta}{\varrho^2} \int_0^t \sin^2 \varrho (t - \xi) d\xi = \frac{\theta t}{2\varrho^2} + \text{periodical terms.}$$

Thus we find

$$\overline{u^2} = \frac{\theta t}{2\varrho^2} + \text{periodical terms} \quad . \quad . \quad . \quad . \quad . \quad (13)$$

This formula shows that $\overline{u^2}$ increases indefinitely with the time, while it is evident according to statistical mechanics that $\overline{u^2}$ must approach $\frac{kT}{m}$.

Consequently if the equation (9) is treated as a differential equation we arrive at results which are not right [1]).

§ 3. Miss Dr. SNETHLAGE and Prof. Dr. J. v. D. WAALS JR. have observed, that the theory of the Brownian motion must be in accordance with a general theorem of statistical mechanics. For the case that we consider a particle, that under the influence of the impacts of the molecules of the liquid executes a movement, the force which the molecules exercise does not depend upon the velocity, but only upon the coordinates. Consequently the product of force and velocity must on the average be zero, as well in a canonical as in a microcanonical as in a time ensemble. Now they are of opinion that EINSTEIN's formula comes into conflict with this. I shall demonstrate that this is only the case to a certain extent.

If we assume in a canonical (or micro-canonical ensemble) all systems selected in which the velocity of our particle at a point of time 0 is equivalent to u_0 and if then we follow this group of particles,

[1]) An analogous question is treated by M. PLANCK (Ann. der Phys. 1912. Bd. 37 p. 462) where it is demonstrated that the energy of a resonator subjected to the irregular field of black radiation increases in proportion to the time; the f of PLANCK agrees here with v. D. WAALS' u.

we can work out an average of every arbitrary quantity for the group of systems which after a time t has developed from the group considered at first. The value of the quantity considered varies for the different systems of our group (part ensemble), because the systems where at $t = 0$ the velocity of the particle is u_0, may still show considerable differences, so that e.g. the impulses which the particle gets will be widely different. I shall call this average the *case-average with a given initial velocity*. Moreover the velocity u_0 may be varied and again the case-average may be worked out and then by making u_0 run through all possible values all systems are taken into consideration in determining the average at the time t. If now the case-average of a quantity $g(u)$ for u_0 is $g(u_0)$ and if the number of systems of the group is $N(u_0)$ then — if N represents the total number of systems in the ensemble — the quantity

$$\frac{\Sigma\, N(u)\, g(u)}{N}$$

is the case-average for the entire ensemble.

However, as the ensemble is stationary the case average for a quantity is equal to the average of the corresponding quantity in the ensemble. If in particular $g(u)$ for every u is equal to zero, the average in the ensemble is also zero. I shall now demonstrate that if we start from EINSTEIN's formula the case average of Ku for every initial velocity u_0 is zero, and from this it follows immediately that EINSTEIN's formula does *not come into conflict* with the theory of the ensembles, more particularly that $\overline{Ku}^e = 0$, ($\overline{}^e$ means determining the average of an ensemble, which $\overline{}$ is used everywhere here for the case average).

EINSTEIN's equation comes into conflict with the theory of the ensembles if we select at $t = 0$ a group of particles with a given velocity u_0 from the ensemble. For if we determine the average of the equation it yields $m\overline{\frac{du}{dt}} = \overline{K} = -wu_0$, whilst according to the theory of ensembles $\overline{K}$ is independent of the velocity. If however we leave the selected group to itself and if we apply to its motion EINSTEIN's equation which is not right in the first moment, it is evident that in the long run the group moves in such a way that in the long run EINSTEIN's equation can be applied to it. Moreover from the group with particles with a given velocity u_0 those systems can be selected to which (1) applies. From what follows it becomes apparent that for this group in the long run the usual relations with regard to the averages in a canonic ensemble become the right ones.

Instead of $\overline{Ku} = 0$ we can also write $m \overline{\frac{du}{dt} u} = 0$ or $\overline{u \frac{du}{dt}} = 0$. If now we multiply the EINSTEIN-HOPF equation by u and work out the case average, we get

$$\overline{u \frac{du}{dt}} = - \beta \overline{u^2} + \overline{u F}$$

Further we shall demonstrate that for sufficiently long periods the second member is identically zero.

