**Physics.** — *"On the shape of small drops and gas-bubbles"*. By J. E. VERSCHAFFELT. Supplement N°. 42*c* to the Communications from the Physical Laboratory at Leiden. (Communicated by Prof. H. KAMERLINGH ONNES).

(Communicated in the meeting of June 29, 1918).

§ 1. It is well known that the meridian-section of a liquid drop or gas-bubble (which we shall suppose to be bodies of revolution) cannot be represented by a finite equation by means of known functions. The differential equation to the section

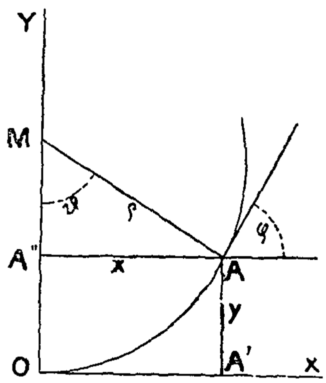


Fig 1.

$$\frac{1}{R_1}+\frac{1}{R_2}=\frac{1}{x}\frac{d}{dx}\left(\frac{xy'}{\sqrt{1+y'^2}}\right)=k(h+y)\text{ [1]}\quad(1)$$

has as a first integral the equation

$$x \sin \varphi = \tfrac{1}{2} khx^2 + \frac{k}{2\pi} u, \quad . \quad (2)$$

where $\varphi$ represents the angle which the tangent forms with the $x$-axis (fig. 1; $OY$ is the axis of revolution) and $u = 2\pi \int_0^x xy\,dx$ [2]), but the computation of $u$ and consequently

[1]) In this equation $k$ stands for the expression $\frac{(\mu_1-\mu_2)g}{\sigma}$, $\sigma$ being the surface tension, $\mu_1-\mu_2$ the difference of the densities below and above the surface in its top, $g$ the acceleration of gravity; $k$ is therefore positive or negative according as the liquid is below the top of the surface, as with a drop resting on a plane, or above it, as with a hanging drop; $y$ is the height of a point of the surface above the tangent plane at the top. $h$ is determined by $kh=\frac{2}{R_0}$, $R_0$ being the radius of curvature at the top; $R_0$ will be reckoned as positive when the surface is hollow upwards, negative in the opposite case.

[2]) $u$ is evidently the volume of the body which is originated by rotation of the surface $OAA'O$ (fig. 1) about the $y$-axis. Equation (2) may be written in the form

$$2\pi x\sigma \sin \varphi = (\mu_1-\mu_2)g\,(\pi x^2 h+u), \quad . \quad . \quad . \quad . \quad (3)$$

the further integration of the differential equation can only be carried out by successive approximations or a development in series.

In the case that the drop or bubble deviates little from the spherical shape, $y$ is small compared to $h$ [1]). In first approximation $y$ may thus be neglected by the side of $h$, i. e. we may put $y = 0$; as a second approximation a circular meridian section is then obtained; if the expression for $y$ corresponding to this as a function of $x$ is substituted in $u$, a first deviation from the sphere is found as a third approximation, etc. [2]).

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which is also found directly, when, for instance by applying the so called "weight-method", the rise in a capillary tube is calculated. The contradiction found by A. FERGUSON (Phil. Mag., (6), 28, (1914) p. 128) between the result of the integration of the differential equation and that of the application of the weight-method is merely due to an error of computation in the approximation of equation (2), owing to which FERGUSON's formula (7) is incorrect.

Equation (2) can also be written as follows

$$x \sin \varphi = \tfrac{1}{2} kx^2 (h+y) - \frac{k}{2\pi} v, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2')$$

where $v = \pi x^2 y - u$ represents the volume arising by the rotation of the surface $OAA''O$. (2) gives:

$$2\pi x\sigma \sin \varphi = \pi (\mu_1 - \mu_2) gx^2 (h + y) - (\mu_1 - \mu_2) gv, \quad . \quad . \quad (3')$$

which expresses for instance, that the resultant of the forces acting along the edge of a section of a hanging drop makes équilibrium with the hydrostatic pressure on the section and the weight of the portion below it, in other words the surface tension does not balance the weight of a hanging drop alone, a fact which may also be derived from a simple consideration of the equilibrium (cf. on this point TH. LOHNSTEIN, Ann. d. Phys., (4) 20 (1906) p. 238).

