Mathematics. — *"Null-Systems in the Plane"*. By Prof. JAN DE VRIES.

(Communicated in the meeting of January 26, 1918).

1. In a null-system $\mathfrak{N}(\alpha, \beta)$ a group of α straight lines n passing through a point N is associated to that point; to a straight line n belongs a group of β points N lying on n. A point is called *singular*, when it is null-point of ∞ null-rays; a straight line is called *singular* if it has ∞ null-points.

The null-systems, for which α or β is equal to 1 (*linear null-systems*) are characterized by the fact that they always have singular null-points if $\alpha = 1$, always show singular null-rays if $\beta = 1$. Considerations concerning the case $\alpha = 1$ are to be found in my papers "On plane Linear Null-Systems" (These Proceedings vol. XV, page 1165) and "Lineare ebene Nullverwandtschaften" (Bull. de l'Acad. des Slaves du Sud de Zagreb, July 1917, Auszug aus der im Rad. Bd. 215, S. 122 veröffentlichten Abhandlung).

That a non-linear null-system does not necessarily possess singular elements, appears among others from the consideration of the null-system $\mathfrak{N}(3, 3n-6)$ formed by the points of inflection and their tangents appearing in a general net of curves of order n [1]). Only for $n = 3$ we have in general a group of 21 singular null-rays, viz. the straight parts of the binodal figures.

2. Let us suppose that a $\mathfrak{N}(\alpha, \beta)$ possesses σ singular points S, which are singular null-points on each ray drawn through them, and σ_* singular points S_*, which replace two null-points on each ray [2]). We further suppose that there are $\bar{\sigma}$ singular rays s and $\bar{\sigma}_*$ singular rays s_*; the latter are characterized by the fact that they represent two coinciding null-rays for each of their points.

If the straight line n is caused to revolve round the point P, the β null-points N describe a curve (P) of *order* $(\alpha + \beta)$, which has an α-fold point in P.

Analogously the null-rays n, which have a null-point N on the

[1]) See my paper *"Two null-systems determined by a net of cubics"* (These Proceedings vol. XIX, page 1124)

[2]) In the linear null-system formed by the tangents and their points of contact of a pencil (c^n) the base-points are singular points S_*, the nodes singular points S.

straight line p, envelop a curve (p) of *class* $(\alpha+\beta)$, of which p is a β-fold tangent.

Through a point S pass $(\alpha+\beta)$ tangents of (p); from this it is evident that the null-points on the rays of the pencil S form a curve $(S)^{\alpha+\beta}$. Now, S is always one of the null-points, so that an arbitrary ray of the pencil bears only $(\beta-1)$ points N outside S. Consequently $(S)^{\alpha+\beta}$ has an $(\alpha+1)$-fold point in S.

Analogously we find that $(s)_{\alpha+\beta}$ has the straight line s as $(\beta+1)$-fold tangent, while a straight line s_* is a $(\beta+2)$-fold tangent of the curve $(s_*)_{\alpha+\beta}$.

3. The curve $(P)^{\alpha+\beta}$ is of class $(\alpha+\beta)(\alpha+\beta-1)-\alpha(\alpha-1)$. Through P pass therefore $(2\alpha+\beta)(\beta-1)$ more tangents, which touch it elsewhere. To them belong evidently the straight lines PS_*, as S_* represents two coinciding null-points. *Consequently the null-rays bearing a double null-point envelop a curve of class* $(2\alpha+\beta)(\beta-1)-\sigma_*$.

The complete enveloping figure contains moreover the σ_* *class-points* S_*.

It is of course possible that the enveloped curve breaks up. This e. g. happens with the null-system that arises if each tangent of a pencil (c^n) is associated to the $(n-2)$ points, in which it moreover intersects the c^n (satellite points of the point of contact).

We have to distinguish then between the envelope of the stationary tangents, which each bear *one* double null-point, and the envelope of the bitangents, which each contain *two* double null-points. The curve (P) is now the so-called *satellite-curve* [1]).

In a similar way we find: *The locus of the points N, for which two of the null-rays n have coincided is a curve of order* $(\alpha+2\beta)(\alpha-1)-\overline{\sigma}_*$.