For according to (5) we have got for the first term $-\frac{\vartheta}{2\beta}\beta$ or $-\frac{\vartheta}{2}$. We shall now determine the second average For this purpose we multiply (2) by F and work out the average, and arrive at

$$\overline{uF} = \overline{u_0 e^{-\beta t} F} + \overline{e^{-\beta t} F \int_0^t e^{\beta t} F(t)\, dt}$$

In this formula the first average in the second member is zero. To determine the second average we must consider that F for the integral refers to a definite time t. And so only those parts of the integral where the argument differs only slightly from t yield distributions to the average. In the exponent we can again take the argument equivalent to t, so that we can write for the second term

$$\int_{t-\eta}^{t} \overline{F(t)\, F(t-\eta)}\, d\eta \quad \text{or} \quad \int_{-\infty}^{0} \overline{F(\xi)\, F(\xi-\eta)}\, d\eta$$

Now this integral is just half of the integral of (4*a*), as it covers half the region of integration, whilst the integrand is of course symmetrically with regard to ξ.

And so in the end we find for the second member identically zero as the two averages first neutralize each other. If now we take (5) into consideration for shorter periods the second member becomes

$$u_0{}^2 e^{-2\beta t} - \frac{\vartheta}{2\beta} e^{-2\beta t}$$

This result is also obtained by differentiating (4) with respect to the time.

The case average of $u \frac{du}{dt}$ for finite times not large with reference to $\frac{1}{\beta}$ is consequently not strictly zero. Now we can however determine the average according to the initial velocity u_0 and so find the ensemble average of $u\frac{du}{dt}$. Then we obtain

$$\overline{u\frac{du}{dt}}^e = \left(\overline{u_0^2}^e - \frac{\vartheta}{2\beta}\right)e^{-2\beta t} = 0$$

as $\overline{u_0^2}^e$ possesses the equipartition value. Thus it is proved that also for short periods the average value of the case average does not come into conflict with statistical mechanics.

4. I shall now deduce the law of frequention for distribution of velocity. If we integrate the equation (1) for a short time τ, we can write for it

$$u - u_0 = -\beta u_0 \tau + x \quad \text{or} \quad u = u_0(1-\beta\tau) + x \quad . \quad . \quad (14)$$

where $x = \int_0^\tau F(t)\,dt$ and $\overline{x^2} = \vartheta\tau$.

Now there is for x a law of frequention $\varphi(x)$, so that

$$\int_{-\infty}^{+\infty}\varphi(x)\,dx = 1 \quad , \quad \int_{-\infty}^{+\infty}x\varphi(x)\,dx = 0 \text{ and } \int_{-\infty}^{+\infty}x^2\varphi(x)\,dx = \vartheta\tau \quad . \quad (15)$$

If now a particle starts with a given velocity u_0, the number of particles, for which in the time t the velocity lies between u and $u + du$, may be represented by

$$f(u_0, u\, t)\,du$$

or shorter

$$f(u, t)\,du$$

Let us now consider the distribution of velocity at the time $t+\tau$ and again fit our attention on the particles whose velocity lies between u and $u + du$. These particles have had at t a velocity u' in such a way that

$$u'(1-\beta\tau) = u - x$$

or

$$u' = u(1+\beta\tau) - x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

whilst an interval $du' = (1+\beta\tau)du$ corresponds to the interval du). The number of particles that is at t in du' and at $t+\tau$ in du consequently amounts to

$$f(u', t)\,\varphi(x)\,dx\,du'$$

and thus we get

$$f(u, t + \tau)\, du = (1 + \beta\tau)\, du \int f(u', t)\, \varphi(x)\, dx \;. \qquad (17)$$

If now we work this out and retain the terms up to the first order in τ and if we take (15) into consideration and if after division by τ, we make τ approach zero, we obtain

$$\frac{df(u, t)}{\partial t} = \beta \frac{\partial}{\partial u}(u \,.\, f) + \frac{\vartheta}{2}\frac{d^2 f}{\partial u^2} \quad . \; . \; . \; . \; . \qquad (18)$$

To the function of frequency for the velocity the extended equation of diffusion is thus applicable, where ϑ plays the part of coefficient of diffusion. The equation is quite of the same form as that for the Brownian motion under the influence of a quasi-elastic force ($-u$ or $-s$) (cf. also § 4). If we apply (18) to determine the stationary condition we have

$$0 = \beta \frac{\partial}{\partial u}(uf) + \frac{\vartheta}{2}\frac{\partial^2 f}{\partial u^2}$$

from which follows

$$f = C_1 e^{-\frac{\beta}{\vartheta}u^2} + C_2 e^{-\frac{\beta}{\vartheta}u^2}\int_0^u e^{\frac{\beta}{\vartheta}u^2} du.$$

This last term becomes infinite for $u = \infty$, consequently the integration constant must be taken $c_2 = 0$.