[1]) Hence $R_0$ is also small compared to $h$ or to $\frac{2}{kR_0}$, that is $kR_0^2$ is a small number.

[2]) Cf. for instance A. WINKELMANN, Handb. der Physik, 2e Aufl. I (2), 1143—1144, 1908.

Putting $y = R_0 - \sqrt{R_0^2 - x^2} + z$, where $z$ is considered infinitely small as compared to $y$, and supposing that $z'$ is also infinitely small compared to $y'$, $\sin \varphi = \frac{y'}{\sqrt{1+y'^2}}$ may be developed in a series, which gives, if $z_1$ represents the first approximation of $z$:

$$z_1 = \tfrac{1}{6} k \frac{R_0^4}{\sqrt{R_0^2 - x^2}} - \tfrac{1}{6} kR_0^3 + \tfrac{1}{3} kR_0^3 \log \frac{R_0 + \sqrt{R_0^2 - x^2}}{2R_0},$$

as is also found by FERGUSON (loc. cit.) although in a somewhat circuitous manner. This expression, however, does not hold near $x = R_0$, as $z'_1$ is there no longer infinitely small with respect to $y'$, but of the same order of magnitude (viz. of the order $(kR_0^2)^{-\frac{1}{2}}$; this fact has been overlooked by FERGUSON (loc. cit.).

§ 2. The introduction of polar coördinates, choosing as origin the centre of curvature at the top $M$ (fig. 1), gives the advantage that there is no discontinuity at $\vartheta = \frac{\pi}{2}$; in that case

$$x = \varrho \sin \vartheta \quad \text{and} \quad y = R_0 - \varrho \cos \vartheta \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and the equation (1) becomes

$$\frac{\varrho \sin \vartheta - \varrho' \cos \vartheta}{\varrho \sin \vartheta \, (\varrho^2 + \varrho'^2)^{1/2}} + \frac{\varrho^2 + 2\varrho'^2 - \varrho\varrho''}{(\varrho^2 + \varrho'^2)^{3/2}} = \frac{2}{R_0} + k(R_0 - \varrho \cos \vartheta). \quad . \quad (5)$$

If we now put

$$\varrho = R_0 (1 - \tau) \quad \text{and} \quad \tau = \tau_1 + \tau_2 + \tau_3 + \dots, \quad . \quad . \quad . \quad . \quad (4')$$

where $\tau_1$, $\tau_2$, etc. represent the successive approximations to the infinitely small quantity $\tau$, we can, as long as $\tau$ and $\tau'$ are infinitely small, separate equation (5) into a series of other ones, the first of which being

$$\tau''_1 \sin \vartheta + \tau'_1 \cos \vartheta + 2\tau_1 \sin \vartheta = kR_0^2 (1 - \cos \vartheta) \sin \vartheta \, ; \quad . \quad . \quad (5')$$

hence [1])

$$\tau_1 = \tfrac{1}{6} kR_0^2 \left[ (1 - \cos \vartheta) + 2 \cos \vartheta \log \left( \frac{1 + \cos \vartheta}{2} \right) \right], \quad . \quad . \quad (6)$$

an expression which remains valid from $\vartheta = 0$ to $\vartheta = \pi$ throughout.

§ 3. The result of the third approximation is as follows

$$u = \tfrac{1}{3}\pi R_0^3 (1 - \cos \vartheta)^2 (1 + 2 \cos \vartheta) + \tfrac{1}{3}\pi k R_0^5 (1 - \cos \vartheta)^2 \cos^2 \vartheta +$$
$$+ \tfrac{1}{3}\pi k R_0^5 \sin^2 \vartheta \, (1 - 2 \cos \vartheta + 2 \cos^2 \vartheta) \log \left( \frac{1 + \cos \vartheta}{2} \right) \quad . \quad . \quad (7)$$

and

$$v = \tfrac{1}{3}\pi R_0^3 (1 - \cos \vartheta)^2 (2 + \cos \vartheta) - \tfrac{1}{6} \pi k R_0^5 (1 - \cos \vartheta)^3 (2 + \cos \vartheta) -$$
$$- \tfrac{1}{3}\pi k R_0^5 \sin^4 \vartheta \log \left( \frac{1 + \cos \vartheta}{2} \right). \quad . \quad . \quad . \quad . \quad . \quad (7')$$

