Physics. — *"On the rotational oscillations of a cylinder in an infinite incompressible liquid".* By D. COSTER. (Communicated by Prof. J. P. KUENEN).

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The method to be followed in the discussion of the problem will be in the main the same as that used by Prof. VERSCHAFFELT in the analogous case of the sphere [1]). We consider the rotational swings about its axis of an infinitely long cylinder which executes a forced vibration. Our object will be to ascertain the motion in the liquid which will establish itself after an infinite time (in practice after a relatively short time [2])) in order to compute the frictional moment of forces exerted on the cylinder by the liquid. For the sake of simplicity the calculations will be referred to a height of 1 cm.

The motion of the cylinder may be represented by $\alpha = a \cos pt$ where α is the angle of rotation. An obvious assumption to be made is that the liquid will be set in motion in coaxial cylindrical shells each of which will execute its oscillations as a whole. On this assumption it is not difficult to establish the differential equation for the motion of the liquid.

Let ϱ be the density of the liquid.
μ the viscosity of the liquid.
ω the angular velocity of a cylindrical shell.
r the radius of the shell.

The frictional force per unit surface of one of the shells will then be $F = r\mu \frac{\partial \omega}{\partial r}$ and the frictional couple on a cylindrical surface of radius $r : 2\pi r^3 \mu \frac{\partial \omega}{\partial r}$.

Taking a shell of thickness dr its equation of motion will be

$$2\pi r^3 \, dr \, \varrho \frac{\partial \omega}{dt} = \frac{\partial}{\partial r} \left\{ 2\pi r^3 \mu \frac{\partial \omega}{\partial r} \right\} dr,$$

which reduces to

$$\frac{\varrho}{\mu} \frac{\partial \omega}{\partial t} = \frac{\partial^2 \omega}{\partial r^2} + \frac{3}{r} \frac{\partial \omega}{\partial r} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

1) Comp. Proceedings 18 p. 840. Sept. 1915. Comm. Leiden 148*b*.
2) Comp. Comm. 148*b*. pag. 22 footnote.

It is important to note that equation (1) may also be deduced from the general equation of hydrodynamics without its being necessary to neglect the second power of the velocities, as is the case in many problems of that kind. For an infinitely long time of vibration i. e. for uniform rotation (1) simplifies to

$$0=\frac{d^2\omega}{dr^2}+\frac{3}{r}\frac{d\omega}{dr} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The solution of (2) is $\omega=\frac{c_1}{r_2}+c_2$, c_1 and c_2 being integration-constants. If the solid cylinder (radius R) rotates with uniform speed Ω in an infinite liquid, the result will be $\omega=\frac{R^2\Omega}{r^2}$, giving for the frictional couple as is well known the expression

$$-4\pi\mu R^2\Omega . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2')$$

In order to arrive at a possible solution of (1) we have to make our assumption regarding the motion of the liquid a little more definite by assuming that the angular displacement of each shell is represented by

$$\alpha_r=f(r)\cos\left(pt-\varphi(r)\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

We may also consider (3) as the real part of the complex function ue^{ipt}, where u is a function of r the module of which gives the amplitude of the oscillation and the argument the phase-shift $\varphi(r)$. Remembering that $\omega=\frac{\partial\alpha}{\partial t}$ equation (1) may be reduced to

$$\frac{d^2u}{dr^2}+\frac{3}{r}\frac{du}{dr}-\frac{i\varrho pu}{\mu}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Equation (4) is closely related to the differential equation of the cylindrical functions. Indeed by the substitution $y=zv$ BESSEL's equation of the 1st order $\frac{d^2y}{dz^2}+\frac{1}{z}\frac{dy}{dz}+\left(1-\frac{1}{z^2}\right)y=0$, changes to

$$\frac{d^2v}{dz^2}+\frac{3}{z}\frac{dv}{dz}+v=0.$$

It follows that the general solution of equation (4) is

$$u=\frac{1}{r}\{A\,J_1(cr)+B\,N_1(cr)\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where $c=\sqrt{\frac{-i\varrho p}{\mu}}$, A and B being complex integration-constants.

