Mathematics. — "*Linear Null-Systems in the Plane*". By Professor JAN DE VRIES.

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1. A linear null-system $\mathfrak{N}$ $(1,m)$ may be determined by two equations of the form

$$\xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3 = 0$$
$$\xi_1 A_1 + \xi_2 A_2 + \xi_3 A_3 = 0,$$

where A_k indicates a function of order m, in x_k.

When the straight line n revolves round the point $P(y_k)$, its m null-points N, viz. the intersections of $\xi_x = 0$ with the curve $\Sigma\xi_k A_k = 0$, describe a curve of order $(m+1)$. As $\xi_y = 0$, this *null-curve* $(P)^{m+1}$ has as equation,

$$\begin{vmatrix} y_1 & y_2 & y_3 \\ x_1 & x_2 & x_3 \\ A_1 & A_2 & A_3 \end{vmatrix} = 0.$$

The curves $(P)^{m+1}$ form a net that is represented on the point-field by the points P; for each netcurve belongs to a definite point P.

The net has (m^2+m+1) *base-points*. For, if for the sake of brevity its equation is written in the form

$$y_1 B_1 + y_2 B_2 + y_3 B_3 = 0,$$

it appears that the curves $B_1 = 0$ and $B_2 = 0$ have in the first place the points indicated by $x_3 = 0$, $A_3 = 0$ in common, which, however, do not lie on the curve $B_3 = 0$. For the (m^2+m+1) points S_k, which they have moreover in common, we have the relation

$$A_1 : x_1 = A_2 : x_2 = A_3 : x_3.$$

These points lie consequently at the same time on $B_3 = 0$.

Each of the base-points S_k bears ∞^1 null-rays n, is therefore a *singular point* of the null-system.

Two null-curves $(P)^{m+1}$ and $(Q)^{m+1}$ have in the first place the m null-points of the straight line PQ in common; the remaining intersections must be singular as they bear each two null-rays; they are therefore identical with the (m^2+m+1) *singular points* S.

If the point O_3 is laid in one of the singular points we have to write $A_k = a^{(k)} x_3^{m-1} + \ldots$, where $a^{(k)}$ indicates a linear function of x_1 and x_2.

We find then for null-curve of O

$$(x_1\, a^{(2)} - x_2\, a^{(1)})\, x_2^{m-1} + \ldots = 0,$$

from which it is evident that the *null-curve* σ_k^{m+1} of S_k has a node in S_k.

This result was to be expected, but of course holds good only in the case of S being single null-point for an arbitrary ray passing through S.

2. If a point N describes the straight line p, its null-ray n envelops a curve of class $(m+1)$, which will be indicated by the symbol $(p)_{m+1}$. For the null-curve of an arbitrary point Q intersects p in $(m+1)$ points N, of which the null-rays pass through Q. Evidently p is an m-fold tangent of $(p)_{m+1}$.

The *null-curves* $(p)_{m+1}$ and $(q)_{m+1}$ have a common tangent in the null-ray of the point pq. Each of the remaining common tangents is a straight line n, of which one of the null-points N lies on p, another null-point N' on q. If N describes the straight line p, the remaining null-points N' describe consequently a curve (N') of order $(m^2 + 2m)$.

Each of the null-points of p is to be considered $(m-1)$ times as point N', so that (N') in those null-points has $m(m-1)$ points in common with p. In each of the remaining $3m$ intersections of p with (N') a point N' coincides with a point N into a *double null-point* $N^{(2)}$ of the corresponding straight line n.

In a double null-point the curves (P) of a pencil have a common tangent, one of the pencil-curves has a node there. The locus of the double null-points (*curve of coincidence*) coincides with the *Jacobiana* of the net of the curves (P). As the latter is in general a curve of order $3m$, the conclusion may be drawn from the above made statement that the null-system possesses in general *no singular straight lines.* For, if a straight line has each of its points as null-point, it is common tangent of null-curves $(p)_{m+1}$ and $(q)_{m+1}$.

The *curve of coincidence* γ^{3m} has, as Jacobiana, $(m^2 + m + 1)$ nodes S_k.