§ 4. Between the angles $\vartheta$ and $\varphi$ the following relation holds:

$$\sin \varphi = \frac{\varrho . \sin \vartheta - \varrho' \cos \vartheta}{\sqrt{\varrho^2 + \varrho'^2}} = \sin \vartheta + \tau' \cos \vartheta + \dots ;$$

[1]) In order to integrate these equations we have to bear in mind, that

$$\cos \vartheta \, (\tau'' \sin \vartheta + \tau' \cos \vartheta + 2\tau \sin \vartheta) = \frac{d}{d\vartheta} \left[ \sin \vartheta \, (\tau \sin \vartheta + \tau' \cos \vartheta) \right]$$

and

$$\tau \sin \vartheta + \tau' \cos \vartheta = \cos^2 \vartheta \frac{d}{d\vartheta} \left( \frac{\tau}{\cos \vartheta} \right).$$

The integration does not offer any special difficulties, but the calculations are long, that of $\tau_2$ being already very laborious; for that reason we have confined ourselves to $\tau_1$.

It is easily seen, that $R_0 \tau_1 = z_1 \cos \vartheta$.

putting therefore

$$\varphi = \vartheta + \psi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

we find in first approximation (for $\vartheta < \pi$)

$$\psi = \tau'_1 = \tfrac{1}{6} kR_0^{\,2} \sin\vartheta \left[\frac{1 - \cos\vartheta}{1 + \cos\vartheta} - 2 \log\left(\frac{1 + \cos\vartheta}{2}\right)\right] \quad . \quad (8')$$

Hence, as long as $\varphi$ is not too near $\pi$, equation (2) in connection with (7) gives

$$x = R_0 \sin\varphi - \tfrac{1}{6} kR_0^{\,3} \sin\varphi \frac{(1 - \cos\varphi)(1 + 2\cos\varphi)}{1 + \cos\varphi} -$$

$$- \tfrac{1}{36} k^2 R_0^{\,5} \sin\varphi \frac{1 - \cos\varphi}{(1 + \cos\varphi)^2} (1 - 3\cos\varphi + 6\cos^2\varphi + 8\cos^3\varphi) -$$

$$- \tfrac{1}{6} k^2 R_0^{\,5} \sin\varphi \log\left(\frac{1 + \cos\varphi}{2}\right)^{1)} . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

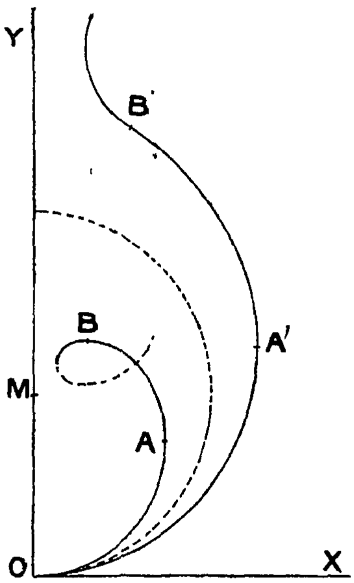

Fig. 2.

§ 5. In fig. 2 $OAB$ represents the meridian section of the capillary surface for $k > 0$, $OA'B'$ gives the section for $k < 0$; both have been drawn for a positive $R_0$ (for $R_0$ negative the diagram must be turned upside down about the $x$-axis); the dotted curve between the two is the circle with radius $R_0$ (corresponding to $k = 0$). In both cases $x$ goes through a maximum (in $A$ and $A'$ respectively), but, whereas in the first case the curvature keeps the same sign all the way, so that $y$ passes a maximum (in $B$), $x$ a minimum etc. (§ 7), the curve in the second case has a point of inflection (in $B'$), beyond which $x$ becomes minimum etc.

