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Mathematics. — "*On the direct analyses of the linear quantities belonging to the rotational group in three and four fundamental variables*". By Prof. J. A. SCHOUTEN. (Communicated by Prof. CARDINAAL).

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Quantities and direct analyses.

By a (geometric or algebraic) quantity existing with a definite transformation-group we mean, according to F. KLEIN, any complex of numbers (characteristic numbers of the quantity), that is transformed *into itself*[1]) by the transformations of that group. Quantities only have any signification and only exist with definite transformation-groups and may be "disturbed" as such with other groups, whose transformations do not transform the characteristic numbers into themselves. They are completely determined by their *mode of orientation*, i.e. the mode of transformation of their characteristic numbers. The variables of the group are called *fundamental variables* and are the characteristic numbers of a *fundamental element*. If the group is the linear homogeneous one in n variables, the simplest quantities are those, whose characteristic numbers are transformed as the determinants in a matrix of p fundamental elements independent of each other, $p = 1, \ldots, n$. With a homogeneous interpretation of the fundamental variables they correspond to the linear R_{n-p}-complexes in R_{n-1}, provided with a number-factor. All the quantities, whose characteristic numbers are transformed in that way under the transformations of the *rotational* group, we call *linear quantities.*

By a *direct analysis* we mean a system of an addition and some multiplications by means of which we can express the relations among quantities of a definite kind left invariant under the transformations of a definite group. Every quantity is in the analysis a higher complex number. Till recently suchlike analyses were brought about by choosing for multiplications some characteristically distributive combinations conspicuous in geometry or mechanics, and uniting them into a system as well as might be. Owing to the great number of existing combinations of this kind arbitrariness could not fail to arise, and this led to the formulation of many systems, the adherents of which have been involved in a violent polemic for these twenty five years.

1) e. g F. KLEIN, Elementarmathematik vom höheren Standpunkte aus. Leipzig (09) II p. 59.

Application of KLEIN's *Principle of Classification.*

The author of this paper observed in 1914[1]) that it follows from the application of KLEIN's principle of classification to analyses belonging to definite quantities, that to a given group of transformations and given quantities belongs a completely determined system, which may simply be computed. This was practically done for $n = 3$, the rotational group, and quantities up to the second order inclusive. In a more exhaustive investigation contemplating four different sub-groups of the linear homogeneous group the same was executed for arbitrary values of n and for quantities of an arbitrary degree[2]). We shall briefly state some results of this investigation bearing on linear quantities, in particular for $n = 3$ and $n = 4$, founded on the:

rotational group ($a_1^2 + \ldots + a_n^2$ invariant, det. $= + 1$)

and availing ourselves of the:

orthogonal group ($a_1^2 + \ldots + a_n^2$ invariant, det. $= \pm 1$)
special-affin. group (lin. hom. with det. $+ 1$)
equivoluminar group (lin. hom. with det. ± 1)
linear homogeneous group

for further classification of the quantities existing with the rotational group.

General symmetrical and alternating multiplication.

Three multiplications of fundamental elements exist with all the sub-groups of the linear homogeneous group and for all the values of n, viz. the *general*, the *symmetrical* and the *alternating* one.

The *general product* of p fundamental elements has n^p characteristic numbers, being the products of the characteristic numbers of the factors. Their mode of transformation is entirely determined by this definition. We express the product in this manner:

$$\mathbf{a}_1 \circ \mathbf{a}_2 \circ \ldots \circ \mathbf{a}_p = \mathbf{a}_1 \overset{\circ}{\ldots} \mathbf{a}_p \quad \ldots \quad (1)$$

By *isomers* of $\mathbf{a}_1 \overset{\circ}{\ldots} \mathbf{a}_p$ we mean all the general products that can be formed by permutation of the factors from $\mathbf{a}_1 \overset{\circ}{\ldots} \mathbf{a}_p$. An even respectively odd isomer is concomitant with an even resp. odd permutation. The *symmetrical product* of $\mathbf{a}_1 \ldots \mathbf{a}_p$, is the sum total of all the isomers divided by their number $p!$:

$$\mathbf{a}_1 \smile \mathbf{a}_2 \smile \ldots \smile \mathbf{a}_p = \mathbf{a}_1 \overset{\smile}{\ldots} \mathbf{a}_p = \frac{1}{p!} \Sigma\, \mathbf{a}_{i_1} \overset{\circ}{\ldots} \mathbf{a}_{i_p} \quad . \; . \quad (2)$$

The *alternating product* is the sum of all the even isomers dimi-

[1]) Grundlagen der Vektor- und Affinoranalysis, Leipzig (14).
[2]) Ueber die Zahlensysteme der rotationalen Gruppe. Nieuw Archief voor Wiskunde 1919.

nished by the sum of all odd ones divided by $p!$ and may be expressed as Cayleyan determinant:

$$a_1 \frown a_2 \frown \ldots \frown a_p = a_1 \ldots \overset{\frown}{\ldots} \ldots a_p = \frac{1}{p!}\begin{vmatrix} a_1 & \ldots & a_p \\ \vdots & & \vdots \\ a_1 & \ldots & a_p \end{vmatrix} \text{(to be developed according to rows)} \quad (3)$$

The alternating product of p fundamental elements is a linear quantity for $p \leq n$. For $p > n$ it is zero. A symmetrical product is never a linear quantity.

The Associative Systems R_n.

Classifying up to the lin. homog. group inclusive, the system belonging to the linear quantities is R_n^l, which is an associative system, entirely determined by the rules:

$$e_i \dashv e_j = - e_j \dashv e_i = e_{ij} \text{ [1]} \qquad e'_i \dashv e'_j = - e'_j \dashv e'_i = e'_{ij}$$

$$e_i \dashv e_i = k \qquad e'_i \dashv e'_i = k'$$

$$e_i e_j \ldots \overset{\dashv}{e_l} = e_{ij \ldots l} \qquad e'_i e'_j \ldots \overset{\dashv}{e'_l} = e'_{ij \ldots l}$$

$$e_{12 \ldots n} = I \qquad e'_{12 \ldots n} = I'$$

$$e_1 = \varkappa^n e'_2 \ldots \overset{\dashv}{\ldots} e'_n I, \quad e_1 \dashv e'_1 = e'_1 \dashv e_1 = \varkappa, \quad e'_1 = \varkappa^n e_2 \ldots \overset{\dashv}{\ldots} e_n I'$$

$$I I' = I \dashv I' = I' \dashv I = \varkappa^{n-1} \qquad i, j, \ldots, l = 1, \ldots, n.$$

$$\varkappa = (-1)^{\frac{n(n-1)}{2}}.$$

$e_1, \ldots, e_n$ are the covariant *fundamental units,* i.e. units of a fundamental element, and $e'_1, \ldots, e'_n$ are the contravariant fundamental units belonging to characteristic numbers, transforming themselves contragrediently relative to the fundamental variables.

