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Physics. *"On the Evaporation from a Circular Surface of a Liquid"*. By Dr. H. C. BURGER (Communicated by Prof. W. H. JULIUS).

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In a publication recently published Miss N. THOMAS and Dr. A. FERGUSON[1]) communicate observations concerning the evaporation from circular water surfaces. These observations are made under different circumstances viz. in a dark, very quiet room, in a lighted room and in the open air. It appeared, that in every case the quantity of water evaporated in unit time, might be represented by:

$$E = K r^n,$$

in which r is the radius of the water surface and K and n are constants that, except on the external circumstances, also depend upon the distance of the surface of the liquid and the rim of the basin in which this is contained. Now while, as the writers remark, usually in the literature the opinion is found, that the evaporation is proportional to the area of the surface, i. e. that $n = 2$, it was shown by their experiments that this exponent was always between 1 and 2. Now STEFAN[2]) has treated the evaporation from a circular surface of a liquid, supposing that the vapour diffuses in the space above the plane in which the level of the liquid lies, while at the liquid the concentration of the vapour is a constant.

The result of the computation is, that the speed of evaporation is proportional to the radius of the surface. So it is apparent that in the experiments of THOMAS and FERGUSON the conditions that STEFAN supposes in treating the problem, are not fulfilled.

As I have already been engaged for some time upon the theoretical and experimental treatment of the diffusion in a flowing liquid[3]), it was of importance to inquire whether my results agreed with the above mentioned investigations. For this purpose we must extrapolate the values of the exponent n for the case that the surface of the liquid is on a level with the rim of the basin. When this is

1) Phil. Mag. XXXIV p. 308, 1917.

2) Wied. Ann. XVII p. 550, 1882.

3) My principal purpose in this is to investigate whether the solution at the surface of the crystal is saturated or if perhaps, when the solving takes place sufficiently rapidly, an undersaturation arises.

not the case it hardly seems possible to apply the mathematical analysis to the problem.

For the three cases the extrapolated exponent is resp. 1.4, 1.5 à 1.6 and 1.65. In the last case, in which we are most certain that the air above the liquid is in continuous movement, n proves to agree quite sufficiently with the theoretical value $^5/_3 = 1.67$, which will be deduced hereafter, so that therefore in this case we may be sure, that the air-currents effect the evaporation. In experiments in more quiet air, the values of n approach the value $n = 1$ more closely, which value is found by STEFAN.

In the following sections we will give a theoretical treatment of the diffusion in a flowing gas. As the evaporation from an arbitrarily formed surface is easily deduced from that of a rectangular one, we firstly choose this last shape, We imagine the space above the plane $z = 0$ filled with a flowing gas, while the plane $z = 0$ itself is formed by a fixed wall, of which a part consists of a surface of the liquid. Let this part have the shape of a rectangle with its sides parallel to the axes of x and y, situated at positive y and bounded by the axis of x. Further we will choose the velocity of the gas to be parallel to the axis of y and to be proportional to z, so $v_y = az$. As namely the gas at the plane $z = 0$ through external friction must have a velocity equal to zero, we may put:

$$v_y = az + a_2 z^2 + a_3 z^3 + \ldots,$$

and we may neglect the second and following terms of this series when as will generally be the case, the vapour is concentrated in a thin layer above the plane $z = 0$.

When we put c for the concentration of the vapour and D for the coëfficient of diffusion then, as is easily seen, c fulfills the altered equation of diffusion:

$$\frac{\partial c}{\partial t} = D \Delta c - div\,(v\,c) \text{ [1]} \quad . \quad . \quad . \quad . \quad . \quad . \quad (I)$$

Further we suppose that c at the surface of the liquid fulfills the boundary condition :

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[1]) The last term in the second member may be explained in this way: In the element of volume $dx\ dy\ dz$ flows through the element of surface $dy\ dz$ an amount of vapour : $cv_x\ dy\ dz$ inward and $\left\{ cv_x + \frac{\partial}{\partial x}(cv_x) \right\} dy\ dz$ outward. By computing these amounts also for the axes of y and z, we get for the total amount that flows outward $div\,(cv)\ dx\ dy\ dz$, when v is the velocity, considered as a vector.

$$c = C$$ [1])

in which C is the concentration of the saturated vapour.