J_1 is the cylindrical function of the 1st kind and 1st order, N_1 that of the 2nd kind and 1st order [1]).

As regards c an agreement must be come to. We shall choose the root with the negative imaginary part i.e. $c = ke^{-\frac{i\pi}{4}}$, where $k = |c| = \left|\sqrt{\frac{\varrho p}{\mu}}\right|$.

As a first boundary-condition we have $\lim\limits_{r=\infty} r\alpha_r = 0$ [2]). As this relation must hold for all values of t, it follows that $\lim\limits_{r=\infty} ru = 0$.

The cylindrical functions with complex argument all become infinite at infinity with the exception of the so-called functions of the 3rd kind or HANKEL's functions $H_p^{(1)}$ and $H_p^{(2)}$. Of these $H_p^{(1)}$ disappears at infinity in the positive imaginary half-plane and on the contrary becomes infinite in the negative half, whereas the opposite is true for $H_p^{(2)}$. By our choice of c in the negative imaginary half we are led to the function $H_1^{(2)}$. For the integration-constants in equation (5) this gives the relation $B = -iA$ [3]), so that (5) becomes

$$u = \frac{A}{r} H_1^{(2)}(cr) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

For the determination of A we have to use the 2nd boundary-condition $\alpha_R = a \cos pt$, R being the radius of the cylinder. We therefore assume that there is no slipping along the wall.

Hence $A = \dfrac{aR}{H_1^{(2)}(cR)}$,

so that

$$\alpha_r = \mathrm{R}\, \frac{aR}{H_1^{(2)}(cR)} \, \frac{H_1^{(2)}(cr)}{r} \, e^{ipt} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The symbol R is intended to indicate, that the real part has to be taken of the function which stands after it.

If we had chosen for c the root with the positive imaginary part, we should have had to utilize the function $H_1^{(1)}$. It is quite easy to verify that this would not have made any essential change in the solution (7).

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[1]) Comp. JAHNKE u. EMDE. Funktionentafeln pp. 90 and 93.
NIELSEN. Cylinderfunktionen. Instead of N NIELSEN uses the symbol Y.

[2]) Prof. VERSCHAFFELT puts $\lim\limits_{r=\infty} \alpha_r = 0$, which in my opinion is not quite correct, as the *linear* velocity has to disappear at an infinite distance. Comm. 148*b* p. 22.

[3]) Between J, N, and H a linear relation holds. Comp. J. u E. p. 95.

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For large values of x (real, positive) $H_1^{(2)}(x\sqrt{-i})$ approaches asymptotically to

$$-\frac{e^{-\frac{x}{\sqrt{2}}}}{\sqrt{\frac{1}{2}\pi x}} e^{-i\left(\frac{x}{\sqrt{2}}-\frac{\pi}{8}\right)};$$

therefore for $(k\ R)$ sufficiently large:

$$\alpha, \approx -\frac{aR}{|H_1^{(2}(cR)|} \frac{e^{-\frac{kr}{\sqrt{2}}}}{\sqrt{\frac{1}{2}\pi k}\, r^{1\frac{1}{2}}} \cos\left(pt-\frac{kx}{\sqrt{2}}+\frac{\pi}{8}-\varphi\right) \quad . \quad (8)$$

where $\varphi =$ argument $H_1^{(2)}(c\ R)$.

From (8) it appears that damped waves are propagated from the cylinder to infinity, the velocity of propagation being

$$v=\frac{p}{k/\sqrt{2}}=\frac{p\sqrt{2}}{k}=\sqrt{\frac{2p\mu}{\varrho}}$$

and the wave-length

$$\lambda = vT = \frac{2\pi v}{p} = \frac{2\pi\sqrt{2}}{k} = 2\pi\sqrt{\frac{2\mu}{\varrho p}}. \quad . \quad . \quad . \quad . \quad (8')$$

The frictional moment on the wall of the vibrating cylinder is $2\pi\mu R^3\left[\frac{\partial\omega}{\partial r}\right]_R$ where $\omega = \frac{\partial\alpha}{\partial t}$. First we determine $\left[\frac{\partial\alpha r}{\partial r}\right]_R$ from (7)

$$\left[\frac{\partial\alpha,}{\partial r}\right]_R = \mathrm{R}\left[-\frac{a}{R}e^{ipt} + ac\frac{H_1^{(2)\prime}(cR)}{H_1^{(2)}(cR)}e^{ipt}\right] \quad . \quad . \quad . \quad . \quad (9)$$