When classifying up to the equiv. group incl., the system R_n^a is constituted, being obtained from the preceding one by the identification

$$I = I'$$

and being entirely determined by the rules:

$$\left.\begin{array}{l} e_i \dashv e_j = - e_j \dashv e_i = e_{ij} \\ e_i \dashv e_i = k \\ e_i e_j \ldots \overset{\dashv}{\ldots} e_l = e_{ij \ldots l} \\ e_{12 \ldots n} = I \end{array}\right\} i, j, \ldots, l = 1, \ldots, n. \quad (4)$$

$$I^2 = I \dashv I = \varkappa^{n-1}$$

Quantities, whose units, apart from an eventual factor **I**, do not contain two equal fundamental units as factors, exist unlike the

[1]) In a more exhaustive investigation "Die direkte Analysis zur neueren Relativitätstheorie", Verhand der Kon. Akad. v. Wet. Sectie I Deel XII N°. 6 we consider also not linear quantities and we write $e_i e_j = e_{ij}$ and $e_i \dashv e_j = \frac{e_i e_j - e_i e_j}{2} = e_{\overline{ij}}$ etc. For more convenience we write here $e_i \dashv e_j = e_{ij}$.

others with the lin. homog. group too, and are called *projective quantities.* Then they are of the *sub-degree* (Dutch: ondertrap, German: Unterstufe) p, when the number of the factors of the units is p, $p=1, \ldots, 2n$, and we write them $_p\mathbf{a}$. The others are called *orthogonal quantities.* All linear quantities may be composed of projective ones and powers of **k**.

When classifying up to the special affin. group inclusive, for n odd the system R_n^s is obtained from the preceding one by the identification:

$$\mathbf{I}=\varkappa \quad \ldots \ldots \ldots \ldots \quad (5)$$

The sub-degree p, $p \gtreqless n$ coincides with the sub-degree $(n+p)$ and forms the *degree* (trap, Stufe) p. For n even no system is feasible here, because

$$\mathbf{I} \dashv \mathbf{e}_i = - \mathbf{e}_i \dashv \mathbf{I}, \quad \ldots \ldots \ldots \quad (6)$$

hence identification of I with an ordinary number is impossible.

When classifying up to the orth. group inclusive, R_n^0 arises out of R_n^a, by the identification

$$\mathbf{k}=\varkappa \quad \ldots \ldots \ldots \ldots \quad (7)$$

The system makes no difference between projective and non-projective quantities. The sub-degree p, $p \leqq n$ coincides with the sub-degree $(2n-p)$ and forms the *by-degree* (neventrap, Nebenstufe) p.

When classifying up to the rotational group inclusive, for n odd, R_n^r arises out of R_n^a by the identification

$$\mathbf{I}=\mathbf{k}=\varkappa \quad \ldots \ldots \ldots \ldots \quad (8)$$

Neither does this system make any difference between projective and non-projective quantities. The sub-degrees p, $(n-p)$, $(n-p)$ and $(2n-p)$ coincide and constitute the *principal degree* (hoofdtrap, Hauptstufe) p; $p \leqq n'$, $n' = \frac{n-1}{2}$ for n odd and $n' = \frac{n}{2}$ for n even. In all these systems the associative product of dissimilar fundamental units is equal to the alternating one.

The systems R_n are the products of *original systems* and *principal rows* [1]) according to the general formulae:

$$\left.\begin{aligned} R_n^r &= O_2^{\frac{n-1}{2}} \\ R_n^s &= H_n\, O_2^{\frac{n-1}{2}} \\ R_n^o &= H_1\, O_2^{\frac{n-1}{2}} \\ R_n^a &= H_1\, H_n\, O_2^{\frac{n-1}{2}} \end{aligned}\right\} \quad \ldots \ldots \quad (9)$$

[1]) Cf. Grundl. pages 11—18.

for n odd and

$$\left.\begin{aligned} R_n^o &= O_2^{\frac{n}{2}} \\ R_n^a &= H_n\, O_2^{\frac{n}{2}} \end{aligned}\right\} \quad \ldots\ldots \quad (10)$$

for n even, where O_i denotes an original system of the order i and H_i a principal row of the order i. But for some divergence in + and — signs the systems R_n^0 are identical with CLIFFORD's n-way algebras [1]).

If none of the units is privileged the choice of the numbers occurring in the identifications is altogether determined by the dualities existing in the different groups. There are four altogether, and we shall call them:

$\mathbf{a} - {}_{n-1}\mathbf{a}$	$\alpha - \beta$
$\mathbf{a} - {}_{n+1}\mathbf{a}$	$\alpha - \gamma$
$\mathbf{a} - {}_{2n-1}\mathbf{a}$	$\alpha - \delta$
$\mathbf{a} - \mathbf{a}'$	$\alpha - \varepsilon$

From the mode of transformation we conclude for the existence of these dualities as subjoined:

Duality:	$\alpha - \beta$		$\alpha - \gamma$		$\alpha - \delta$		$\alpha - \varepsilon$
Group:	n even	n odd	n even	n odd	n even	n odd	
linear homog.	—	—	—	—	—	—	+
equivolumin.	+ for $n=2$ identity	—	+ for $n=2$ identity	—	+	+	$=\alpha-\delta$
special-affin	+	+	identity	identity	$=\alpha-\beta$	$=\alpha-\beta$	$=\alpha-\beta$
orthogon.	+ $=\alpha-\gamma$	—	+ $=\alpha-\beta$	-	identity	identity	identity
rotation	identity	identity	identity	identity	identity	identity	identity

+ = existing, — = not existing. [2])

[1]) CLIFFORD's systems have been worked out by J. JOLY, Proc. Roy. Ir. Acad. 5 (98) 73—123, A manual of quaternions (05) 303—309. He gives geometrical applications after the manner of the quaternion-theory without decomposition of the product. A. M'AULAY has elaborated this matter as well, Proc. Roy. Soc Edinb. 28 (07) 503—585. These papers do not aim at a foundation on the theory of invariants or a closer investigation of the fundamental groups.

[2]) The squares of the dualities not founded on contragredience have been indicated by blacker demarcation. These dualities only exist when n is even.

The associative Systems R_3 and R_4.