4. The curves $(p)_{\alpha+\beta}$ and $(q)_{\alpha+\beta}$ have the α null-rays of the point pq in common. To the remaining common tangents the singular rays s and s_* evidently belong [2]). There are therefore $(\alpha+\beta)^2-\alpha-\overline{\sigma}-\overline{\sigma}_*$ rays n, a null-point N of which lies on p, another null-point N' on q.

This number has another meaning yet. If N describes the straight

[1]) Cf. my paper "*On linear systems of algebraic plane curves*" (These Proceedings vol. VII, page 712) or "*Faisceaux de courbes planes*" (Archives Teyler, série II, t. XI, p. 101).

[2]) If $\beta=1$, (p) and (q) have, besides the α null-rays of pq, *only* singular rays in common; consequently we have $\overline{\sigma}+\overline{\sigma}_*=\alpha^2+\alpha+1$. The tangents and points of contact of a tangential pencil provide an example of this.

line p, the remaining null-points N' of the null-rays n borne by N will describe a curve $(N')_p$. Its order is evidently equal to the number of rays n, which have a null-point on p and another on q.

Let us now consider the points that $(N')_p$ has in common with p. Each of the β null-points of p is associated to each of the remaining $(\beta-1)$ null-points, and therefore is a $(\beta-1)$-fold point of the curve (N'). The remaining points N' lying on p are evidently double null-points on one of the null-rays determined by them. Hence:

The locus of the double null-points is a curve (N_2) *of order* $\alpha^2+2\alpha\beta-\alpha+\beta-\overline{\sigma}-\overline{\sigma}_{*}$.

The consideration of the curves (P) and (Q) produces analogously:

The double null-rays envelop a curve (n_2) *of the class* $\beta^2+2\alpha\beta+$ $+\alpha-\beta-\sigma-\sigma_{*}$.

5. By means of an arbitrary conic φ^2 another null-system may be derived from a given null-system. Let N be one of the null-points of the ray n, N^* the intersection of n with the polar line of N with regard to φ^2. A new null-system arises now if on each straight line n the null-points N are replaced by the corresponding points N^*[1]). The number β remains intact. In order to find what α passes into, we observe that the null-rays n of the new null-point N^* must have one of their old null-points N on the polar line p of N^*. The null-rays n of the points of p envelop the curve $(p)_{\alpha+\beta}$. On each of the $(\alpha+\beta)$ tangents which it sends through N^* is N^* one of the new null-points.

By the harmonical transformation $\mathfrak{N}(\alpha,\beta)$ *is therefore transformed into a* $\mathfrak{N}^*(\alpha+\beta,\beta)$.

If N lies on φ^2 while one of its null rays touches at φ^2, N^* becomes an arbitrary point of n, and n a singular straight line of $\mathfrak{N}^*$.

In order to determine the number of these singular rays, we associate to each tangent n of φ^2 the β tangents p, which meet n in its β null-points N.

The envelop $(p)_{\alpha+\beta}$ determined by p has evidently $2(\alpha+\beta)$ tangents in common with φ^2. Besides the straight line p, which, as β-fold tangent of the envelope (p), replaces β common tangents, $(2\alpha+\beta)$ rays n are associated to p. The correspondence between p and n has $2(\alpha+\beta)$ coincidences; on φ^2 lie therefore $2(\alpha+\beta)$ points N, of which one of the rays n touches at φ^2. In other words $\mathfrak{N}^*(\alpha+\beta,\beta)$ *has* $2(\alpha+\beta)$ *singular rays more than* $\mathfrak{N}(\alpha,\beta)$.

---

[1]) The "*harmonical*" transformation dually corresponding to this I applied formerly to a $\mathfrak{N}(1,\beta)$ (vide "*Plane Linear Null-Systems*").

By the dual transformation $\mathfrak{N}(\alpha,\beta)$ passes into a $\mathfrak{N}^*(\alpha,\alpha+\beta)$, which has $2(\alpha+\beta)$ *singular points* more than $\mathfrak{N}$.

6. The harmonical transformation may be replaced by a more general transformation in the following way.

The polar curve π of a point N with regard to a given curve Φ^{m+1} intersects the null-ray n in m points N^*, which we shall consider as new null-points of n. In the new null-system $\mathfrak{N}^*$ each straight line has then $m\beta$ null-points N^*.