This may be confirmed as follows. Through P pass $(m^2 + m - 2)$ tangents of $(P)^{m+1}$: their points of contact are double null-points, consequently points of γ^{3m}. The remaining $3m(m+1) - (m^2 + m - 2)$ intersections of (P) with γ must lie in the singular points, but then γ must have a node in each point S.

3. Let us now consider the locus $\varkappa$ of the groups of $(m-2)$ null-points, lying on the null-rays t, which possess a double null-point.

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Through each point S pass (m^2+m-6) tangents of the null-curve σ^{m+1} of S; as they bear a double null-point each, S is an (m^2+m-6)-fold point of the *complementary curve* $\varkappa$. Besides the points S, $\varkappa$ has moreover the groups of $(m-2)$ null-points in common with $(P)^{m+1}$; these points lie on the (m^2+m-2) straight lines t, which meet in P. The two curves have consequently in common $(m^2+m+1)(m^2+m-6)+(m^2+m-2)(m-2)$ points. For the order of $\varkappa$ we find from this $(m^4+3m^3-5m^2-9m-2):(m+1)$, i. e. m^3+2m^2-7m-2, or $(m-2)(m+4m+1)$.

4. The straight lines t envelop a curve τ of the *class* $(m+2)(m-1)$.

If a curve c^{m+1} of the net has a node D, DP replaces two of the rays t meeting in P; P is then a point of τ and PD the tangent in P at that curve.

If P lies on a *binodal* c^{m+1}, with nodes D and D', PD and PD' replace each two straight lines t and are tangents in a node of τ.

If a c^{m+1} has a *cusp* in K, PK replaces three straight lines t, and P is a cusp of τ.

Now the net $[c^{m+1}]$ contains according to a well-known proposition $\frac{3}{2}m(m-1)(3m^2+3m-11)$ *binodal* and $12m(m-1)$ *cuspidal* curves.

If we moreover take into consideration that the base-points S are nodes of τ, it appears that τ possesses $\frac{1}{2}(9m^4-40m^2+35m+2)$ *nodes* and $12m(m-1)$ cusps.

We can now determine the remaining characteristic numbers of τ.

From the formula $\nu=n(n-1)-2d-3r$ it ensues at once that the order of τ is $3m^2$.

From $3n-r=3\nu-\varrho$ we deduce for the number of *points of inflexion* $3(m-2)(2m+1)$.

The *genus* of τ is equal to that of γ^{3m}, viz. equal to $\frac{1}{2}m(7m-11)$.

And we now finally arrive from

$$g=\tfrac{1}{2}(\nu-1)(\nu-2)-(\delta+\varrho)$$

at the number $\frac{1}{2}(m-2)(m-3)(m^2+7m+4)$ of *bitangents*.

It appears from the results arrived at that $\mathfrak{N}(1,m)$ has $3(m-2)(2m+1)$ *rays with triple null-point* $N^{(3)}$ *and* $\frac{1}{2}(m-2)(m-3)(m^2+7m+4)$ *rays that have two double null-points each.*

By means of these two numbers it would be possible to determine again the order of the complementary curve. For the curves γ and $\varkappa$ will touch in the triple null-points and must intersect in the coupled double null-points; they have further in each singular point

$2(m^2+m-6)$ points in common. Taking this into account we find indeed for the order of $\varkappa$ the number arrived at above.

5. Till now we have supposed that the singular points are all single and different, but moreover that each point S is *single null-point* on a ray arbitrarily drawn through S. An example of a $\mathfrak{N}(1,m)$, of which the singular points are partly *double null-points*, is furnished by a pencil of curves c^r, when each straight line is associated to its points of contact with curves of the pencil. A ray passing through a base-point of (c^r) is touched outside that point by $2(r-2)$ curves, while an arbitrary straight line has $2(r-1)$ null-points; so each base-point is to be considered as double null-point. The remaining singular points of this null-system $\mathfrak{N}(1,2r-2)$ lie in the nodes of the nodal curves c^r; they are evidently *single null-points* on the straight lines drawn through them.