If we call the unities of the sub-degrees $(n-1)$, $(n+1)$, and $(2n-1)$ corresponding to $e_\iota : e_\iota'$, $\overline{e}_\iota$ and $\overline{e}_\iota'$ and the contragrediënt unities e_ι', the rules of calculation for $n = 3$ are:

R_3^l	$e_1 =$	$-e'_{23}I$		
	$e_{23} =$	$-e'_1 I$		
	$e_{123} = I$			
		$e'_{123} = I$		$\alpha-\beta$: $-$
	$-e_1 I' =$	e'_{23}		$\alpha-\gamma$: $-$
	$-e_{23} I' =$	e'_1		$\alpha-\delta$: $-$
	$II' =$	$I'I$	$=+1$	$\alpha-\varepsilon$: $-$
	$e_{11} = k$	$e'_{11} = k'$		$\varkappa = -1$
	$e_1 \dashv e'_1 =$	$e'_1 \dashv e_1$	$=-1$	cycl., 1, 2, 3[1]).

R_3^a	$e_1 =$	$-e'_{23} I =$	$-\overline{e}_1 I =$	$-\overline{e}'_{23} I$		
	$e_{23} =$	$e'_1 =$	$\overline{e}_{23} =$	$-\overline{e}'_1 I$		
	$e_{123} =$	$-e'_{123} I =$	$-\overline{e}_{123} I =$	$\overline{e}'_{123}$	$=I$	$\alpha-\beta$: $-$ (1
	$-e_1 I =$	$e'_{23} =$	$\overline{e}_1 =$	$\overline{e}'_{23}$		$\alpha-\gamma$: $-$
	$-e_{23} I =$	$-e'_1 I =$	$-\overline{e}_{23} I =$	$\overline{e}'_1$		$\alpha-\delta$: $+$ $\alpha-\varepsilon = \alpha-\delta$
	$e_{123} I =$	$-e'_{123} =$	$-\overline{e}_{123} =$	$\overline{e}'_{123} I$	$= I^2 = -k^3 = +1$	cycl. 1, 2, 3
	$e_{11} = k$	$e'_{11} = -k^2$	$\overline{e}_{11} = k$	$\overline{e}'_{11} = -k^2$		

R_3^s	$-k\, e_{23} =$	$e_1 =$	$e_{23}' =$	$-k\, e_1'$		
	$k^2 e_1 =$	$e_{23} =$	$e_1' =$	$k^2 e'_{23}$		$\alpha-\beta$: $+$
		$e_{123} =$	$e_{123}' =$		$= I = k^3 = -1$,	$\alpha-\gamma$: identity (12)
		$e_{11} = k$	$e_{11}' = -k^2$			$\alpha-\delta = \alpha-\beta$. $\alpha-\varepsilon = \alpha-\gamma$ cycl. 1, 2, 3.

R_3^o		$e_{123}\, I =$	$-\overline{e}_{123} =$		$= -k^3 = +1$	
	$-\overline{e}_{23} I =$	$e_1 =$	$-\overline{e}_1\, I =$	$-e_{23} I$		$\alpha-\beta$: $-$
	$-e_1\, I =$	$e_{23} =$	$\overline{e}_{23} =$	e_1		$\alpha-\gamma$: $-$ (13)
		$e_{123} =$	$-\overline{e}_{123} I =$		$= I$	$\alpha-\delta$: identity.
		$e_{11} = -1$	$\overline{e}_{11} = -1$			$\alpha-\varepsilon$: identity. cycl. 1, 2, 3.

1) "Cycl $1, 2, 3, \ldots; n$" means that the numbers $1, \ldots, n$ may be substituted by any even permutation of these numbers.

R^r_3	$e_1 = e_{23}$	α–β identity
	$e_{123} = -1$	α–γ „ . . . . (14)
	$e_{11} = -1$	α–δ „
		δ–ε „
		cycl. 1, 2, 3.

With a non-homogeneous rectangular interpretation of the fundamental variables $\mathbf{e}_1$ is a polar vector, $\mathbf{e}'_1$ an axial bivector, $\overline{\mathbf{e}}_1$ an axial vector, $\overline{\mathbf{e}'}_1$ a polar bivector [1]), $\mathbf{I}$ a projective, and $\mathbf{k}$ an orthogonal "pseudoscalar", $\mathbf{k}\mathbf{e}_1$ a polar, and $\mathbf{k}^2\mathbf{e}_{23}$ an axial versor (quaternion with tensor 1) without scalar part. R^a_3 includes and discriminates all these quantities, R^s_3 identifies polar quantities with axial ones and $\mathbf{I}$ with an ordinary number, R^o_3 identifies all the polar quantities and all the axial ones as well, and $\mathbf{k}$ with a common number, whereas in R^r_3 only the difference between vectors and ordinary numbers exists.

The rules of calculation for $n = 4$ are:

	$e_1 =$	$e'_{234}\ I$		
	$e_{12} =$	$-e'_{34}\ I$		
	$e_{34} =$	$-e'_{12}\ I$		
	$e_{234} =$	$e'_1\ I$		
	$e_{1234} =$	e'_{1234}		$\alpha-\beta$: —
	$e_1 I' =$	e'_{234}		$\alpha-\gamma$: —
R^l_4	$-e_{12} I' =$	e'_{34}		$\alpha-\delta$: —
	$-e_{34} I' =$	e'_{12}		$\alpha-\varepsilon$: +
	$e_{234} I' =$	e'_1		$\varkappa = +1$
	$I I' =$	$I' I$	$= +1$	cycl. 1, 2, 3, 4.
	$e_{11} = k$	$e'_{11} = k'$		
	$e_1 \dashv e'_1 =$	$e'_1 \dashv e_1$	$= +1$	

[1]) In space these quantities have the symmetry-properties of a line-part with direction, a plane-part with rotative direction, a line-part with rotative direction and a plane-part with + and — side, all conceived as parallel removable with respect to themselves. For n odd it holds good that polar quantities change their sign, when the + direction of all axes is inverted, and that axial ones do not change their signs.

R_4^a

$e_1 =$	$-i e'_{234} =$	$-i\bar{e}_1\ I=$	$\bar{e}'_{234}\ I$
$e_{12} =$	$-e'_{34}\ I=$	$\bar{e}_{12} =$	$\bar{e}'_{34}\ I$
$e_{34} =$	$-e'_{12}\ I=$	$\bar{e}_{34} =$	$-\bar{e}'_{12}\ I$
$e_{234} =$	$-i e'_1 =$	$-i\bar{e}_{234}\ I=$	$\bar{e}'_1\ I$
$e_{1234} =$	$e'_{1234} =$	$\bar{e}_{1234} =$	$\bar{e}'_{1234}$
$e_1\ I=$	$-i e'_{234}\ I=$	$-i\bar{e}_1 =$	$\bar{e}'_{234}$
$-e_{12}\ I=$	$e'_{34} =$	$-\bar{e}_{12}\ I=$	$\bar{e}'_{34}$
$-e_{34}\ I=$	$e'_{12} =$	$-\bar{e}_{34}\ I=$	$\bar{e}'_{12}$
$e_{234}\ I=$	$-i e'_{234} =$	$-i\bar{e}_{234} =$	$\bar{e}'_1$
$e_{1234} I=$	$e'_{1234} I=$	$\bar{e}_{1234} I=$	$\bar{e}'_{1234} I$
$e_{11} = k$	$e'_{11} = k^3$	$\bar{e}_{11} = k$	$\bar{e}'_{11} = k^3$