The maximum value of $x$ is obtained by putting $\varphi = \frac{\pi}{2}$ in (9), the result being:

$$x_A\,(x_{A'}) = R_0 - \tfrac{1}{6} kR_0^{\,3} + \tfrac{1}{36} k^2 R_0^{\,5} (6 \log 2 - 1)\,, \quad . \quad . \quad (10)$$

1) This degree of approximation (the 4th) is one higher than what is obtained by simply using the relation (4).

and correspondingly

$$\vartheta_A(\vartheta_{A'}) = \frac{\pi}{2} - \psi_A = \frac{\pi}{2} - \tfrac{1}{6} kR_0^2 (2 \log 2 + 1)\,^{1)} \quad . \quad . \quad (10')$$

and

$$y_A(y_{A'}) = R_0 - R_0\, \psi_A = R_0 - \tfrac{1}{6} kR_0^3 (2 \log 2 + 1) \quad . \quad (10'')$$

§ 6. From the equations (6) and (8′) it follows that in the neighbourhood of $\vartheta = \pi$, putting $\vartheta = \pi - \varepsilon$,

$$\tau_1 = -\tfrac{1}{3} kR_0^2 \left(\log \frac{\varepsilon^2}{4} - 1\right) \text{ and } \tau'_1 = \tfrac{2}{3} kR_0^2 \frac{1}{\varepsilon}; \quad . \quad . \quad (11)$$

in order therefore that these equations may still be valid in that region, seeing that $\tau'_1$ has to be small, it is necessary, that $\varepsilon$ must remain large with respect to $kR_0^2$. This is still the case in $B$, where $y$ has its maximum, for (comp. 4, 6 and 11)

$$y = 2R_0 - \tfrac{1}{2} R_0 \varepsilon^2 + \tfrac{1}{3} kR_0^3 \left(\log \frac{\varepsilon^2}{4} - 1\right), \quad . \quad . \quad . \quad . \quad (11')$$

so that it follows from $\dfrac{dy}{d\varepsilon} = 0$ that

$$\varepsilon_B = \sqrt{\tfrac{2}{3} kR_0^2},\ y_B = 2R_0 \left[1 + \tfrac{1}{6} kR_0 \log \left(\tfrac{1}{6} kR_0^2\right) - \tfrac{1}{3} kR_0^2\right] \quad . \quad (12)$$

and, also to a third approximation,

$$x_B = R_0 \varepsilon_B = R_0 \sqrt{\tfrac{2}{3} kR_0^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (12')$$

These coordinates are only real, when $k$ is positive.

If $k$ is negative, $\varphi$ has a maximum in $B'$ (fig. 1) corresponding to a value of $\varepsilon$ which is determined by $0 = \dfrac{d\varphi}{d\vartheta} = 1 + \dfrac{d\psi}{d\vartheta}$ (see eq. 8); this gives:

$$\varepsilon_{B'} = \sqrt{-\tfrac{2}{3} kR_0^2}\,^{2)}; \text{ hence } y_{B'} = 2R_0 \left[1 + \tfrac{1}{6} kR_0^2 \log \left(-\tfrac{1}{6} kR_0^2\right)\right] \quad . \quad (13)$$

$$x_{B'} = R_0 \sqrt{-\tfrac{2}{3} kR_0^2} \text{ and } \varphi_B = \pi - 2\sqrt{-\tfrac{2}{3} kR_0^2} \quad . \quad . \quad (13')$$

§ 7. It is possible to go a step further in the analysis of the meridian section of the capillary surface. Close to $\vartheta = \pi$ the curve has a sharp bend (fig. 3): $BCD$ with a double point $E$ for $k > 0$,

[1]) Obviously this expression is also found by putting $\dfrac{dx}{d\vartheta} = 0$.

[2]) $\varepsilon_B'$ may also be found by putting $0 = \dfrac{d^2y}{dx^2} = \dfrac{1}{R_0^2} \dfrac{d^2y}{d\varepsilon^2}$, (see eq. 11′).

$B'C'D'$ with two points of inflexion $B'$ and $D'$ for $k < 0$; the dotted line (two circular arcs) represents the transition between the two cases for $k = 0$.

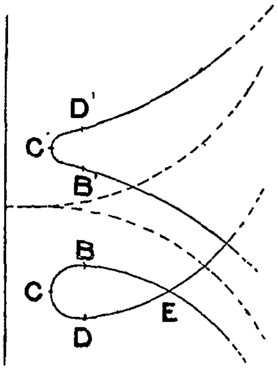

Fig. 3.

