Mathematics. — *"On the evaluation of* $\zeta(2n+1)$*"*. By Prof. J. C. KLUYVER.

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By means of a characteristic and very general method MARKOFF [1]) transformed the very slowly convergent series $\sum_1^\infty n^{-2}$ and $\sum_1^\infty n^{-3}$ into other series, that converge more rapidly, and J. G. VAN DER CORPUT [2]) described a special method of transformation applicable to the series $\sum_1^\infty n^{-(2h+1)}$ for larger values of h. I propose to deal anew with the transformation of these series and to add a few results to those previously obtained. In order to appreciate the increase of convergence resulting from the transformation, I will consider D'ALEMBERT's ratio for the transformed expansion, which I will call its index. For the series given by MARKOFF the index is $\frac{1}{27}$, and I will show that a lower index can be attained.

In the first place I base my deductions on the properties of the function

$$\varphi_k(z) = \sum_{n=1}^{n=\infty} \frac{z^n}{n^k},$$

where k denotes a positive integer. In order to uniformise $\varphi_k(z)$, it is convenient to regard the right line $(+1, +\infty)$ as a cut in the complex z-plane, and with this convention we may enunciate the following properties of $\varphi_2(z)$ in substance deduced by ABEL:

$$\left.\begin{aligned}
&\varphi_2(z) + \varphi_2(1-z) = -\log z \log(1-z) + \zeta(2),\\
&\varphi_2(z) + \varphi_2\left(\frac{1}{z}\right) = -\tfrac{1}{2}\log^2 z + \pi i \log z + 2\,\zeta(2)\\
&\varphi_2(z) - \varphi_2\left(\frac{z-1}{z}\right) = \tfrac{1}{2}\log^2 z - \log z \log(1-z) + \zeta(2)\\
&\varphi_2(z) + \varphi_2(-z) = \tfrac{1}{2}\,\varphi_2(z^2)
\end{aligned}\right\} \quad . \quad . \quad . \quad (1)$$

Obviously in these formulae we have to take real values for $\log z$ and for $\log(1-z)$, when z itself is real and between 0 and 1.

[1]) Comptes rendus, t. 109, p. 934.
[2]) These Proc. XIX, p. 489.

As in general we have

$$\frac{d}{dz}\varphi_{k+1}(u) = \varphi_k(u)\frac{d\log u}{dz},$$

it is possible to extend partially the above relations to the function $\varphi_3(z)$, and indeed it follows that

$$\left.\begin{aligned} &\varphi_3(z) - \varphi_3\left(\frac{1}{z}\right) = -\tfrac{1}{6}\log^3 z + \tfrac{1}{2}\pi i \log^2 z + 2\zeta(2)\log z, \\ &\varphi_3\left(\frac{z-1}{z}\right) + \varphi_3(1-z) + \varphi_3(z) = \tfrac{1}{6}\log^3 z - \tfrac{1}{2}\log^2 z \log(1-z) + \\ &\qquad\qquad + \zeta(2)\log z + \zeta(3), \\ &\varphi_3(z) + \varphi_3(-z) = \tfrac{1}{4}\varphi_3(z^2). \end{aligned}\right\} \quad (2)$$

For $k > 3$ a linear relation between $\varphi_k(z)$, $\varphi_k(1-z)$ and $\varphi_k\left(\frac{z-1}{z}\right)$ no longer exists, there only remains, besides the relation

$$\varphi_k(z) + \varphi_k(-z) = \frac{1}{2^{k-1}}\varphi_k(z^2)$$

an equation of the form

$$\varphi_k(z) + (-1)^k \varphi_k\left(\frac{1}{z}\right) = -(2\pi i)^k g_k\left(\frac{\log z}{2\pi i}\right),$$

where $g_k(u)$ denotes the differential coefficient of BERNOULLI's polynomial $f_k(u)$.