For the reduction of the 2nd part on the right hand side of (9) we make use of the well-known recursion-formula of the cylindrical functions:

$$\frac{dH_1^{(2)}(z)}{dz} = H_0^{(2)}z - \frac{1}{z}H_1^{(2)}(z)$$

By its application (9) obtains the form

$$\left[\frac{\partial\alpha,}{\partial r}\right]_R = \mathrm{R}\left[-\frac{2a}{R}e^{ipt} + ac\frac{H_0^{(2)}(cR)}{H_1^{(2)}(cR)}e^{ipt}\right]. \quad . \quad . \quad . \quad (10)$$

giving for the frictional couple

$$K = 2\pi\mu R^3\left[\frac{\partial\omega}{\partial r}\right]_R = -4\pi\mu R^2\omega + \mathrm{R}\frac{d}{dt}\left[2\pi\mu R^3 ac\frac{H_0^{(2)}(cR)}{H_1^{(2)}(cR)}e^{ipt}\right] \quad (11)$$

For an infinite time of swing, i. e. $p=0$, but with a rotational velocity differing from 0, $|c| = \sqrt{\frac{\varrho p}{\mu}}$ becomes 0. In that case the second term on the right of (11) disappears on two grounds: (1) because $c = o$ (2) $\lim\limits_{cR=0} \frac{H_0^{(2)}(cR)}{H_1^{(2)}(cR)} = 0$; only the first term then remains, which agrees with (2′).

Moreover

$$\lim_{cR=\infty} \frac{H_0^{(2)}(cR)}{H_1^{(2)}(cR)} = -i. \quad {}^{1)}$$

It appears from the accompanying graphs [2]) of the module and argument of $\frac{H_0^{(2)}(cR)}{H_1^{(2)}(cR)}$ that this limiting value is practically reached at

$$\left.\begin{aligned} |cR| &= k \,.\, R = 10 \\ |c| &= k = \frac{2\pi\sqrt{2}}{\lambda} \quad (\text{cf. } 8') \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The condition $|cR| \geq 10$ means, that the radius of the cylinder must be about equal to or larger than the wave-length. If R is small compared with λ the second part of the frictional couple is negligible. For $|cR| \geq 10$ the 2[nd] term on the right-hand side of (10) becomes

$$-aci\, e^{ipt} = -ak\, e^{i\left(pt+\frac{\pi}{4}\right)} \quad \left(\text{since } c = k\, e^{-\frac{i\pi}{4}}\right)$$

Hence equation (11) now becomes:

$$K = -4\pi\mu R^2 \omega - 2\pi\mu k R^3 \frac{d}{dt}\left(a \cos\left(pt + \frac{\pi}{4}\right)\right) \quad . \quad (13)$$

where

$$\omega = \frac{d}{dt}(a \cos pt).$$

The frictional couple thus divides into two parts, one which does not contain the density of the liquid and another, in which it occurs and which therefore refers to the emission of waves. In the transition to the limit of uniform rotation the first part only remains.

In the discussion of the 2[nd] part of the frictional moment the quantity $k = \sqrt{\frac{p\varrho}{\mu}}$ is an important factor. If we take a time of oscillation of 2π seconds, so that $p = 1$, we have $k = \sqrt{\frac{\varrho}{\mu}}$.

This gives the following values for k.