$= I$ $\alpha-\beta: +$, $\alpha-\gamma: +$ } (complicated); $\alpha-\delta: +$; $\alpha-\varepsilon = \alpha-\delta$; cycl. 1, 2, 3, 4. (15)

$= I^2 = k^4 = +1$

R_4^o

	$e_{1234}\ I =$	$\bar{e}_{1234} I=$	
$e_{234} I=$	$e_1 =$	$-i\bar{e}_{234} =$	$-i\bar{e}_1\ I$
$-e_{34}\ I=$	$e_{12} =$	$\bar{e}_{12} =$	$-\bar{e}_{34}\ I$
$-e_{12}\ I=$	$e_{34} =$	$\bar{e}_{34} =$	$-\bar{e}_{12}\ I$
$e_1\ I=$	$e_{234} =$	$-i\bar{e}_1 =$	$-i\bar{e}_{234} I$
	$e_{1234} =$	$\bar{e}_{1234} =$	
	$e_{11} = +1$	$\bar{e}_{11} = +1$	

$= I^2 = +1$

$\alpha-\beta: +$ (complicated); $\alpha-\gamma = \alpha-\beta$; $\alpha-\delta$: identity; $\alpha-\varepsilon$: identity (16)

$= I$ cycl. 1, 2, 3, 4.

The dualities $\alpha-\beta$ are complicated ones in this case, i. e. dualising leads say for $\alpha-\beta$ from e_i to e'_i, from e'_i to $-e_i$, from $-e_i$ to $-e'_i$, and from $-e'_i$ again to e_i. This complicated duality always exists for n even [1]), as long as one of the units is not privileged. If one of the units is privileged, or, to put it otherwise, if we derive the system belonging to the group, leaving invariant the quadratic form

$$-a_0^2 + a_1^2 + \ldots + a_{n-1}^2$$

we find, when classifying up to the orthogonal groups inclusive, the system:

[1]) The complicated duality exists also in GRASSMANN's Ausdehnungslehre for n even.

$$
\left.\begin{array}{lll}
\mathbf{e}_0 = \bar{\mathbf{e}}_{123} & & \bar{\mathbf{e}}_0 = \mathbf{e}_{123} \\
\mathbf{e}_1 = -\bar{\mathbf{e}}_{023} & & \bar{\mathbf{e}}_1 = -\mathbf{e}_{023} \\
 & \mathbf{e}_{01} = -\bar{\mathbf{e}}_{01} & \\
 & \mathbf{e}_{12} = +\bar{\mathbf{e}}_{12} & \\
\mathbf{e}_{00} = +1 & & \bar{\mathbf{e}}_{00} = +1 \\
R_4^o\ \mathbf{e}_{11} = -1 & & \bar{\mathbf{e}}_{11} = -1 \\
\mathbf{e}_{0123} = \mathbf{I} & & \bar{\mathbf{e}}_{0123} = -\mathbf{I} \\
\mathbf{e}_0\,\mathbf{I} = \bar{\mathbf{e}}_0 & & -\bar{\mathbf{e}}_0\,\mathbf{I} = \mathbf{e}_0 \\
\mathbf{e}_1\,\mathbf{I} = +\bar{\mathbf{e}}_1 & & -\bar{\mathbf{e}}_1\,\mathbf{I} = +\mathbf{e}_1 \\
\mathbf{e}_{01}\,\mathbf{I} = \mathbf{e}_{23} = \bar{\mathbf{e}}_{23} & & -\bar{\mathbf{e}}_{01}\,\mathbf{I} = \bar{\mathbf{e}}_{23} = \mathbf{e}_{23} \\
\mathbf{e}_{12}\,\mathbf{I} = -\mathbf{e}_{03} = \bar{\mathbf{e}}_{03}, & \mathbf{I}^2 = -1, & -\bar{\mathbf{e}}_{12}\,\mathbf{I} = -\bar{\mathbf{e}}_{03} = \mathbf{e}_{03}
\end{array}\right|
\begin{array}{l}
\alpha\text{–}\beta : + \\
\alpha\text{–}\gamma = \alpha\text{–}\beta \\
\alpha\text{–}\delta : \text{identity.} \\
\alpha\text{–}\varepsilon : \text{identity.} \\
\text{cycl. } 1, 2, 3.
\end{array}
\qquad (17)
$$

with non-complicated duality. This system may also be obtained from the preceding system R_4^o (page 334) by the transition $\mathbf{e}_1 \to \mathbf{e}_0$, $-i\,\mathbf{e}_2 \to \mathbf{e}_3$, etc., $\bar{\mathbf{e}}_1 \to \mathbf{e}_0$, $i\,\bar{\mathbf{e}}_2 \to \mathbf{e}_2$, etc. It it noteworthy that, for $n = 4$ the theory of relativity (for the space-element) exactly corresponds to this more simple system.

For non-homogeneous rectangular interpretation of the fundamental variables $\mathbf{e}_1$, and $\mathbf{e}_{123}$ are a vector, resp. a trivector of the first kind and $\mathbf{I}\,\mathbf{e}_1$, and $\mathbf{I}\,\mathbf{e}_{123}$ are the corresponding quantities of the second kind [1]). $\mathbf{I}$ is a projective and $\mathbf{k}$ an orthogonal pseudoscalar. R_4^a contains and distinguishes all these quantities. R_4^o identifies a vector resp. a trivector of the first kind with a trivector resp. a vector of the second kind and $\mathbf{k}$ with an ordinary number.

Decomposition of the Associative Product.

The associative product of two projective quantities of the subdegrees p' and q' and the principal degrees p and q, $p', q' \leqq n$, $p \leqq q$, consists in the most general case of $p + 1$ parts, each of which being a product of a projective quantity with a certain number of factors $\mathbf{k}$. As a distributive combination each of these parts is a product itself. The number of factors $\mathbf{k}$ is called the transvection-number of this product and this number is at most equal to the smallest of the numbers p' and q'. We call these products, if p' and q' are both $\leqq$ or both $\geqq n'$, beginning from the lowest and otherwise beginning from the highest in sequence:

[1]) The customary distinction for n odd between polar and axial quantities does not hold good for n even.

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(first) vectorial product $\times$
second ,, ,, $\underset{2}{\times}$

(only for p even) α-th middle product, $\alpha = \frac{p}{2} + 1$.

second scalar product $\underset{2}{\cdot}$
first scalar product $\smile$.