Another expansion valid for all positive and integer values of k is the following:

$$\varphi_k(e^{-y}) = \frac{(-y)^{k-1}}{(k-1)!}\left(\tfrac{1}{1} + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \frac{1}{k-1} - \log y\right) + \\ + \sum_{n=0}^{n=\infty}{}' \frac{(-y)^n}{n!}\zeta(k-n) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Here the right line $(0, -\infty)$ must be considered as a barrier in the complex y-plane and $\log y$ is real for positive values of y. The accent in Σ' denotes that the term with the index $n = k-1$ is to be excluded. As for the numerical values that the ζ-function takes for the value zero and for negative integer values of the argument, we have

$$\zeta(0) = -\tfrac{1}{2}\ ,\ \zeta(-2h) = 0\ ,\ \zeta(1-2h) = (-1)^h \frac{B_h}{2h}.$$

Therefore after a certain stage the coefficients in the expansion at the right hand side of (3) are expressible by BERNOULLI's numbers and the radius of convergence of the expansion is evidently 2π,

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as we might expect since equation (3) may be established by integrating repeatedly both sides of the equation

$$\sum_{n=1}^{n=\infty} \frac{e^{-ny}}{n} = -\log y + \tfrac{1}{2} y + \sum_{n=1}^{n=\infty} \frac{(-1)^n B_n}{2n\,!} \cdot \frac{y^{2n}}{2n} \cdot (0 < y < 2\pi).$$

By substituting $z = \frac{1}{2}$ in the formulae (1) and (2) we find

$$\varphi_2 (\tfrac{1}{2}) = -\tfrac{1}{2} \log^2 2 + \tfrac{1}{2} \zeta(2),$$

$$\varphi_3 (\tfrac{1}{2}) = \tfrac{1}{6} \log^3 2 - \tfrac{1}{2} \zeta(2) \log 2 + \tfrac{7}{8} \zeta(3),$$

and then, remembering that

$$\log\log 2 = -\tfrac{1}{2} \log 2 + \sum_{n=1}^{n=\infty} \frac{(-1)^n B_n}{2n\,!} \cdot \frac{(\log 2)^{2n}}{2n},$$

we deduce from (3) by taking $y = \log 2$

$$\zeta(2) = \tfrac{1}{2} \log^2 2 + 2 \left\{ \frac{\log 2}{1} + \sum_{n=1}^{n=\infty} \frac{(-1)^{n-1} B_n}{2n\,!} \; \frac{(\log 2)^{2n+1}}{2n+1} \right\},$$

$$\zeta(3) = \tfrac{2}{3} \log^3 2 + 4 \left\{ \frac{\log^2 2}{2} + \sum_{n=1}^{n=\infty} \frac{(-1)^{n-1} B_n}{2n\,!} \cdot \frac{(\log 2)^{2n+2}}{2n+2} \right\}.$$

The index of these expansions is about $\frac{1}{82}$ and the error involved in neglecting terms beyond the second does not affect the fifth decimal.

It is possible to connect these expansions with the equation

$$\frac{u}{2} \operatorname{Coth} \frac{u}{2} = 1 + \sum_{n=1}^{n=\infty} \frac{(-1)^{n-1} B_n}{2n\,!} u^{2n}$$

and to deduce in this way definite integrals representing $\zeta(2)$ and $\zeta(3)$. Thus we arrive in the first place at the well-known formula

$$\zeta(2) = 2 \int_0^1 \frac{\log (1 + \xi)}{\xi} d\xi,$$

but we get also the less familiar result

$$\zeta(3) = 4 \int_0^1 \frac{\log^2 (1 + \xi)}{\xi} d\xi = 8 \sum_{n=2}^{n=\infty} \frac{(-1)^n}{n^2} \left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} \dots + \frac{1}{n-1} \right).$$

Further, we may prove Euler's result

$$\zeta(3) = \tfrac{1}{2} \int_0^1 \frac{\log^2 (1 - \xi)}{\xi} d\xi = \sum_{n=2}^{n=\infty} \frac{1}{n^2} \left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n-1} \right),$$

and, finally, we may show that

$$\zeta(3) = -\tfrac{8}{5}\int_0^1 \frac{\log(1-\xi)\log(1+\xi)}{\xi}\, d\xi =$$

$$= \tfrac{4}{5}\sum_{n=1}^{n=\infty} \frac{1}{n^2}\left(\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{1}{2n-1}\right).$$