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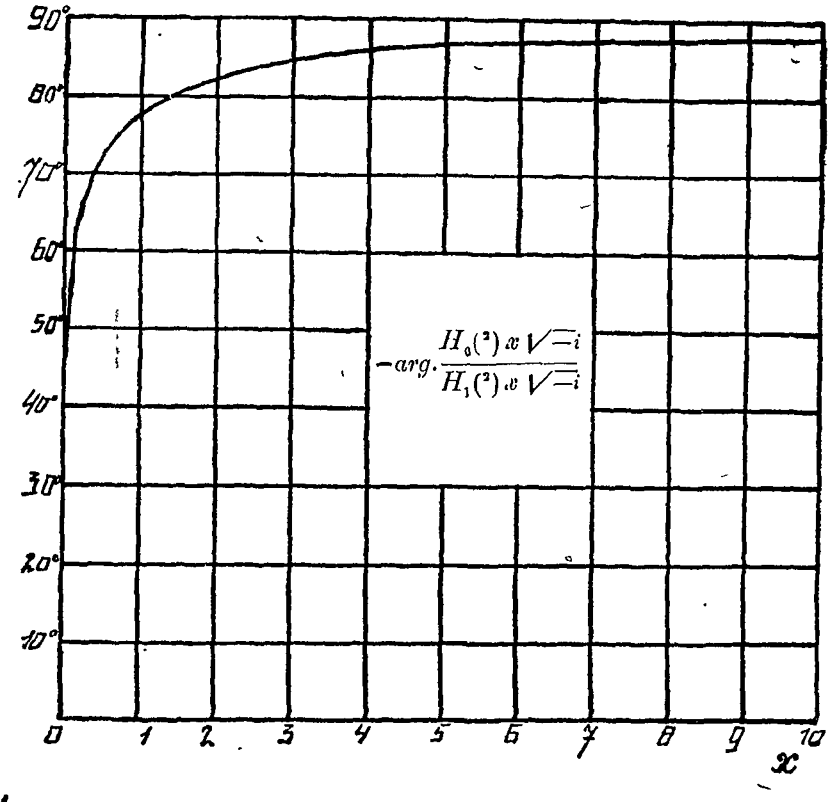
1) Comp. J. u. E. l. c.

2) Tables for $H_0(1)$ and $H_0(2)$ will be found J. u. E. p. 139, 140.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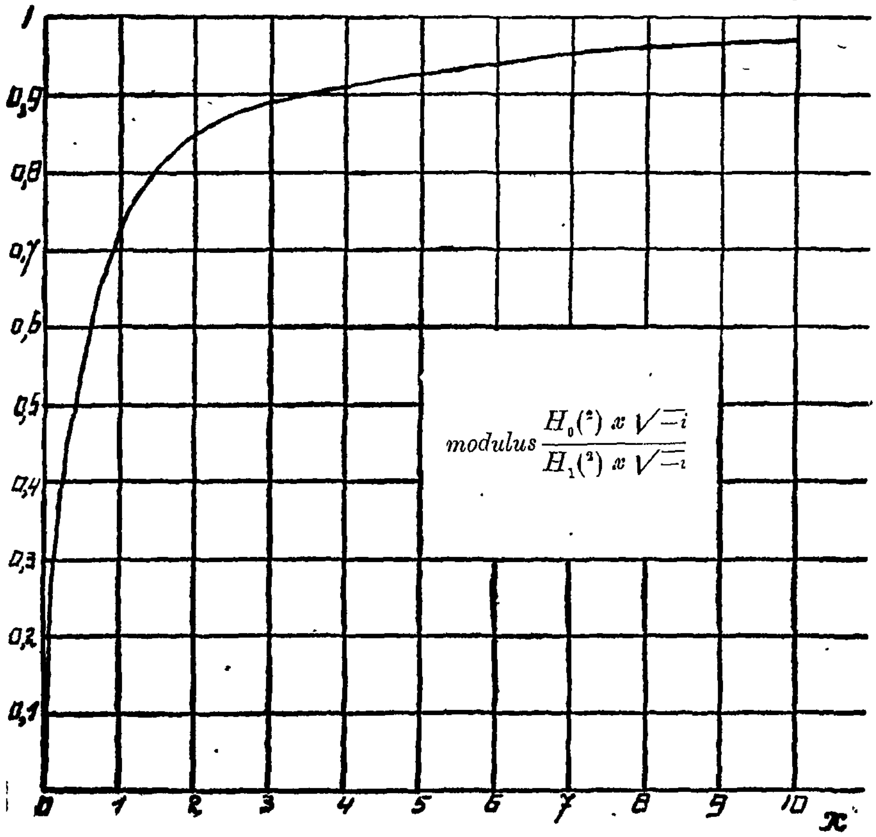
x
$-arg.\dfrac{H_0^{(2)}\, x\sqrt{-i}}{H_1^{(2)}\, x\sqrt{-i}}$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x
$modulus\,\dfrac{H_0^{(2)}\, x\sqrt{-i}}{H_1^{(2)}\, x\sqrt{-i}}$