Now we will suppose that the state is stationary, i. e. that $\frac{\partial c}{\partial t} = 0$.

Then c satisfies the equation:

$$D\left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} + \frac{\partial^2 c}{\partial z^2}\right) = az\,\frac{\partial c}{\partial y} \quad . \; . \; . \; . \; . \; . \quad (II)$$

while $v_x = v_z = 0$ and $v_y = az$. In this equation we will further take $\frac{\partial^2 c}{\partial x^2} = 0$. Of course this is only approximately true for the values of x that concern points within the rectangle. For points beside the rectangle c will be very small only when $\sqrt{\frac{D}{a}}$ is small with respect to the dimensions of the rectangle, which we will always suppose. So we will treat the problem as a twodimensional one, i. e. as if the rectangle has an infinite breadth in the direction of X [2]). So we will neglect $\frac{\partial^2 c}{\partial x^2}$.

Finally we remark that $\frac{a}{D}$ being large, consequently $D\,\frac{\partial^2 c}{\partial y^2}$ may be neglected with respect to $az\,\frac{\partial c}{\partial y}$. One might object to this when c is zero or very small, but then is $c = C$ or at least then c is approximately a constant, so all terms of the differential equation are zero or very small, and so it will be allowed to omit $\frac{\partial^2 c}{\partial y^2}$. That $D\,\frac{\partial^2 c}{\partial z^2}$ may not be neglected, notwithstanding the small factor D, is caused by the fact that the evaporated substance will be concentrated in a thin layer, so that c varies rapidly with z; $\frac{\partial c}{\partial z}$ and $\frac{\partial^2 c}{\partial z^2}$ therefore are large.

After these simplifications the differential equation for c becomes:

$$\frac{\partial^2 c}{\partial z^2} = \frac{a}{D}\,z\,\frac{\partial c}{\partial y} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (III)$$

As, in consequence of a sufficiently rapid stream, a diffusion

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[1]) When by the rapid evaporation an undersaturation arises, this will probably be proportional to the speed of evaporation.

[2]) Experiments with crystals that solve in a flowing liquid, have confirmed this supposition.

against the stream is impossible, we suppose with regard to the fact that the arriving gas is free from vapour that $c = 0$ for $y = 0$. For the same reason we may further assume that the surface of the liquid extends from $y = 0$ to $y = \infty$, while for arbitrary y the concentration will not be influenced by the presence of liquid at the boundary $z = 0$ at greater values of y. As was already said we take for $z = 0$ the boundary-condition $c = C$, while for $z = \infty$ c of course must be zero.

Problems of this kind may be solved in a general way by making the range in which z may vary finite, further by constructing a solution with the aid of a series of proper functions, and finally by going to the limit, whereby the range is made infinite. I hope to explain this method at length in my dissertation; here however it may suffice to give a much simpler treatment, because the purpose is only to find how the quantity of liquid evaporated in unit time depends upon the length of the rectangle, i. e. upon y.

When we introduce in (III) as a new variable:

$$\zeta = z \sqrt[3]{\frac{a}{D}},$$

this equation assumes the form:

$$\frac{\partial^2 c}{\partial \zeta^2} = \zeta \frac{\partial c}{\partial y} \quad \text{. . . . . } (IIIa)$$

The boundary conditions of c are here:

$$c = 0 \quad \text{for} \quad y = 0$$
$$c = C \quad \text{,,} \quad \zeta = 0$$
$$c = 0 \quad \text{,,} \quad \zeta = \infty$$

The solution of the transformed equation will not contain a or D, because these quantities occur neither in the differential equation nor in the boundary conditions.