Quite other expansions of the quantities $\zeta(2)$ and $\zeta(3)$ are obtained by substituting in (1) and (2)

$$z = \tfrac{1}{2}(\sqrt{5}-1) = a.$$

As we have

$$1 - a = a^2 \text{ and } \frac{a-1}{a} = -a$$

we readily get from (1)

$$\varphi_2(a) = -\log^2 a + \tfrac{3}{5}\zeta(2)$$
$$\varphi_2(a^2) = -\log^2 a + \tfrac{2}{5}\zeta(2)$$

and from (2)

$$\varphi_3(a^2) = -\tfrac{2}{3}\log^3 a + \tfrac{4}{5}\zeta(2)\log a + \tfrac{4}{5}\zeta(3).$$

Now writing $-2\log a = \alpha$, we have

$$\log \alpha = \sum_{n=1}^{n=\infty} \frac{(-1)^n B_n}{2n!}\cdot\frac{\alpha^{2n}}{2n},$$

and substituting $y = \frac{\alpha}{2}$ and also $y = \alpha$ in (3), in order to obtain expansions of $\varphi_2(a)$, $\varphi_2(a^2)$ and $\varphi_3(a^2)$, we infer that

$$\zeta(2) = \tfrac{5}{2}\left\{\left(\frac{\alpha}{2}\right) + \tfrac{3}{4}\left(\frac{\alpha}{2}\right)^2 + \sum_{n=1}^{n=\infty}\frac{(-1)^{n-1}B_n}{2n!}\cdot\frac{\left(\frac{\alpha}{2}\right)^{2n+1}}{2n+1}\right\},$$

$$\zeta(2) = \tfrac{5}{3}\left\{\alpha + \sum_{n=1}^{n=\infty}\frac{(-1)^{n-1}B_n}{2n!}\cdot\frac{\alpha^{2n+1}}{2n+1}\right\},$$

$$\zeta(3) = \tfrac{5}{2}\left\{\frac{\alpha^2}{2} + \sum_{n=1}^{n=\infty}\frac{(-1)^{n-1}B_n}{2n!}\cdot\frac{\alpha^{2n+2}}{2n+2}\right\}.$$

The index of the first series representing $\zeta(2)$ is about $\frac{1}{170}$, for the other two series it is less than $\frac{1}{42}$.

Again it is possible to convert the series into definite integrals. From the power-series in α we find

$$\zeta(2) = \tfrac{10}{3}\int_0^{\frac{1}{2}} \frac{d\xi}{\xi}\log(\xi + \sqrt{1+\xi^2}),$$

$$\zeta(3) = 10\int_0^{\frac{1}{2}} \frac{d\xi}{\xi}\log^2(\xi + \sqrt{1+\xi^2}).$$

If we wish to expand $\zeta(k)$ for $k > 3$, we must proceed in a different manner. We shall use the general identity

$$\varphi_s(z) + \varphi_s(z\theta) + \varphi_s(z\theta^2) + \ldots + \varphi_s(z\theta^{p-1}) = \frac{1}{p^{s-1}}\varphi_s(z^p) \quad . \quad (4)$$

where p is an integer and $\theta = e^{\frac{2\pi i}{p}}$
Denoting the series

$$\sum_{n=1}^{n=\infty} \frac{\cos\frac{mn\pi}{3}}{n^s}$$

by U_m, we get by substituting in (4)

$$p = 3\,,\; z = e^{\frac{\pi i}{3}}$$

$$p = 2\,,\; z = 1$$

$$2U_1 + U_3 = \frac{1}{3^{s-1}}U_3$$

$$U_0 + U_3 = \frac{1}{2^{s-1}}U_0$$

and hence

$$U_1 = \sum_{n=1}^{n=\infty}\frac{\cos\frac{n\pi}{3}}{n^s} = \tfrac{1}{2}\left(1 - \frac{1}{2^{s-1}}\right)\left(1 - \frac{1}{3^{s-1}}\right)\zeta(s).$$