	ϱ	μ	$k = \sqrt{\frac{\varrho}{\mu}} (p = 1).$
Water 16°	1	0.011	9.5
Atm. air 0°	0.0013	0.000171	2.8
Air 0.01 atm. [1]			0.28
Air 0.001 atm. [1]			0.09
Hydrogen 1 atm. 0°	0.0000898	0.000085	1

From this table it appears that, except for dilute gases, R has to be relatively small in order that the 2nd part may be neglected with respect to the first. For instance for atmospheric air with $R = 0.5$ c.m. $kR = 1.4$ and $\left|\frac{H_0^{(2)}(cR)}{H_1^{(2)}(cR)}\right| = 0.80$, so that the amplitude of the 2nd term of the frictional couple is still 56 % of that of the first (see equation (11)), every thing calculated for a time of oscillation of 2π seconds.

There is a further special limiting case of equation (13), which is of some interest. Let R become infinite, and let a at the same time disappear, in such a manner that Ra converges to a finite limit b. We thus approach the one-dimensional problem of the oscillation of an unlimited flat plate in its own plane in an infinitely extended liquid. The frictional force per unit of surface is found from (13) to be

$$F = -\mu k \frac{d}{dt}\left(b \cos\left(pt + \frac{\pi}{4}\right)\right) \quad . \quad . \quad . \quad . \quad (14)$$

a formula which is well-known from hydrodynamics [2]). A term analogous to $-4\pi\mu R^2 \omega$ does not occur in the one-dimensional problem, the reason evidently being that with a uniform translation of the plate a condition of equilibrium does not arise, until the whole liquid away to infinity proceeds with the velocity of the plate.

Finally it is of importance to ascertain for what frequency the amplitude of the forced vibration becomes a maximnm, in other words to what frequency the system cylinder-liquid resounds, if the cylinder is urged back to the position of equilibrium by a quasi-elastic force.

[1]) At these pressures μ has not become much smaller. Comp. KUNDT u. WARBURG. Pogg. Ann. 1875 Band CLV.

[2]) Comp. LAMB. Hydrodynamics, 3rd edition 1905, p. 559.

The differential equation for the forced oscillation in complex notation is as follows:

$$\theta \frac{d^2 \alpha}{dt^2} + L \frac{d\alpha}{dt} + M\alpha = E e^{ipt} \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Here in our case L is a complex quantity $L = L' + iL''$, where

$$L' = (4\pi \mu R^2 + \sqrt{2}\, \pi \mu k R^3)$$
$$L'' = \sqrt{2}\, \pi \mu k R^3.$$

If we only concern ourselves with the particular solution of (15) which gives the forced oscillation, we can also write (15) in the form:

$$\left(\theta + \frac{L''}{p}\right) \frac{d^2 \alpha}{dt^2} + L' \frac{d\alpha}{dt} + M\alpha = E e^{ipt}. \quad . \quad . \quad . \quad . \quad (16)$$

We see therefore that in consequence of the motion of the liquid an apparent increase of the moment of inertia arises.

Putting

$$\theta + \frac{L''}{p} = \theta'$$

the particular solution of (16) becomes:

$$\alpha = \frac{E}{\sqrt{(M - \theta' p^2)^2 + L'^2 p^2}} e^{i(pt - \varphi)}$$

in which the phase-angle φ is determined by the constants of the differential equation.

Resonance occurs for $M - \theta' p^2 = 0$

or

$$\theta p^2 + L'' p - M = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Now L'' is proportional to k and $k = \sqrt{\frac{p\varrho}{\mu}}$, so that we may conveniently write $L'' = N p^{\frac{1}{2}}$, N being a constant.