We shall now suppose that $\mathfrak{N}(1,m)$ has s_2 singular points $S^{(2)}$, which are *double null-points* of their rays. As a ray passing through $S^{(2)}$ outside that point bears $(m-2)$ null-points the *null-curve* $\sigma^{(2)}$ has a *triple* point in $S^{(2)}$. The complementary curve now consists of the s_2 null-curves $\sigma^{(2)}$ and a curve $\varkappa^*$ of *order* $(m-2)(m^2+4m+1)$ — $-(m+1)s_2$, while the curve τ has been replaced by a curve τ^* of *class* $(m+2)(m-1)-s_2$ and the s_2 class-points $S^{(2)}$.

If it is taken into consideration that $\sigma_k^{(2)}$ contains all singular points $S_l^{(2)}$ and S_m it is found that $\varkappa^*$ passes through each point S with $(m^2+m-6-s_2)$ branches and with $(m^2+m-8-s_2)$ branches through each point $S^{(2)}$.

6. In order to arrive at a determination of the number of triple null-points $N^{(3)}$, we associate to each point $N^{(2)}$ of a ray t the $(m-2)$ null-points N' of t, and consider the correspondence which arises in consequence of this in a plane pencil with centre T. As the points $N^{(2)}$ lie on the curve γ^{3m}, the points N' on the curve $\varkappa'$, the characteristic numbers of this correspondence are evidently $3m(m-2)$ and $(m-2)(m^2+4m+1)-(m+1)s_2$, while any ray t passing through T produces an $(m-2)$-fold coincidence. The number of the remaining coincidences amounts to

$3m(m-2)+(m-2)(m^2+4m+1)-(m+1)s_2-(m+m-2-s_2)$ $(m-2)$ i.e. $(m-2)(6m+3)-3s_2$.

There are consequently $3(m-2)(2m+1)-3s_2$ *null-rays with a triple null-point.*

In order to find the number of coupled double null-points $N^{(2)}$ we associate to each point N' of a ray t each of the remaining null-points N'' of t. The involutory relation which arises in consequence of this in the plane pencil T has as characteristic number $[(m-2)(m^2+4m+1)-(m+1)s_2](m-3)$; any ray t passing through T represents now $(m-2)(m-3)$ coincidences. The remaining coincidences to the number of $2(m-3)[(m-2)(m^2+4m+1)-(m+1)s_2]-(m^2+m-2-s_2)(m-2)(m-3)$ form pairs of double null-points.

There are consequently $\frac{1}{2}(m-2)(m-3)(m^2+7m+4)-\frac{1}{2}(m-3)(m+4)s_2$ *rays which each bear two double null-points.*

A null-system $\mathfrak{N}(1,m)$ with (m^2+m+1) simple singular points has therefore $3(m-2)(2m+1)$ null-rays with a triple null-point and $\frac{1}{2}(m-2)(m-3)(m^2+7m+4)$ null-rays with two double null-points.

With this the results of § 4 are confirmed.

For the null-system $\mathfrak{N}(1, 2r-2)$ mentioned above $s_2=r^2$; the number of triple null-points amounts therefore to $3(7r^2-22r+12)$. For $r=3$ we find from this 27. For each pencil (c^3) each base-point is point of inflexion on three curves c^3; the number 27 consequently arises from the fact that the 9 base-points serve each on three null-rays as triple null-point. As this observation holds good for each pencil (c^r) the number of points $N^{(3)}$ outside the base-points will be equal to $3(6r^2-22r+12)$. In such a point a c^r has four coinciding points in common with its tangent. In general a pencil (c^r) has therefore $6(r-3)(3r-2)$ curves that have a *point of undulation* [1]).

7. If the curves $A_k=0$ (§ 1) have an r-fold point in O_3, $S_0 \equiv O_3$ is an *r-fold null-point* on each of its rays. Outside the singular null-point S_0 there are then moreover $(m^2+m+1)-r^2$ simple singular null-points S.

The null-curve of S_0 has as equation $A_1x_2-A_2x_1=0$; hence it has in S_0 an *$(r+1)$-fold point.*

The null-curve $(P)^{m+1}$ has in S_0 an r-fold point, consequently sends through P $(m^2+m-2)-(r^2-r)$ tangents t, of which the points of contact lie on the *curve of coincidence* γ. The latter has nodes in the points S; so of its intersections with $(P)^{m+1}$ there lie in S_0 $3m(m+1)-(m^2+m-2-r^2+r)-2(m^2+m+1-r^2)=$ $=(3r-1)r$ points.