In that region the equation to the surface may be written in the form

$$\frac{1}{R_1} + \frac{1}{R_2} = k(h_C + \eta) \quad . \quad . \quad (7')$$

where $h_C = h + y_C$, $y_C$ being the ordinate of $C$ (or $C'$), and $\eta = y - y_C$. In the region under consideration, however, $\eta$ is small compared to $R_0$, so that in second approximation $\eta$ may be neglected with respect to $h_c$ and therefore with the same degree of accuracy to which hitherto the deviation from the circular shape was calculated we may write:

$$\frac{1}{R_1} + \frac{1}{R_2} = kh_C = \text{constant} \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

In third approximation $BCD$ is thus a part of the curve which was called nodoid by PLATEAU, $B'C'D'$ a part of an onduloïd.

The equations of these curves are known [1]; but in our case they may be materially simplified. Putting $kh_C = \frac{2}{r_0}$ [2]) the first integral of (14) in the case of the nodoid ($\sin\varphi = 0$ for $x = x_B$) will be

$$r_0 x \sin\varphi = x^2 - x_B{}^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

If $x_1$ and $x_2$ ($= x_C$) are the maximum- and minimum-values of $x$ corresponding to $\sin\varphi = 1$ and $\sin\varphi = -1$ we have approximately since $x_B$ is very small with respect to $r_0$ (see eq. 12')

$$x_1 = r_0 = R_0 \;^{3)} \qquad x_2 \text{ of } x_C = \frac{x_B}{r_0} = \frac{2}{3} kR_0{}^2 \quad . \quad . \quad (16)$$

Further it follows from (15), as long as $x$ is small with respect to $r_0$ [4])

$$\pm\eta = x_2 \log \frac{x + \sqrt{x^2 - x_2{}^2}}{x_2} - \frac{1}{2R_0} x \sqrt{x^2 - x_2{}^2} \quad . \quad . \quad . \quad (17)$$

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[1]) See for instance WINKELMANN, loc. cit., p 1150.

[2]) In first approximation $r_0 = R_0$; in second approximation $kh_C = k(h + 2R_0) = \frac{2}{R_0} + 2kR_0 = \frac{2}{R_0}(1 + kR_0)$, so that $r_0 = R_0(1 - kR_0{}^2)$.

[3]) Here $x_1$ belongs to the nodoid and has thus not the same meaning as $x_A$ in § 5.

[4]) Since in that case

$$\pm\frac{d\eta}{dx} = \frac{x_B{}^2 - x^2}{\sqrt{r_0{}^2 x^2 - (x_B{}^2 - x^2)^2}} = \frac{x_B{}^2 - x^2}{\sqrt{r_0{}^2 x^2 - x_B{}^4}} = \frac{R_0 x_2 - x^2}{R_0 \sqrt{x^2 - x_2{}^2}}.$$

This gives for $x = x_B$

$$\eta_B = -\tfrac{1}{3} kR_0^3 \log(\tfrac{1}{6} kR_0^2) - \tfrac{1}{3} kR_0^3. \quad . \quad . \quad . \quad . \quad (18)$$

whence

$$y_C = y_B - \eta_B = 2R_0 + \tfrac{2}{3} kR_0^3 \log(\tfrac{1}{6} kR_0^2) - \tfrac{1}{3} kR_0^3 \text{ }^{1)}, \quad . \quad (16')$$

and similarly, if $x_D$, $y_D$ and the coördinates of $D$,

$$x_D = x_B = R_0 \sqrt{\tfrac{2}{3} kR_0^2} \qquad y_D = y_C - \eta_B = 2R_0 + kR_0^3 \log(\tfrac{1}{6} kR_0^2) \quad (19)$$

§ 8. In the case of the ondnloid, where $\sin\varphi$ goes through a minimum in $B'$, we have

$$r_0\, x \sin\varphi = x^2 + x_B{}^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

The maximum- and minimum-values of $x$ ($\sin\varphi = 1$) are now approximately

$$x_1 = R_0 \qquad x_2 = x_C = \frac{x_{B'}{}^2}{R_0} = -\tfrac{2}{3} kR_0^3 \quad . \quad . \quad . \quad (21)$$