From equation (3) we deduce by taking $k = 2h+1, y = iv$

$$\sum_{n=1}^{n=\infty}\frac{\cos nv}{n^{2h+1}} = \frac{(-1)^h v^{2h}}{2h\,!}\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2h} - \log v\right) +$$

$$+ \sum_{n=0}^{n=\infty}{}' \frac{(-1)^n v^{2n}}{2n\,!}\zeta(2h+1-2n) \quad . \; . \quad (5)$$

hence writing

$$1 - \tfrac{1}{2}\left(1 - \frac{1}{2^{2h}}\right)\left(1 - \frac{1}{3^{2h}}\right) = A_{2h+1}\,,$$

we may conclude to

$$A_{2h+1}\,\zeta(2h+1) = \frac{(-1)^{h-1}v^{2h}}{2h\,!}\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2h} - \log v\right) +$$

$$+\sum_{n=1}^{n=h-1}\frac{(-1)^{n-1}v^{2n}}{2n!}\zeta(2h+1-2n)+(-1)^{h-1}\sum_{n=1}^{n=\infty}\frac{B_n}{(2n+2h)!}\cdot\frac{v^{2n+2h}}{2n},\ (6)$$

where v now stands for $\frac{\pi}{3}$.

By means of this equation $\zeta(2h+1)$ is expressed in terms of $\zeta(3), \zeta(5), \ldots, \zeta(2h-1)$ and taking successively $h=1,2,3\ldots$, we get $\zeta(3, \zeta(5), \zeta(7), \ldots$ expressed by a linear combination of expansions the index of which is $\frac{1}{36}$ [1]).

A slight transformation of (6) is possible. By using

$$\log v=\sum_{n=1}^{n=\infty}\frac{Bn}{2n!}\cdot\frac{v^{2n}}{2n},$$

and by effecting some other reductions, it may be shown that (6) can take the form

$$A_{2h+1}\,\zeta(2h+1)=\left(1-\frac{A_{2h-1}}{h(2h-1)}\right)\zeta(2h-1)\cdot\frac{v^2}{2}+$$

$$+\sum_{n=2}^{n=h-1}\frac{(h-n)(2h+2n-1)}{h(2h-1)}\cdot\frac{(-1)^{n-1}v^{2n}}{2n!}\zeta(2h+1-2n)+$$

$$+\frac{(-1)^{h-1}v^{2h}}{2h(2h-1)}\left\{\frac{4h-1}{2h!}-\sum_{n-1}^{n=\infty}\frac{Bn}{(2n+2h)!}\cdot(2n+4h-1)v^{2n}\right\}.$$

$$\left(v=\frac{\pi}{3}\right).$$

If we put $h=1$ and $h=2$, it will be seen that

$$\zeta(3)=\tfrac{3}{4}\left\{\frac{3}{2!}v^2-\sum_{n=1}^{n=\infty}\frac{B_n}{(2n+2)!}(2n+3)v^{2n+2}\right\},$$

$$\zeta(5)=\tfrac{9}{58}\left\{\frac{137}{4!}v^4-\sum_{n=1}^{n=\infty}\frac{B_n}{(2n+4)!}[4(2n+3)^2(2n+4)-(2n+7)]\,v^{2n+4}\right\}.$$

I will now proceed to show that for each of the quantities $\zeta(2h+1)$ there exists a linear combination of expansions with an index less than $\frac{1}{36}$. For this purpose I use again the identity (4) and writing

$$w=\frac{\pi}{15}\quad,\quad U_m=\sum_{n=1}^{n=\infty}\frac{\cos mnw}{n^s}$$

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[1]) Similar results were deduced by Mr. VAN DER CORPUT in the paper quoted. However, in the fundamental expansion of the quantity $I(n,a)$ on p. 1464 by a slight inadvertence the factor 2^{2k} has been omitted in the general term, hence in all the subsequent expansions the general term should be multiplied by 2^{2k} and the index of the series on p. 1470 is $\frac{1}{36}$ and not $\frac{1}{144}$.

I make in (4) the substitutions

$$p=2\ ,\quad z=e^{iw}\ \text{ and }\ z=e^{3iw},$$
$$p=3\ ,\quad z=e^{4iw},$$
$$p=5\ ,\quad z=1.$$