From this it ensues that γ has in S_0 a *$(3r-1)$-fold* point.

In order to determine the order of the complementary curve, we consider two pencils of null-curves (c_1^{m+1}) and (c_2^{m+1}), and associate

[1]) Another deduction of this number I gave in "Faisceaux de courbes planes". (Archives Teyler, sér. II, t. XI, p. 99).

to each c_1^{m+1} the $(m^2 + m - 2 - r^2 + r)$ curves c_2^{m+1}, which it intersects on γ^{3m}, outside the points S. The figure produced by the pencils coupled in this way consists of twice the curve γ, of $(m^2 + m - 2 - - r^2 + r)$ times the curve c^{m+1}, which belongs to both pencils and of the complementary curve $\varkappa_0$. We now find as its order $(m^2 + m - 2 - r^2 + r)(m + 1) - 6m$ i. e. $(m - 2)(m^2 + 4m + 1) - (m + 1)r (r - 1)$.

With regard to § 3 we conclude from this that the null-curve of S_0 is to be considered $r(r-1)$ times as component part of $\varkappa$.

Applying the method of § 6 again, we now find the number of *triple null-points* from

$$3m(m-2) + (m-2)(m^2+4m+1) - (m+1)r(r-1) - (m-2)(m^2+m-2-r^2+r)$$

i.e.

$$(m-2)(6m + 3) - 3r(r-1).$$

Analogously we find for the number of null-rays with *two double null-points*

$$\tfrac{1}{2}(m-2)(m-3)(m^2+7m+4) - \tfrac{1}{2}(m-3)(m+4)r(r-1).$$

8. A very particular linear null-system is obtained by supposing that the functions A_k (§ 1) only contain x_1 and x_2 In that case

$$\xi_1 A_1 + \xi_2 A_2 + \xi_3 A_3 = 0$$

represents an involution of rays of the second rank, of which the ∞^2 groups, each of m rays, correspond projectively to the straight lines of the plane.

The null-curves have now in $S_0 \equiv O_3$ an m-fold point, are consequently rational; the null-curve of S has degenerated into $(m + 1)$ rays, which each contain one of the simple singular null-points S.

If the derivatives of A_k with regard to x_1 and x_2 are indicated by $(A_k)_1$ and $(A_k)_2$, we find for the *locus of the double null-points* the equation

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ (A_1)_1 & (A_2)_1 & (A_3)_1 \\ (A_1)_2 & (A_2)_2 & (A_3)_2 \end{vmatrix} = 0$$

This curve of order $(2m - 1)$ has in S_0 a $(2m - 2)$-fold point. By the $(m + 1)$ rays S_0S_k it is completed into the Jacobiana of the net of the null-curves.

The rays t with the double null-points envelop a curve τ of class $(2m - 1)$; for $(P)^{m+1}$ is now of class $(m + 1)m - m(m - 1) = 2m$.

The triple rays of the above mentioned involution are indicated by

$$\begin{vmatrix} (A_1)_{11} & (A_2)_{11} & (A_3)_{11} \\ (A_1)_{12} & (A_2)_{12} & (A_3)_{12} \\ (A_1)_{22} & (A_2)_{22} & (A_3)_{22} \end{vmatrix} = 0.$$

Their number amounts therefore to $3(m-2)$.

There are consequently $3(m-2)$ null-rays with *triple null-point*; they are evidently *stationary tangents* of the curve τ enveloped by the null-rays t.

Analogously the *bitangents* of that curve are intersected in their points of contact by the pairs of double rays that occur in the groups of the involution. Their number, as is known, amounts to $2(m-2)(m-3)$.

For the order of τ we find now m; it has no cusps, but $\frac{1}{2}(m-1)(m-2)$ nodes. It is, just as γ^{2m-1}, rational.

The involution has $\frac{1}{2}(m-1)(m-2)$ neutral pairs. Each pair belongs to ∞^1 groups and corresponds projectively to a plane pencil of null-rays. In connection with this the null-curve of the centre of that pencil consists in the corresponding neutral pair of rays and a curve of order $(m-1)$, which has an $(m-2)$-fold point in S_0.

The null-curve of a singular point S_k consists of the ray S_kS_0 and a curve of order m with $(m-1)$-fold point S_0.