Mathematics. — "*Null-Systems determined by two linear congruences of rays*". By Professor JAN DE VRIES.

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1. A twisted curve α^p intersected by a straight line a in $(p-1)$ points, determines a linear congruence $(1,p)$, of which each ray u rests on a and on α^p. Analogously a curve β^q intersected by the straight line b in $(q-1)$ points determines a congruence $(1,q)$, of which the rays v rest on b and β^q.

Through the point N pass in general *one* ray u and *one* ray v. If the plane $\nu \equiv u\,v$ is associated as null-plane to N a null-system arises in which a plane ν has in general $p\,q$ null-points, viz. the intersections of the p rays of u with the q rays of v.

If N describes a straight line l, the rays u and v describe two ruled surfaces, which are successively of order $(p+1)$ and order $(q+1)$, and intersect along a curve (l) of order $(p\,q+p+q)$. An arbitrary plane ν passing through l has with (l) the $p\,q$ null-points of ν in common, and moreover $(p+q)$ points lying on l, which belong each as null-point to a definite plane ν. In other words, the straight line l is $(p+q)$ times *null-ray*. In R. STURM's notation the null-system has therefore the characteristic numbers $\alpha=1, \beta=p\,q$, $\gamma=p+q$, may consequently be indicated by $\mathfrak{N}(1, p\,q,\ p+q)$.

2. If v coincides with u, any point of that straight line has any plane passing through that straight line as null-plane. Now, the congruences $(1,p)$ and $(1,q)$ have in general $(p\,q+1)$ rays in common. There are consequently $(p\,q+1)$ *singular straight lines s*.

The curves α^p and β^q are also *loci of singular points*. Through a point A^* of α^p passes a ray v^* and a plane pencil of rays u. In any plane passing through v^* lies *one* ray v; so A^* is null-point to any plane of a pencil that has v as axis. The straight lines v^* form a *ruled surface* of order $p(q+1)$; for a plane passing through b contains p rays v^* and a point of b bears $p\,q$ rays v^*. Finally the points of a and b too are *singular null-points*. A point A_* of a bears *one* ray v_* and ∞^1 rays u, which form a cone of order p with $(p-1)$-fold generatrix. Any plane passing through v_* contains p rays u, so that A_* is to be considered as p-fold null-point. The rays v_* form a *ruled surface* of order $(q+1)$. A straight line u

(or v) is null-ray to any of its points; in connection with this the curve (l) degenerates for $l \equiv u$ or $l \equiv v$.

3. If a plane ν continues to pass through the point P, its null-points describe a surface (P) of order $(p+q+1)$. For a straight line l passing through P bears $(p+q)$ points N, which send their null-plane through P.

The straight lines u and v, which intersect in P, lie on (P); for each of their points sends its null-plane through P.

On (P) lie further the $(pq+1)$ singular rays s and the singular curves α^p, β^q, while the singular straight line a is evidently a p-fold line, the singular straight line b a q-fold line. The surfaces (P) and (Q) have, in connection with this, the singular lines s, a, b, α and β in common and intersect further along the curve (l), which belongs to $l \equiv PQ$.

4. As the straight line l intersects the ruled surface (v^*) in $p(q+1)$ points, the curve (l) contains evidently $p(q+1)$ singular null-points A^* and thus $q(p+1)$ singular null-points B^*.

There are further $(q+1)$ planes passing through l, which bear a p-fold null-point A_* each, and consequently $(p+1)$ planes each with a q-fold null-point B_*.

Let R be a point outside the straight line l. To the intersections of the surface (R) with the curve (l) belong in the first place the pq null-points of the plane lR. Further the $p(q+1)$ points A^* and the $q(p+1)$ points B^*. The remaining common points to the number of $(p+q+1)(p+q+pq) - pq - p(q+1) - q(p+1)$ i.e. $p^2(q+1) + \alpha q^2(p+1)$ must be lying in the $(q+1)$ points A_* and the $(p+1)$ points B_*. As a on (R) is a p-fold line each of the $(q+1)$ points A_* must be a p-fold point of the curve (l). Analogously has (l) in each of the $(p+1)$ points B_*, a q-fold point. The curve α^p is rational, sends consequently $2(p-1)$ tangent planes through l. In each of these tangent planes two rays u coincide, so there are q double null-points, so that the plane is q-fold tangent plane of (l). Analogously β^q sends through l $2(q-1)$ tangent planes which are p-fold tangent planes of the curve (l). As l is intersected by (l) in $(p+q)$ points, the *rank* of l is equal to $2(p-1)q + 2(q-1)p + 2(p+q)$, i. e. $4pq$.

5. Let us inquire in how far the results arrived at are altered when the congruence of rays $(1,q)$ is replaced by the congruence $(1,3)$ of the bisecants v of a twisted cubic β^3.

Let B^* be a point of β^3, u^* the ray which the congruence $(1,p)$ sends through that point. Any plane passing through u^* contains two straight lines v, which intersect in B^*; B^* is consequently a double null-point.

The surface $(P)^{p+4}$ has consequently β^3 as *nodal curve;* it further cointains the curve $\omicron^p$, the $(3p+1)$ singular straight lines s and passes p times through the singular straight line a.

The ruled surface (v^*) is of order $4p$, the ruled surface (u^*) of order $(3p+3)$, while the straight lines v_*, as bisecants of β^3, form a ruled surface of the fourth order.

If the congruence $(1,p)$ is also replaced by the congruence (1,3) of the bisecants of a curve α^3, a null-system $\mathfrak{N}(1, 9, 6)$ arises. The surface $(P)^7$ has α^3 and β^3 as nodal curves and contains 10 singular straight lines s; $(P)^7$ and $(Q)^7$ have moreover a curve $(l)^{15}$ in common. The ruled surfaces (u^*) and (v^*) are of order 12.

6. For $p=1$, $q=1$ we have a bilinear null-system $\mathfrak{N}$ (1, 1, **2**), in which the rays u rest on two straight lines a, a', the rays v on two straight lines b, b'.

The singular figure consists then of the straight lines a, a', b, b' and their two transversals s, s'. For each singular point the null-planes form a pencil; the axes of those pencils form four quadratic systems of generatrices. The surface $(P)^3$ has a triple tangent plane [1]) in the null-plane of P.

[1]) Cf. my paper "On bilinear null-systems" (These Proceedings, vol. XV, p. 1160).