With this notation, which is in agreement with the existing dualities, products *that are identical with the rotational group obtain the same name and the same symbol.* Owing to the identification of **I** and **k** with common numbers the first middle-product is identical with the product of ordinary numbers mutually and with other quantities, hence its symbol may be omitted as being customary.

The rule of transvection.

If each factor is an alternating product of fundamental elements:

$$_{p'}\mathbf{a} = \mathbf{a}_1 \overset{\frown}{\ldots\ldots} \mathbf{a}_{p'}$$
$$_{q'}\mathbf{b} = \mathbf{b}_1 \overset{\frown}{\ldots\ldots} \mathbf{b}_{q'}$$

we can form the combination:

$$(\mathbf{a}_{p'} \,.\, \mathbf{b}_1)\ (\mathbf{a}_{p'-1} \,.\, \mathbf{b}_2) \ldots\ldots (\mathbf{a}_{p'-i+1} \,.\, \mathbf{b}_i)\ \mathbf{a}_1 \ldots\ldots \overset{\frown}{\mathbf{a}_{p'-i}}\, \mathbf{b}_{i+1} \ldots\ldots \mathbf{b'}_q,$$

repeat the same for all $p\,!$ resp. $q'!$ modes of notation of $_{p'}\mathbf{a}$ and $_{b'}\mathbf{b}$ and add the results.

The sum then consists of $p'!\ q'!$ terms, equivalent to each other in groups of $(p'-i)!\ (q'-i)!\ i!$. This sum divided by $(p'-i)!\ (q'-i)!\ i!$, or, stated more briefly, the sum of $(p'_i)\ (q'_i)\ i!$ arbitrary *different* terms, is called the *i-fold-combination* of $_{p'}\mathbf{a}$ and $_{q'}\mathbf{b}$. The i-fold combination is now equal to the product with the transvection-number i. The transvection-number of a product being known, we can hence write it down from memory by this rule.

The free rules for R_3 and R_4.

Hence the free rules for R_3^l, R_3^a, R_3^s, R_3^o and R_3^r are:

Transv. numb.:		
0	$\mathbf{a} \times \mathbf{b} =$ quantity of the second sub degree.	(18)
1	$\mathbf{a} \,.\, \mathbf{b} =$ scalar in **k** resp. 1.	
0	$\mathbf{a} \,.\, (\mathbf{b} \times \mathbf{c}) = \mathbf{a} \times \mathbf{b} \,.\, \mathbf{c} =$ scalar in **I** resp. 1.[1]	
1	$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \,.\, \mathbf{b})\, \mathbf{c} - (\mathbf{a} \,.\, \mathbf{c})\, \mathbf{b}$	
1	$\mathbf{a}\, (\mathbf{b} \times \mathbf{c} \,.\, \mathbf{d}) = (\mathbf{a} \,.\, \mathbf{b})\, (\mathbf{c} \times \mathbf{d}) + (\mathbf{a} \,.\, \mathbf{c})\, (\mathbf{d} \times \mathbf{b}) + (\mathbf{a}.\mathbf{d})\, (\mathbf{b} \times \mathbf{c})$	
1	$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = (\mathbf{b} \,.\, \mathbf{c})\, (\mathbf{a} \times \mathbf{d}) - (\mathbf{b} \,.\, \mathbf{d})\, (\mathbf{a} \times \mathbf{c}) + \ldots\ldots$	
2	$(\mathbf{a} \times \mathbf{b}) \,.\, (\mathbf{c} \times \mathbf{d}) = (\mathbf{b} \,.\, \mathbf{c})\, (\mathbf{a} \,.\, \mathbf{d}) - (\mathbf{b} \,.\, \mathbf{d})\, (\mathbf{a} \,.\, \mathbf{c})$	
2	$(\mathbf{a} \times \mathbf{b})\, (\mathbf{c} \times \mathbf{d} \,.\, \mathbf{e}) = (\mathbf{b} \,.\, \mathbf{c})\, (\mathbf{a} \,.\, \mathbf{d})\, \mathbf{e} - (\mathbf{b} \,.\, \mathbf{d})\, (\mathbf{a} \,.\, \mathbf{c})\, \mathbf{e} + \ldots\ldots$	
3	$(\mathbf{a} \times \mathbf{b} \,.\, \mathbf{c})\, (\mathbf{d} \times \mathbf{e} \,.\, \mathbf{f}) = (\mathbf{c} \,.\, \mathbf{d})\, (\mathbf{b} \,.\, \mathbf{e})\, (\mathbf{a} \,.\, \mathbf{f}) + (\mathbf{c} \,.\, \mathbf{e})\, (\mathbf{b} \,.\, \mathbf{f})\, (\mathbf{a} \,.\, \mathbf{d}) + \ldots$	

[1]) In alternating products the brackets have been omitted for the association (..)., so that we write the alternating product of $\mathbf{a}_1, \ldots, \mathbf{a}_p$:

$$\mathbf{a}_1 \times \mathbf{a}_2 \times \ldots\ldots \times \mathbf{a}_{n'} \times \mathbf{a}_{n'+1} \,.\, \mathbf{a}_{n'+2} \,.\, \ldots\ldots \,.\, \mathbf{a}_p \,.$$

The four systems differ only by the different signification attached to **I** and **k**. R_3^r is the common vector-analysis, in which no difference is made between polar quantities and axial ones and between vectors and bivectors. R_3^o distinguishes between polar quantities and axial ones. In GIBBS's form of this vector-analysis, owing to the groundlessly introduced + sign in $e_1 . e_1 = -1$, the formulae acquire apparently irregular changes of + and — signs and the transvection-rule becomes ineffectual, so that the formulae stand side by side independent of one another and can be used only by means of a table When applied to units the rules for R_3^o and R_3^r are:

$$R_3^o \left.\begin{array}{ll} e_1 \times e_2 = - e_2 \times e_1 = e_{12} & e_{23} \times e_{12} = e_{31} \\ e_1 \, . \, e_1 = -1 & e_{12} \, . \, e_{12} = -1 \\ e_1 \, . \, e_{23} = -I & e_{12} \, I = I \, e_{12} = - e_3 \\ e_1 \times e_{12} = - e_2 & I^2 = +1 \\ e_1 \, I = I \, e_1 = - e_{23} & \end{array}\right\} \text{cycl. } 1, 2, 3.$$

$$R'_3 \quad \left.\begin{array}{l} e_1 \times e_2 = - e_2 \times e_1 = e_3 \\ e_1 \, . \, e_1 = -1 \end{array}\right\} \text{cycl. } 1, 2, 3. \quad . \quad . \quad (20)$$

The rules (18) and (20) can be dualised according to all existing dualities as given in the table.