Therefore is:

$$c = \varphi(\zeta, y) = \varphi\left(z \sqrt[3]{\frac{a}{D}}, y\right).$$

The quantity of the liquid that evaporates in unit time from the part of the surface between $y = 0$ and y is found by computing the quantity of substance that flows through a plane perpendicular to the axis of y. As the velocity of the gas is az, the quantity of vapour that flows in unit time through a unit surface perpendicular to the axis of y, is azc; so the total mass of vapour that flows away per unit breadth in the direction of x, amounts to:

$$E = \int_0^\infty az\,c\,dz = a\int_0^\infty z\,\varphi\left(z\sqrt[3]{\frac{a}{D}},\,y\right)dz.$$

When now we introduce again $\zeta = z\sqrt[3]{\frac{a}{D}}$, this quantity becomes:

$$E = a\,.\left(\sqrt[3]{\frac{D}{a}}\right)^2\int_0^\infty \zeta\,\varphi(\zeta, y)\,d\zeta = a^{1/3}\,D^{2/3}\,\psi(y) \quad . \quad . \quad (IVa)$$

We may transform (III) also by putting:

$$\eta = y\,.\,\frac{D}{a}.$$

Then we get the equation:

$$\frac{\partial^2 c}{\partial z^2} = z\frac{\partial c}{\partial \eta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (IIIb)$$

To this belong the boundary conditions:

$$c = 0 \quad \text{for} \quad \eta = 0$$
$$c = C \quad ,, \quad z = 0$$
$$c = 0 \quad ,, \quad z = \infty.$$

Here again the solution will be independent of a and D viz.:

$$c = f(z, \eta) = f\left(z, y\frac{D}{a}\right).$$

From this we find:

$$E = \int_0^\infty a\,z\,c\,dz = a\int_0^\infty z\,f\left(z, y\frac{D}{a}\right)dz = aF\left(y\frac{D}{a}\right). \quad . \quad (IVb)$$

When now we compare the found values of E, (IVb) proves to agree with (IVa) only when:

$$F(p) = A\,.\,p^{2/3},$$

where A is a constant. So E becomes:

$$E = a\,.\,A\frac{y^{2/3}\,D^{2/3}}{a^{2/3}} = A\,a^{1/3}\,D^{2/3}\,y^{2/3}. \quad . \quad . \quad . \quad . \quad (IVc)$$

Of this result the fact that in the first place interests us is that E proves to be proportional to $y^{2/3}$.

To deduce from the acquired result what E becomes for a surface of an arbitrary shape we imagine that this surface is divided into narrow strips with the long sides parallel to the axis of y i. e. to the current. As the breadth of these strips may not be taken too small when we wish to apply the acquired results, but on the other hand may not be too broad when we want to consider them as rectangles, it proves that the circumference of the surface of the liquid may not be too irregular and also that the linear dimensions of this surface may not be too small with respect to $\sqrt{\frac{D}{a}}$.

Then however for each of these rectangles E is proportional to the breadth and to the $^2/_3$rd power of the length. The total value of E is found by integrating over the whole surface, and it is easily seen that this quantity for conform figures is proportional to the $^5/_3$rd power of the linear dimensions, of which this exponent $^2/_3$ as it were refers to the length and $^3/_3$ to the breadth.

As all circles are conform it is proved by this that the evaporation from a circular surface of a liquid is proportional to the $^5/_3$rd power of the radius as is also found by Miss THOMAS and Dr. FERGUSON, when the circumstances were in agreement with those that are used at the theoretical treatment given above.

The theory that is given here I have found confirmed by experiments of the solving of crystals in a flowing liquid, which will be treated in my dissertation. The quantity of the solved substance proved to be proportional to the $^1/_3$rd power of the velocity of the liquid, with the breadth and with the $^2/_3$rd power of the length.

Institute for theoretical Physics.

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