(17) is now replaced by

$$\theta p^2 + N p^{1\frac{1}{2}} - M = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

This equation which is bi-quadratic in $\sqrt{p}$ determines the frequencies to which the system resounds. On closer examination there appears to be but one resonance-frequency. Naturally we are only concerned with the real roots p of equation (18). There are found to be two of such, one for which $\sqrt{p}$ is positive, and another for which $\sqrt{p}$ is negative. Now it follows from our calculation that we have assumed $\sqrt{p}$, which occurs in k to be essentially positive. For if we substitute a negative value for $\sqrt{p}$ in our equations, we obtain a system of waves which moves from infinity towards the cylinder.

But the amplitude of this system is infinite at infinity, so that our first boundary-condition would not be satisfied.

We may also choose our boundary-conditions differently. We may for instance imagine the liquid limited on the outside by a second cylinder co-axial with the first and at rest. It is then advisable to write the general solution of equation (4) in the following form

$$u = \frac{1}{r}\{C\, H_1^{(2)}(cr) + D\, H_1^{(2)}(cr)\} \quad . \quad . \quad . \quad . \quad . \quad (19)$$

At a sufficient distance from the axis of the cylinders two systems of waves then arise, one of which is propagated outwards and the other inwards. At the surface of the exterior cylinder we obtain reflection with reversal of phase, so that the liquid there is at rest. For the determination of the integration-constants C and D we obtain comparatively complicated relations which may be omitted here as they do not yield anything of further interest.

The problem of the free oscillation does not now give any further special difficulties.

We must now seek a solution of equation (1) of the form

$$a_r = f_{(r)}\, e^{-k't} \cos(k''t - \varphi(r)),$$

which for $r = R$ becomes $a_R = a\, e^{-k't} \cos k''t$. Again we may write $a = u\, e^{nt}$, where $n = -k' + ik''$.

The same method of solution may now be followed. Instead of (7) we obtain:

$$a_r = \frac{aR}{H_1^{(2)}(c'R)} \frac{H_1^{2}(c'r)}{r} e^{nt}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

where $c' = \sqrt{-\frac{n\varrho}{\mu}}$, if for c' the root with the negative imaginary part is chosen. Hence

$$\left[\frac{da_r}{dr}\right]_R = -\frac{2a}{R} e^{nt} + ac' \frac{H_0^{(2)}(c'R)}{H_1^{(2)}(c'R)} e^{nt}. \quad . \quad . \quad . \quad . \quad (21)$$

$$\lim_{|c'R| = \infty} \frac{H_0^{(2)}(c'R)}{H_1^{(2)}(c'R)} = -i$$

Therefore:

$$\left[\frac{da_r}{dr}\right]_R = -\frac{2a}{R} e^{nt} - a\sqrt{\frac{n\varrho}{\mu}}\, e^{nt}, \quad . \quad . \quad . \quad . \quad (22)$$

if for $\sqrt{\frac{n\varrho}{\mu}}$, we take the root with the positive real term.

The frictional moment now becomes:

$$2\,\pi\,\mu\,R^2 \left[\frac{d\dot{a}}{dr}\right]_R = -\,2\,\pi\,\mu\,R^2 \left[\frac{2}{R} + \sqrt{\frac{n\varrho}{\mu}}\right] \dot{a}_R$$

The differential equation for the free vibration is:

$$K\frac{d^2 a}{dt^2} + L\frac{da}{dt} + Ma = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

giving for the natural frequencies of the system the equation

$$K n^2 + L n + M = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

The quantity L here contains $\sqrt{n}$.

If we put $L = P + Q\sqrt{n}$, where

$$P = 4\pi\,\mu R^2 \text{ and } Q = 2\pi\,\mu R R^2 \sqrt{\frac{\varrho}{\mu}},$$

(24) assumes the form:

$$K n^2 + (P + Q\sqrt{n})\,n + M = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

Equation (25) is bi-quadratic in $z = \sqrt{n}$. On further examination it is found to have 2 complex roots z in the right hand portion of the complex plane and 2 in the left portion, only the former of which we can use (comp. equation (22)); hence the system has but one natural frequency. Further $z^2 = n$ is found to contain a negative real term, as indeed could not be expected otherwise.