Moreover in that case

$$\pm \eta = x_2 \log \frac{x + \sqrt{x^2 - x_2{}^2}}{x_2} + \frac{1}{2R_0} x\sqrt{x^2 - x_2{}^2}. \quad . \quad . \quad (22)$$

whence

$$\eta_B = -\tfrac{1}{3} kR_0^3 \log(-\tfrac{1}{6} kR_0^2) + \tfrac{1}{3} kR_0^3 \quad . \quad . \quad . \quad . \quad (23)$$

$$y_{C'} = y_{B'} - \eta_{B'} = 2R_0 + \tfrac{2}{3} kR_0^3 \log(-\tfrac{1}{6} kR_0^2) - \tfrac{1}{3} kR_0^3 \quad . \quad (21')$$

$$x_{D'} = R_0 \sqrt{-\tfrac{2}{3} kR_0^2} \quad , \quad y_{D'} = 2R_0 + kR_0^3 \log(-\tfrac{1}{6} kR_0^2) - \tfrac{2}{3} kR_0^3 \quad (24)$$

§ 9. It follows from (7′) that the volume of a drop from the top to the horizontal plane passing through $B$ or $B'$ ($\vartheta = \pi - \varepsilon$), in second approximation is given by

$$v = \tfrac{4}{3} \pi R_0^3 (1 - kR_0^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

With the same degree of approximation this is also the volume of a hanging drop up to the level of the neck; indeed the volume

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[1]) If $x$ is large with respect to $x_2$ we have

$$\pm \eta = x_2 \log \frac{2x}{x_2} - \frac{x^2}{2R_0}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17'')$$

so that the equation to the branch $CBE$ (fig. 3) is

$$y = y_c + \eta_+ = 2R_0 - \tfrac{1}{3} kR_0^3 + \tfrac{1}{3} kR_0^3 \log \frac{x^2}{4R_0^2} - \frac{x^2}{2R_0}$$

in agreement with (11′) $\left(\text{since } \varepsilon = \dfrac{x}{R_0}\right)$.

From this the abscissa of the node $E$ ($y_E = y_C$) is found to be

$$x_E^2 = -\tfrac{2}{3} kR_0^4 \log kR_0^2.$$

between the planes passing through point of inflection and neck is found to contribute a negligible amount to the total.

In connection with this it follows from eq. (2′) in fourth approximation:

$$x_B = \sqrt{\tfrac{2}{3}kR_0^3}\,(1-kR_0^2), \text{ en } x'_c = \pm kR_0^3\,(1-\tfrac{5}{3}kR_0^2) \quad . \quad , \quad (19')$$

the upper sign corresponding to the upper index.

§ 10. Starting from the points $D$ and $D'$ (fig. 3) the analysis may be further continued in a manner similar to the one used above. Indeed the meridian curve of the complete capillary surface consists approximately of a series of nearly semi-circular arcs connected each time by parts of an onduloid or nodoid [1]. The centres of these arcs are situated at the heights $R_0$, 3 $R_0$, 5 $R_0$, etc. successively; with each ($n^{th}$) arc we therefore place the origin in the corresponding ($n^{th}$) centre and as in § 2 write:

$$x = \varrho \sin \vartheta \quad , \quad y = (2n-1)\,R_0 - \varrho \cos \vartheta \quad , \quad \varrho = R_0\,(1-\tau) \quad . \quad (26)$$

$\tau$ is determined by:

$$\tau'' \sin \vartheta + \tau' \cos \vartheta + 2\tau \sin \vartheta = kR_0^2\,(2n-1-\cos \vartheta) \sin \vartheta, \qquad (26')$$

whence it follows, introducing the condition that the arcs and intermediate pieces form a continuous curve:

$$\tau = [\tfrac{1}{6} + \tfrac{1}{3}(n-1) - \{\tfrac{1}{6} + \tfrac{1}{3}\log 2 + \tfrac{1}{3}n(n-1) - \Sigma \frac{4(n-1)}{3} \log (n-1) -$$

$$- \tfrac{2}{3}n(n-1) \log (\pm \tfrac{1}{6}kR_0^2)\} \cos \vartheta + \frac{n}{3} \cos \vartheta \log (1+\cos \vartheta) -$$

$$- \frac{2(n-1)}{3} \cos \vartheta \log (1-\cos \vartheta)]\,kR_0^2, \quad . \quad . \quad . \quad . \quad (27)$$

For the connecting curves equations (17), (22) and $x_2 = x_C = \dfrac{x^2_B}{R_0}$ are each time satisfied.