These substitutions give the four equations

$$U_1+U_{14}=\frac{1}{2^{s-1}}U_2,$$

$$U_3+U_{12}=\frac{1}{2^{s-1}}U_6,$$

$$U_4+U_6+U_{14}=\frac{1}{3^{s-1}}U_{12},$$

$$U_3+2U_6+2U_{12}=\frac{1}{5^{s-1}}U_0.$$

and eliminating U_6, U_{12}, U_{14} we get

$$\left(1-\frac{1}{5^{s-1}}\right)\left(1-\frac{1}{6^{s-1}}\right)U_0+2\left(1+\frac{1}{2^{s-1}}\right)U_1-\frac{1}{2^{s-2}}\left(1+\frac{1}{2^{s-1}}\right)U_2-$$
$$-2\left(1+\frac{1}{3^{s-1}}\right)U_3-2\left(1+\frac{1}{2^{s-1}}\right)U_4=0.$$

Now, taking $s=2h+1$, we may expand U_1, U_2, U_3, U_4 by applying equation (5) and we will get $U_0=\zeta(2h+1)$ expressed in terms of the quantities $\zeta(3), \zeta(5), \ldots, \zeta(2h-1)$ and of four power-series in w, the indices of which are respectively $\frac{1}{900}, \frac{1}{225}, \frac{1}{100}, \frac{4}{225}$. Since the formulae become somewhat complicated, I will consider only the simplest case $h=1$. Then we have

$$336\,\zeta(3)=-900\,U_1+225\,U_2+800\,U_3+900\,U_4$$

and hence

$$689\,\zeta(3)=450w^2(36-24\log w-33\log 2-8\log 3)-900\sum_{n=1}^{n=\infty}\frac{B_n}{(2n+2)!}\cdot\frac{w^{2n+2}}{2n}+$$
$$+225\sum_{n=1}^{n=\infty}\frac{B_n}{(2n+2)!}\cdot\frac{(2w)^{2n+2}}{2n}+800\sum_{n=1}^{n=\infty}\frac{B_n}{(2n+2)!}\cdot\frac{(3w)^{2n+2}}{2n}+$$
$$+900\sum_{n=1}^{n=\infty}\frac{B_n}{(2n+2)!}\cdot\frac{(4w)^{2n+2}}{2n}.$$

I will end with remarking that the calculation of the sum of the series

$$\eta(2h)=\frac{1}{1^{2h}}-\frac{1}{3^{2h}}+\frac{1}{5^{2h}}-\frac{1}{7^{2h}}+\ldots$$

may be conducted along similar lines. Equation (3) gives, if we take $k=2h, y=iu,$

$$\sum_{n=1}^{n=\infty} \frac{\sin nw}{n^{2h}} = \frac{(-1)^{h-1} u^{2h-1}}{(2h-1)!}\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{2h-1}-\log u\right)+$$

$$+\sum_{n=0}^{n=\infty}{}' \frac{(-1)^n u^{2n+1}}{(2n+1)!}\zeta(2h-1-2n) \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and it is possible to express $\eta(2h)$ by means of series of the form $\sum_{n=1}^{n=\infty} \frac{\sin nw}{n^{2h}}$.

Indeed, putting

$$V_m = \sum_{n=1}^{n=\infty} \frac{\sin\frac{mnv}{2}}{n^s} \quad \text{and} \quad v = \frac{\pi}{3}$$

we have by substituting in (4)

$$p = 2 \quad , \quad z = e^{\frac{iv}{2}},$$

$$p = 3 \quad , \quad z = e^{\frac{iv}{2}},$$

$$V_1 - V_5 = \frac{1}{2^{s-1}} V_2,$$

$$V_1 + V_5 - V_3 = \frac{1}{3^{s-1}} V_3$$

whence

$$V_3\left(1+\frac{1}{3^{s-1}}\right) = 2\,V_1 - \frac{1}{2^{s-1}} V_2.$$

Now, taking $s = 2h$, we have $V_3 = \eta(2h)$, and expressing V_1 and V_2 by means of equation (7), we get an expansion for $\eta(2h)$. In the simplest case $h = 1$, I find in this way

$$\eta(2) = \tfrac{3}{8}\, v\left(1+\log\frac{4}{v}\right) + \tfrac{3}{8}\sum_{n=1}^{n=\infty} \frac{B_n}{(2n+1)!}\cdot\frac{v^{2n+1}}{2n}\left(\frac{1}{2^{2n-1}}-1\right).$$

$$\left(v = \frac{\pi}{3}\right)$$

Again the index of the series is $\frac{1}{36}$ and the value of $\eta(2)$ is found to five decimals by taking into account only the first two terms of the expansion.