The free rules for R_4^a and R_4^o are:

Transv. numb.	
0	$a \times b =$ quantity of the second sub-degree
1	$a . b =$ scalar in **k** resp. 1
0	$a \times (b \times c) = a \times b \times c$
1	$a . (b \times c) = (a . b) c - (a . c) b$
0	$a . (b \times c \times d) = a \times b \times c . d =$ scalar in **I** resp. 1
1	$a \times (b \times c \times d) = (a . b)(c \times d) + (a . c)(d \times b) + (a . d)(b \times c)$
1	$a (b \times c \times d . e) = (a . b)(c \times d \times e) - (a . c)(b \times d \times e) + \ldots$
0	$(a \times b) \times (c \times d) = a \times b \times c . d$
1	$(a \times b) \ast (c \times d) = (b . c)(a \times d) - (b . d)(a \times c) + \ldots$ [1]
2	$(a \times b) . (c \times d) = (b . c)(a . d) - (b . d)(a . c)$
1	$(a \times b) . (c \times d \times e) = (b . c)(a \times d \times e) + \ldots$
2	$(a \times b) \times (c \times d \times e) = (b . c)(a . d) e + \ldots$
2	$(a \times b)(c \times d \times e . f) = (b . c)(a . d)(e \times f) + \ldots$
2	$(a \times b \times c) \times (d \times e \times f) = (c . d)(b . e)(a \times f) + \ldots$
3	$(a \times b \times c) . (d \times e \times f) = (c . d)(b . e)(a . f) + \ldots$
3	$(a \times b \times c)(d \times e \times f . g) = (c . d)(b . e)(a . f) g + \ldots$
4	$(a \times b \times c . d)(e \times f \times g . h) = (d . e)(c . f)(b . g)(a . h) + \ldots$

. (21)

independent of the units used, viz. e_1, e_2, e_3, e_4 or e_0, e_1, e_2, e_3.

[1] The index 2 under * is for simplicity omitted.

When applied to units the rules for R_4^0 and for e_1, e_2, e_3, e_4 are

$$\left.\begin{array}{ll}
e_1 \times e_2 = -e_2 \times e = e_{12} & e_{12} * e_{23} = e_{13} \\
e_1 \,.\, e_1 = +1 & e_{12} \,.\, e_{12} = -1 \\
e_1 \times e_{23} = e_{123} = i\,\overline{e_4} & e_{12} \,.\, e_{234} = e_{134} \\
e_1 \,.\, e_{12} = e_2 & e_{12} \times e_{123} = -e_3 \\
e_1 \,.\, e_{234} = I & e_{12}\, I = I\, e_{12} = -e_{34} \\
e_1 \times e_{123} = e_{23} & e_{123} \times e_{234} = -e_{14} \\
e_1\, I = -I\, e_1 = e_{234} = -i\,\overline{e_1} & e_{123} \,.\, e_{123} = -1 \\
e_{12} \times e_{34} = I & e_{234}\, I = -I\, e_{234} = e_1 \\
 & I^2 = +1
\end{array}\right\} \begin{array}{l}\text{cycl. } 1, 2, 3, 4. \\ \text{dual } e - \overline{e} \\ \text{(complicated)}\end{array} \quad (22$$

and for e_0, e_1, e_2, e_3:

(See for formula (23) page 339).

The quantities of an even by-degree form a sub-system with 8 units and the rules:

$$\left.\begin{array}{ll}
i_1 * i_2 = -i_2 * i_1 = i_3 & j_1 * j_2 = -j_2 * j_1 = -i_3 \\
i_1 * j_2 = -j_2 * i_1 = j_3 & j_1 * i_2 = -i_2 * j_1 = j_3 \\
i_1 \,.\, i_1 = -1 & j_1 \,.\, j_1 = +1 \\
I\, i_1 = i_1\, I = -j_1 & I\, j_1 = j_1\, I = +i_1 \\
i_1 \times j_1 = j_1 \times i_1 = I & \begin{pmatrix} i_1 = e_{23} \\ j_1 = e_{01} \end{pmatrix} \\
I^2 = -1 &
\end{array}\right\} \begin{array}{l}\text{cycl.} \\ 1, 2, 3.\end{array} \quad (24$$

But these are the same rules as those for the units $e_1, e_2, e_3, ie_1, ie_2, ie_3$ of R_3' with ordinary complex coefficients, so that the free rules for R_r'' also hold good for quantities of an even by-degree of R_o^4 if, instead of $\times$ and . we introduce the symbols $*$ and $\dot{\times} = . + \times$:

$$\left.\begin{array}{l}
{}_2a * {}_2b = \text{quantity of the second by-degree} \\
{}_2a \dot{\times} {}_2b = \text{scalar in } I \text{ and } 1 \\
{}_2a \dot{\times} ({}_2b * {}_2c) = {}_2a * {}_2b \dot{\times} {}_2c \\
{}_2a * ({}_2b * {}_2c) = ({}_2a \dot{\times} {}_2b)\, {}_2c - ({}_2a \dot{\times} {}_2c)\, {}_2b \\
{}_2a\, ({}_2b * {}_2c \dot{\times} {}_2d) = ({}_2a \dot{\times} {}_2b)\, ({}_2c * {}_2d) + \ldots. \\
({}_2a * {}_2b) * ({}_2c * {}_2d) = ({}_2b \dot{\times} {}_2c)\, ({}_2a * {}_2d) + \ldots. \\
({}_2a * {}_2b) \dot{\times} ({}_2c * {}_2d) = ({}_2b \dot{\times} {}_2c)\, ({}_2a \dot{\times} {}_2d) + \ldots. \\
({}_2a * {}_2b)\, ({}_2c * {}_2d \dot{\times} {}_2e) = ({}_2b \dot{\times} {}_2c)\, ({}_2a \dot{\times} {}_2d)\, {}_2e + \ldots. \\
({}_2a * {}_2b \dot{\times} c)({}_2d * {}_2e \dot{\times} {}_2f) = ({}_2c \dot{\times} d)({}_2b \dot{\times} {}_2e)({}_2a \dot{\times} {}_2f) + \ldots.
\end{array}\right\} . \quad (25$$

Hence these rules may be written down from memory, as well as the others.

The System R_4^o and the theory of relativity (in an element of four dimensional space).