The successive arcs and their connecting curves can only be realised in separate parts, for instance between two horizontal plates or between two vertical coäxial cylinders. Not every surface, however, obtained in that way is a part of the surface whose meridian-section was analysed above by approximation. As an instance, if the surface is formed between two cylinders which are moistened by the liquid, the fraction $\dfrac{(x_C)_{n-1}}{(x_A)_n}$ represents the ratio between the radii of the cylinders and this fraction cannot in the analysis of § 10 assume any arbitrary (small) value, as long as $n$ represents a whole number. Still, putting $2(n-1) = a$ and admitting an arbitrary (positive or

[1]) Cf. WINKELMANN, loc. cit., p. 1141, fig. 404.

negative) value for $\alpha$, the equations (26) and (26′) remain valid and

$$\tau = kR_0^2 \left[a + \tfrac{1}{2}\alpha + b\cos\vartheta + \tfrac{1}{2}(\tfrac{5}{6} - a)\cos\vartheta \log(1+\cos\vartheta) - \right.$$
$$\left. - \tfrac{1}{2}(\tfrac{1}{6} - a)\cos\vartheta \log(1-\cos\vartheta)\right], \quad . \quad . \quad . \quad . \quad . \quad (28)$$

where $a$ and $b$ are integration-constants. $R_0$ is still undetermined, as also $h$, which remains connected with $R_0$ through the relation $kh = \frac{2}{R_0}$; as regards the value of $\alpha$, this may be chosen at will[1]).

With small values of $\vartheta$ the curve shows a minimum for $y$ or a point of inflexion[2]) according as $(\frac{5}{6} - a)k > 0$; for a value of $\vartheta$ which differs but little from $\pi$ the curve has a maximum for $y$, if $(\frac{1}{6} - a)k > 0$ or a point of inflexion, if $(\frac{5}{6} - a)k < 0$. [2])

§ 12. Here again the meridian-section consists of a series of curves which, however, now extends indefinitely upwards as well as downwards. For $k > 0$ the higher curves in the series show maxima and minima for $y$, the lower ones points of inflexion, as represented diagrammatically in fig. 4. For $k > 0$ on the other hand the upper curves have points of inflexion and the lower ones maxima and minima of $y$, which case is obtained by turning fig. 4 upside down. Putting $\frac{1}{6} - a = \beta$ the successive minima and maxima of $x$ satisfy the relations

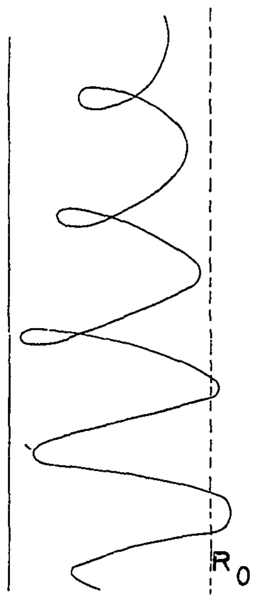

Fig. 4.

$$x_{min.} = \pm\left(\beta + \frac{2n}{3}\right)kR_0^3$$

$$x_{max} = R_0 - \left[\frac{1}{2}\alpha - \beta + \frac{2n+1}{6}\right]kR_0^3 \quad . \quad (29)$$

At the point where $\beta + \frac{2n}{3}$ changes its sign (smallest value of $x_{min}$) is the transition between the two kinds of curves. If accidentally $\beta = \frac{2m}{3}$, $m$ being a whole number, the smallest value of $x_{min}$ becomes zero and the case reduces to that of the meridian-sections discussed in § 10.

1) Supposing for instance the meniscus to be formed between two co-axial cylinders which are moistened by the liquid, the radii of the cylinders being $R$ and $r$, where $r$ has to be small with respect to $R$ $a$ and $R_0$ are determined by the conditions $x_C = r$ and $x_A = R$; $\alpha$ and $b$ may still be chosen at will; one might for instance take $\alpha = 0$, while determining $b$ by putting $y_D = 0$.

2) In general therefore in this case the presence of a minimum or maximum for $y$ is not, as in the section 6 sqq, bound to $k > 0$ or the existence of a point of inflexion to $k < 0$.