For the law of distribution we thus find the MAXWELL division of velocity quite independent of the initial condition. Moreover RAYLEIGH has carefully investigated this question for his particular example. He has deduced a similar equation for a particle in a highly rarefied gas, where only the constants β and $\frac{\vartheta}{2}$ have another meaning (cf. loc. cit.). It goes without saying that if one starts from the equation of v. D. WAALS-SNETHLAGE, one arrives at the conclusion that the division after long periods is *not* that of MAXWELL, and that there does not even exist a stationary division of velocity. And on this point also these investigators thus come into conflict with the statistical mechanics of GIBBS, which is the starting-point of their reasonings.

It may further be observed that for a particle beginning with a velocity zero, as long as u is still small with respect to the velocity of the particles, which collide against it, we get as RAYLEIGH has demonstrated

$$\frac{\partial f}{\partial t} = D\frac{\partial^2 f}{\partial u^2}.$$

For the change in velocity we get then at each impact according to RAYLEIGH

$$u' = u \pm 2q\,v,$$

where q is the relation of the masses of particles and molecules, v the velocity of the molecules. Now the problem treated by RAYLEIGH in this way may be connected directly with the theory of the function of probability for the way in the Brownian motion. If we take the velocity marked as vector, the terminal point is removed $\pm qv$ after every shock. The terminal point of the vector consequently executes a Brownian motion at least according to the scheme which is often given of it (cf. e.g. Mrs. DE HAAS—LORENTZ' dissertation). It is certainly remarkable how Lord RAYLEIGH had already so long ago deduced these results, which came to the foreground only by SMOLUCHOWSKI's work, which opened so many new views.

It may have its advantages now it has become apparent that EINSTEIN's formula is the right one to say something further on the kinetic mechanism. Let us first direct our attention to a single shock of a particle of a great mass with a particle of a small one. If the velocity for the first is before the shock u', after the impact u, the velocity of the small particle v and the relation of the masses q, where we have $q \ll 1$, then we get for every impact:

$$u = u'(1-q) \pm q\,v.$$

If we assume then that again and again after a time τ a collision takes place, then we have

$$\frac{u-u'}{\tau} = -\frac{q}{\tau}u' \pm \frac{q}{\tau}v$$

for every impact. We can only make a differential equation of this equation of differences by taking τ infinitely small. if q is of the same order infinitely small and then we get

$$\frac{du}{dt} = -\beta u \pm \beta v,$$

where F may be written for βv. Thus we see here by a (not very strict approach to the limit) EINSTEIN's equation arise as it were. If now we do not go to the limit, but avail ourselves of the following graphic representation, its meaning becomes even more clearly visible. On one axis we measure out the time (and to make things easier we take again equal intervals between the impacts), on the other the velocity. Between two collisions the velocity is then constant, at an impact the velocity suddenly jumps to another value and this jump consists in every case in two parts; one part proportional to the velocity of the particle before the shock with which the velocity decreases and one part which may be either positive or negative (and in general may posses all sorts of values dependent upon the conditions of the impacts, which in the simple case investigated by RAYLEIGH is $\pm qv$). The velocity-time curve is thus a discontinuous curve. If the velocity has become large it has the tendency to become smaller by shocks owing to the first part, whilst the second part exercises no systematic influence in a contrary sense. If now we imagine a combination of curves drawn starting from a given velocity, EINSTEIN's equation will represent for each of these discontinuous curves the differential equation. At the same time if we introduce the curve $u = u_0 e^{-\beta t}$ into the scheme, this line will at all times be an average of the discontinuous velocity time-curves in the diagram.

§ 4. Finally I will deduce the function of probability for the Brownian motion under the influence of an external force. We take this force km, where k depends upon the place (s).

The equation of motion for our particle is then the following

$$\frac{du}{dt} = -\beta u + F + k \cdot \quad . \quad . \quad . \quad . \quad . \qquad (19)$$

If now a particle has in the time $t = 0$ a velocity u^0, if in a time $t - \tau$ the velocity has become u' and a way s' is accomplished, and if u and s represent these magnitudes in a time t, we get

$$u - u' = -\beta(s - s') + \int_{t-\tau}^{t} F\,dt + \int_{t-\tau}^{t} k\,dt.$$

We now consider the time so small that the way accomplished in that time is small enough to treat K in the last integral which depends upon s as a constant.