Fragments of R_4^o have been used by various authors[1]) on the theory of relativity. With them five products occur and two of these

[1]) H. MINKOWSKI, M. ABRAHAM, A. SOMMERFELD, M. LAUE, PH. FRANK.

and for $\mathbf{e}_0$, $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$:

$$\mathbf{e}_0 \times \mathbf{e}_1 = -\mathbf{e}_1 \times \mathbf{e}_0 = \mathbf{e}_{01} = -\overline{\mathbf{e}}_{01} \qquad \overline{\mathbf{e}}_0 \times \overline{\mathbf{e}}_1 = -\overline{\mathbf{e}}_1 \times \overline{\mathbf{e}}_0 = \overline{\mathbf{e}}_{01} = -\mathbf{e}_{01}$$

$$\mathbf{e}_1 \times \mathbf{e}_2 = -\mathbf{e}_2 \times \mathbf{e}_1 = \mathbf{e}_{12} = \overline{\mathbf{e}}_{12} \qquad \overline{\mathbf{e}}_1 \times \overline{\mathbf{e}}_2 = -\overline{\mathbf{e}}_2 \times \overline{\mathbf{e}}_1 = \overline{\mathbf{e}}_{12} = \mathbf{e}_{12}$$

$$\mathbf{e}_0 \,.\, \mathbf{e}_0 = +1, \quad \mathbf{e}_1 \,.\, \mathbf{e}_1 = -1 \qquad \overline{\mathbf{e}}_0 \,.\, \overline{\mathbf{e}}_0 = +1, \quad \overline{\mathbf{e}}_1 \,.\, \overline{\mathbf{e}}_1 = -1$$

$$\mathbf{e}_1 \times \mathbf{e}_{23} = \mathbf{e}_{123} = \overline{\mathbf{e}}_0, \quad \mathbf{e}_0 \times \mathbf{e}_{12} = \mathbf{e}_{012} = -\overline{\mathbf{e}}_3 \qquad \overline{\mathbf{e}}_1 \times \overline{\mathbf{e}}_{23} = \overline{\mathbf{e}}_{123} = \mathbf{e}_0, \quad \overline{\mathbf{e}}_0 \times \overline{\mathbf{e}}_{12} = \overline{\mathbf{e}}_{012} = -\mathbf{e}_3$$

$$\mathbf{e}_1 \,.\, \mathbf{e}_{12} = -\mathbf{e}_2, \quad \mathbf{e}_1 \,.\, \mathbf{e}_{10} = -\mathbf{e}_0, \quad \mathbf{e}_0 \,.\, \mathbf{e}_{01} = \mathbf{e}_1 \qquad \overline{\mathbf{e}}_1 \,.\, \overline{\mathbf{e}}_{12} = -\overline{\mathbf{e}}_2, \quad \overline{\mathbf{e}}_1 \,.\, \overline{\mathbf{e}}_{10} = -\overline{\mathbf{e}}_0, \quad \overline{\mathbf{e}}_0 \,.\, \overline{\mathbf{e}}_{01} = \overline{\mathbf{e}}_1$$

$$\mathbf{e}_0 \,.\, \overline{\mathbf{e}}_0 = \mathbf{I}, \quad \mathbf{e}_1 \,.\, \overline{\mathbf{e}}_1 = \mathbf{I} \qquad \overline{\mathbf{e}}_0 \,.\, \mathbf{e}_0 = -\mathbf{I}, \quad \overline{\mathbf{e}}_1 \,.\, \mathbf{e}_1 = -\mathbf{I}$$

$$\mathbf{e}_0 \times \overline{\mathbf{e}}_1 = -\mathbf{e}_{23} = -\overline{\mathbf{e}}_{23} \qquad \overline{\mathbf{e}}_0 \times \mathbf{e}_1 = -\overline{\mathbf{e}}_{23} = -\mathbf{e}_{23}$$

$$\mathbf{e}_1 \times \overline{\mathbf{e}}_2 = \mathbf{e}_{03} = -\overline{\mathbf{e}}_{03} \qquad \overline{\mathbf{e}}_1 \times \mathbf{e}_2 = \overline{\mathbf{e}}_{03} = -\mathbf{e}_{03}$$

$$\mathbf{e}_0 \mathbf{I} = -\mathbf{I}\,\mathbf{e}_0 = \mathbf{e}_{123} = \overline{\mathbf{e}}_0 \qquad -\overline{\mathbf{e}}_0 \mathbf{I} = \mathbf{I}\,\overline{\mathbf{e}}_0 = \overline{\mathbf{e}}_{123} = \mathbf{e}_0$$

$$\mathbf{e}_1 \mathbf{I} = -\mathbf{I}\,\mathbf{e}_1 = \mathbf{e}_{023} = -\overline{\mathbf{e}}_1 \qquad -\overline{\mathbf{e}}_1 \mathbf{I} = \mathbf{I}\,\overline{\mathbf{e}}_1 = \overline{\mathbf{e}}_{023} = -\mathbf{e}_1$$

$$\mathbf{e}_{01} \times \mathbf{e}_{23} = \mathbf{I} \qquad \overline{\mathbf{e}}_{01} \times \overline{\mathbf{e}}_{23} = -\mathbf{I}$$

$$\mathbf{e}_{01} * \mathbf{e}_{02} = -\mathbf{e}_{12} = -\overline{\mathbf{e}}_{12}, \quad \mathbf{e}_{01} * \mathbf{e}_{12} = -\mathbf{e}_{02} = \overline{\mathbf{e}}_{02} \qquad \overline{\mathbf{e}}_{01} * \overline{\mathbf{e}}_{02} = -\overline{\mathbf{e}}_{12} = -\mathbf{e}_{12}, \quad \overline{\mathbf{e}}_{01} * \overline{\mathbf{e}}_{12} = -\overline{\mathbf{e}}_{02} = \mathbf{e}_{02}$$

$$\mathbf{e}_{23} * \mathbf{e}_{31} = \mathbf{e}_{12} = \overline{\mathbf{e}}_{12} \qquad \overline{\mathbf{e}}_{22} * \overline{\mathbf{e}}_{31} = \overline{\mathbf{e}}_{12} = \mathbf{e}_{12}$$

$$\mathbf{e}_{01} \,.\, \mathbf{e}_{01} = +1, \quad \mathbf{e}_{12} \,.\, \mathbf{e}_{12} = -1 \qquad \overline{\mathbf{e}}_{01} \,.\, \overline{\mathbf{e}}_{01} = +1, \quad \overline{\mathbf{e}}_{12} \,.\, \overline{\mathbf{e}}_{12} = -1$$

$$\mathbf{e}_{01} \mathbf{I} = \mathbf{I}\,\mathbf{e}_{01} = \mathbf{e}_{23} = \overline{\mathbf{e}}_{23}, \quad \mathbf{e}_{12} \mathbf{I} = \mathbf{I}\mathbf{e}_{12} = -\mathbf{e}_{03} = \overline{\mathbf{e}}_{03}, \quad -\overline{\mathbf{e}}_{01} \mathbf{I} = -\mathbf{I}\,\overline{\mathbf{e}}_{01} = \overline{\mathbf{e}}_{23} = \mathbf{e}_{23}, \quad -\overline{\mathbf{e}}_{12} \mathbf{I} = -\mathbf{I}\,\overline{\mathbf{e}}_{12} = -\overline{\mathbf{e}}_{03} = \mathbf{e}_{03}$$

$$\mathbf{I}^2 = -1 \qquad (-\mathbf{I})^2 = -1$$

cycl. 1, 2, 3. (23)

said products are doubled by introducing the "dual" bivector (dualer sechervektor)[1]. E. WILSON and G. LEWIS have further elaborated the system and obtain all the products, but three[2]. All these conclusions are founded on analogies with the common vector-analysis and the multiplications form no parts of the associative multiplication. Therefore the free calculation-rules cannot immediately be put down from memory according to the transvection-rule, but in so far as they exist they only allow a use by means of a table. The names scalar and vectorial too, have been divided over the existing multiplications by analogy and not in agreement to the duality α—γ.