As N^* lies on the polar curve π^m of N, N belongs to the polar line p of N^* with regard to Φ^{m+1}. Now $(\alpha+\beta)$ tangents of the curve (p) pass through N^*; they are the null-rays of N^* for $\mathfrak{N}^*$. I. e. $\mathfrak{N}(\alpha,\beta)$ is *transformed into a* $\mathfrak{N}^*(\alpha+\beta, m\beta)$ *by the new transformation.*

In opposition to the harmonical transformation this transformation produces *no* new singular straight lines.

7. If we write $\alpha=1$, $\beta=1$, $m=2$, we find from a bilinear null-system a $\mathfrak{N}^*(2,2)$ for which the three singular straight lines of $\mathfrak{N}(1,1)$ are also singular.

We may indicate the bilinear null-system by

$$y_1 : \xi_2\xi_3 = y_2 : \xi_1\xi_3 = y_3 : -2\xi_1\xi_2$$

and the curve Φ^3 by

$$x_1^3 + x_2^3 + x_3^3 + 3x_1x_2x_3 = 0.$$

The polar curve of (y) is then expressed by

$$y_1(x_1^2 + x_2x_3) + y_2(x_2^2 + x_1x_3) + y_3(x_3^2 + x_1x_2) = 0.$$

For the null-system $\mathfrak{N}(2,2)$ we have therefore

$$\left.\begin{aligned} \xi_2\xi_3(x_1^2+x_2x_3) + \xi_1\xi_3(x_2^2+x_1x_3) - 2\xi_1\xi_2(x_3^2+x_1x_2) &= 0 \\ \xi_1x_1 + \xi_2x_2 + \xi_3x_3 &= 0 \end{aligned}\right\} \quad (1)$$

In order to find the equation of the curve $(P)^4$ we have to combine these two equations with

$$p_1\xi_1 + p_2\xi_2 + p_3\xi_3 = 0.$$

Elimination of ξ_k then produces for $(P)^4$

$$(p_1x_3 - p_3x_1)(p_2x_1 - p_1x_2)(x_1^2 + x_2x_3) + (p_3x_2 - p_2x_3)(p_2x_1 - p_1x_2)$$
$$(x_2^2 + x_1x_3) - 2(p_3x_2 - p_2x_3)(p_1x_3 - p_3x_1)(x_3^2 + x_1x_2) = 0.$$

The equations (1) determine the two null-points of the straight line (ξ) as intersections of (ξ) with a conic. As a condition for the coincidence of the two null-points we find after some reduction the equation

$$\xi_1\xi_2\xi_3^4 - (\xi_1^3+\xi_2^3)\xi_3^3 - 3\xi_1^2\xi_2^2\xi_3^2 - 2\xi_1\xi_2\xi_3(\xi_1^3+\xi_2^3) - 4\xi_1^3\xi_2^3 = 0.$$

It shows that the rays that bear two coinciding null-points, envelop a curve of the 6th class.

From this it ensues that the curve $(P)^4$ has no other singularities outside the node P.

Combination of (1) with the equation

$$\pi_1 x_1 + \pi_2 x_2 + \pi_3 x_3 = 0$$

produces for the curve $(p)_4$ by elimination of x_k the equation

$$[(\pi_3 \xi_2 - \pi_2 \xi_3)^2 + (\pi_1 \xi_3 - \pi_3 \xi_1)(\pi_2 \xi_1 - \pi_1 \xi_2)] \xi_2 \xi_3 +$$
$$[(\pi_1 \xi_3 - \pi_3 \xi_1)^2 + (\pi_3 \xi_2 - \pi_2 \xi_3)(\pi_2 \xi_1 - \pi_1 \xi_2)] \xi_1 \xi_3 =$$
$$2 [(\pi_3 \xi_2 - \pi_2 \xi_3)(\pi_1 \xi_3 - \pi_3 \xi_1) + (\pi_2 \xi_1 - \pi_1 \xi_2)^2] \xi_1 \xi_2.$$

This is always satisfied by $\xi_k = 0$, $\xi_l = 0$. This was to be expected as the straight lines $O_1 O_2$, $O_2 O_3$, $O_3 O_1$ must be singular rays.