We have thus

$$u - u' = -\beta(s - s') + \int_{t-\tau}^{t} F\,dt + k\tau \quad . \quad . \quad . \quad . \quad . \qquad (20)$$

Now we want $\overline{\Delta s} = \overline{s - s'}$ and $\overline{\Delta s^2}$. In order to determine these we apply (3), this yields

$$\overline{u} - \overline{u'} = u_0\, e^{-\beta t}(1 - e^{+\beta\tau})$$

as we have to take the mean value of Δs for all possible values of u_0 the average of $\overline{us}$ being zero we get

$$\beta\,\overline{\Delta s} = k\tau \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (21)$$

and in the same way

$$\overline{\Delta s^2} = \vartheta \tau \, . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

In order to arrive then at the differential equation for the function of frequency we reason again in the same way as before. Let $f(s,s_0t)$ represent the chance that a particle that at the time 0 has the coördinate s_0 possesses at the time τ the coördinate s (with margin ds) where we determine the average according to the initial velocity. Now we follow the movement for a short time τ and build the function of frequention at the time $t+\tau$ from that on time s. If Δs again represents the mean deviation during the time τ, and $\varphi(\Delta s)$ the function of frequency, we know for this deviation that we have

$$\int \varphi(\Delta s)\, d\Delta s = 1, \quad \int \Delta s \varphi(\Delta s)\, d\Delta s = k\tau \text{ and} \int (\Delta s)^2 \varphi(\Delta s)\, d\Delta s = \vartheta \tau \quad (22a)$$

We then obtain

$$f(s, s_0, t+\tau)\, ds = \int ds' f(s', s_0\, t)\, \varphi(\Delta s)\, d\Delta s \quad . \quad . \quad . \quad (23)$$

where $s' = s - \Delta s$.

If now we take (20) into consideration we find for the connection of ds' and ds

$$ds' = \left(1 - \frac{1}{\beta}\frac{\partial k}{\partial s}\tau\right) ds.$$

Developing according to (23) up to the first order with respect to τ, we find

$$\frac{\partial f}{\partial t} = -\frac{1}{\beta}\frac{\partial}{\partial s}(kf) + \frac{\vartheta}{2}\frac{\partial^2 f}{\partial s^2} \quad . \quad . \quad . \quad . \quad . \quad (24a)$$

If we introduce the value for ϑ and β we obtain

$$\frac{\partial f}{\partial t} = -\frac{m}{6\pi\mu a}\frac{\partial}{\partial s}(kf) + \frac{m}{6\pi\mu a}\frac{\partial^2 f}{{}^2\partial s} \quad . \quad . \quad . \quad . \quad (24)$$

This equation agrees with that of SMOLUCHOWSKI, if we take D (coefficient of diffusion of the Brownian motion $\frac{kT}{6\pi\mu a}$). The factor $\frac{1}{6\pi\mu a}$ is the ψ of SMOLUCHOWSKI i.e. the factor with which the force mk must be multiplied in order to calculate the velocity which in a stationary condition was caused by this force.

By DEBYE and his pupil Dr. TUMMERS[1]) a differential equation for

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[1]) DEBYE, Zur Theorie der anomalen Dispersion. Verh. Deutsch Phys Ges. X, p. 790.

J. TUMMERS, Over electrische dubbelbreking, Diss. Utr. 1914.

the function of the frequention of the axes has been deduced for the case of molecules (particles) which turn in a liquid, which is acted upon by an external couple and by a couple resulting from the molecular impacts. The deduction of the results obtained there follows immediately from our formulae. We need only split up the couple exerted by the molecular movement into two parts, the one $-\varrho\dot{\alpha}$ ($\dot{\alpha}$ be the angular velocity of the particle) $\varrho = 8\pi\mu a^3$ (a radius of the particle) and into a second part of which the average value (case average) is zero. For the motion of the axis we get then

$$P\ddot{\alpha} = -\varrho\dot{\alpha} + X$$

where P is the moment of inertia of the particle.

If we take $\dot{\alpha} = u, \frac{\varrho}{P} = \beta, \frac{x}{\varrho} = F$, we get for u equation (1), from which appears that the function of frequention can be deduced from a differential equation of the form (24a).

Finally it may be observed that it offers no difficulties to extend our considerations to the Brownian motion of coordinates in systems with an arbitrary number of degrees of freedom.

Utrecht, Dec. 8 1917. *Institute for Theoretical Physics.*