WILSON-LEWIS			SOMMERFELD, LAUE, etc.
$+\mathbf{a}\times\mathbf{b}$	$\mathbf{a}\times\mathbf{b}={}_2\mathbf{c}$		$[\mathbf{a}\,\mathbf{b}]$, vectorial product
$-\mathbf{a}\,.\,\mathbf{b}$	$\mathbf{a}\,.\,\mathbf{b}=c$		$[\mathbf{a}\,\mathbf{b}]$, scalar „
$+\mathbf{a}\times{}_2\mathbf{b}$	$\mathbf{a}\times{}_2\mathbf{b}={}_3\mathbf{c}$		$\mathbf{I}\,\mathbf{c}=[\mathbf{a}\,{}_2\mathbf{b}^*]$, vect. pr. w. dual bivect.
$+\mathbf{a}\,.\,{}_2\mathbf{b}$	$\mathbf{a}\,.\,{}_2\mathbf{b}=\mathbf{c}$		$-[\mathbf{a}\,{}_2\mathbf{b}]$, vect. pr.
$\pm k\mathbf{a}=\pm\mathbf{a}k$	$\mathbf{a}\,\mathbf{I}=-\mathbf{I}\,\mathbf{a}={}_3\mathbf{b}^*$)		
$+\mathbf{a}\times{}_3\mathbf{b}$	$\mathbf{a}\,.\,{}_3\mathbf{b}={}_4\mathbf{c}^*$)		
$-\mathbf{a}\,.\,\overline{\mathbf{b}}$	$\mathbf{a}\times{}_3\mathbf{b}={}_2\mathbf{c}$	selfdualistic	
$+{}_2\mathbf{a}\times{}_2\mathbf{b}$	${}_2\mathbf{a}\times{}_2\mathbf{b}={}_4\mathbf{c}^*$)		$\mathbf{I}\,{}_4\mathbf{c}=({}_2\mathbf{a}\,{}_2\mathbf{b}^*)$, scal. pr.w. dual biv.
	${}_2\mathbf{a} * {}_2\mathbf{b}={}_2\mathbf{c}$		$[{}_2\mathbf{a}\,{}_2\mathbf{b}]$, vector pr. (G. MIE)
$-{}_2\mathbf{a}\,.\,{}_2\mathbf{b}$	${}_2\mathbf{a}\,.\,{}_2\mathbf{b}=c$		$-({}_2\mathbf{a}\,{}_2\mathbf{b})$, scal. pr.
$\pm k\,{}_2\mathbf{a}=\pm{}_2\mathbf{a}k$	${}_2\mathbf{a}\,\mathbf{I}=\mathbf{I}\,{}_2\mathbf{a}={}_2\mathbf{b}^*$)		$-{}_2\mathbf{b}=+{}_2\mathbf{a}^*$
$k\,k=-1$	$\mathbf{I}^2=+1$ *)		
$\pm k\,{}_3\mathbf{a}=\pm{}_3\mathbf{a}\,k$	${}_3\mathbf{a}\,\mathbf{I}=-\mathbf{I}\,{}_3\mathbf{a}=\mathbf{b}^*$)		
	${}_3\mathbf{a}\,.\,{}_2\mathbf{b}={}_3\mathbf{c}$		
$-{}_3\mathbf{a}\,.\,{}_2\mathbf{b}$	${}_3\mathbf{a}\times{}_2\mathbf{b}=\mathbf{c}$		
$+{}_3\mathbf{a}\,.\,\overline{\mathbf{b}}$	${}_3\mathbf{a}\,.\,\mathbf{b}=c$		
	${}_3\mathbf{a}\times\mathbf{b}={}_2\mathbf{c}$		

[1] This is not a proper duality, because in the only duality existing with the orthogonal group, α-γ, a bivector e.g. $\mathbf{e}_{12}$ is not dualistic to the "dual" bivector $\mathbf{I}\mathbf{e}_{12}$, but to $\mathbf{e}_{12}$ itself.

[2] The connection with an associative CLIFFORD algebra and the absence of three products has already been briefly pointed out by J. B. SHAW, "The WILSON and LEWIS Algebra for Four-Dimensional Space" Bull. of the int. ass. for quat. (13) **24**–27.

Therefore this duality does not attain expression, not even in the system of WILSON and LEWIS, though they use units of the kind e_0, e_1, e_2, e_3.

The foregoing table (subjoined p. 340) presents a summary of the products used by various authors.

The table has been arranged dualistically. Each product has been indicated by an example. For the multiplications we used in the columns 1 and 3 the author's own notation, but for the quantities we used all through the notations adopted in this paper. The dual bivector only has been written with the customary asterisk, while the commutative scalar of WILSON and LEWIS has been indicated by k. The products marked with *) do not correspond exactly to the other systems, because these systems do not contain the non-commutative scalar **I**.

The system R_4^0 contains the existing fragments and all the existing multiplications and rules, and owing to the free rules of calculation (21 and 25) it is eminently suited for practical purposes.

The system R_4^0 and the elliptic and hyperbolic geometry in three dimensions.

With a homogeneous interpretation of the fundamental variables R_4^0 corresponds to a projective geometry in three dimensions, a non degenerated quadratic surface being invariant. If the units are selected according to (16) the equation of the absolute surface in point- resp. plane-coordinates is:

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0$$
$$u_1^2 + u_2^2 + u_3^2 + u_4^2 = 0$$

and the geometry is elliptic. If, on the other hand the units are selected according to (17) the geometry is hyperbolic. The free rules of the system are the same for both cases. To a fundamental element a point with a number-value corresponds, to a quantity of the second degree a sum of linear elements (Dyname) and to a quantity of the third degree a planar element. The sub-system of the quantities of the second by-degree is a form of biquaternions, which was first mentioned by CLIFFORD[1]) as a system of linear elements in a non-euclidic three-dimensional space. Hence the system R_4^0 completes these biquaternions to a system which also contains points and planar elements.

[1]) Preliminary sketch of biquaternions. Proc. Lond. Math. Soc. 4 (73) 381—395; Further notes on biquaternions. Coll. Math. Papers (76) 385, 395.