Physics. — "*On the equivalent of parallel translation in non-Euclidean space and on* RIEMANN's *measure of curvature.*" By Dr. A. D. FOKKER. (Communicated by Prof. H. A. LORENTZ.)

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1. *Introduction.* In the following pages I shall try to give a mental picture of some ideas recently developed by prof. J. A. SCHOUTEN before the Mathematical Society at Amsterdam which will help to illustrate the meaning of a "system of axes moving geodesically", and the "geodesic differential", together with a few applications. [1]) The great point will be to realise in a new way wat kind of displacement in non-Euclidean space must be considered to correspond to a parallel translation, this being an operation indispensable in vector-analysis to compare vectors in different points.

One of the characteristic properties of pure translations is this, that all points of a rigid body are thereby transferred over an equally long distance. This property might be used to define a parallel translation, provided the rigid consists of a number of points exceeding a certain minimum. If, for example, in three-dimensional space, we give a prescribed displacement to one of the points of a rigid system consisting of two or three points, it is not enough to demand an equal displacement for the other point or points to define a translated position without ambiguity. But in a Euclidean space of n dimensions other motions than pure translations are excluded, if for a rigid body of no less than $(2n—2)$ points we want all points to run through equal distances.

This will be our starting-point. We know, however, that in general no body of finite dimensions can move in curved space without changing the mutual distances of its points. In order to retain the idea of a rigid body we shall have to confine ourselves to bodies with dimensions of the order of an infinitesimal ε.

Another and more serious difficulty arises from the fact, that we cannot get all points to shift over exactly the same infinitesimal distance Δ. We cannot but leave a margin of the order of $\Delta\varepsilon^2$ for the separate distances. Here the question arises whether in a certain

[1]) Cf. a treatise offered by Prof. SCHOUTEN to be published in the transactions of the Kon. Akademie: "*Die directe Analysis zur neueren Relativitätstheorie*".

direction only one displacement can be effected in which this approximation to the exact equality is realised? This, however, cannot be expected, since in the special case of Euclidean space not only pure translations but screw-displacements too are allowed by leaving this margin. Therefore a second property of pure translations is required, fit to exclude these screw-displacements.

This property is found in the fact that the shifts are not only equal, but also parallel to one another. This amounts to a certain reciprocity between translations in different directions. Consider two translations, by which a point P is transferred to neighbouring points Q and R respectively. The first translation will carry point R to the same place where the second translation will carry point Q. This property indeed excludes screw-displacements.

In the following pages we shall first give a summary of the results arrived at in this paper, and afterwards (§ 6) give the analytical formulae. For examples we will mainly take those of three-dimensional space. The results, however, will hold good, independent of whatever number (n) of dimensions we choose to ascribe to our space.

2. *Geodesic displacement.* Let us define an infinitesimal *rigid* as *an aggregate of particles, which keep their mutual distances unchanged* during their motions. One of these points we may choose as a central, and imagine the other points defined by the ends of infinitesimal vectors from this central point, these vectors having constant lengths (of the order ε) and including constant angles. The number of points must be no less than $(2n-2)$, hence the number of vectors $(2n-3)$, no n of them being situated together in a space of $(n-1)$ dimensions.

We imagine this rigid to execute motions so as to shift the central particle from a starting point P to neighbouring points over distances of the order Δ.

It appears possible (§ 7) to indicate a certain variety of motions in which, firstly *the shifts of all the other points of the rigid,* up to a margin of the order $\Delta\varepsilon^2$, *equal the shift of the central point,* and, secondly, there exists a certain *reciprocity* which becomes apparent when we observe *two arbitrarily chosen motions belonging to the variety,* which shift the central particle, let us say, from P to Q and from P to R, and when we notice the displacements of the particles having their starting points in R and Q respectively. The *particle from R in the motion (PQ) will reach the same point attained by the particle from Q in the other motion (PR).*

The two conditions specified determine without ambiguity a variety of motions which we may call *"geodesic displacements"* of the infinitesimal rigid. They are the substitutes for parallel displacements[1]) in Euclidean extensions. We may assign the name "compass-rigid" to a small rigid body that cannot move but in the geodesic manner defined. It must be understood that a compass-rigid which, after a displacement, returns to its starting-point by the same way, will on arrival be in its starting-position too. If, however, it returns by a circuit, it generally will not be in its starting-position again on arrival.

3. *Geodesic differential.* If we want to compare two vectors in neighbouring points P and Q, we can proceed as follows. We put a compass-rigid with its centre in P and by marking one of its points we delineate the vector in it. Now displacing the compass-rigid to Q it is reasonable to say that the marked point defines the vector displaced geodesically from P to Q. By comparing this vector with the one present in Q we immediately see the meaning of the *geodesic differential* of a vector. If this is known, it is clear what CHRISTOFFEL's covariant differentiation means.

In the same way we can displace our vector-units from P to Q. In general these will differ from the vector-units in Q. A set of geodesically displaced vector-units is what Prof. SCHOUTEN defined as a system of axes moving geodesically.

4. *Geodesic line.* We can easily imagine what we have to do in order to *prolong a given line-element geodesically.* We put the centre of the compass-rigid in the starting-point and mark the end of the line-element by an arrow in the compass-rigid. After the centre has been displaced along the line-element, the arrow will point in another definite direction. This is the geodesic prolongation of the element. Continuing to move the compass-rigid in the direction of the arrow, the centre will gradually describe a geodesic line.

In this case, during displacements along a geodesic line, vectors moving geodesically will continue to include constant angles with the geodesic (cf. LEVI-CIVITA's article), these angles being fixed angles in the compass-rigid.

5. RIEMANN's *measure of curvature.* Let us now suppose that we

[1]) Taking another starting-point, T. LEVI-CIVITA arrives at a definition of parallelism which comes to the same thing: "*Nozione di parallelismo in una varieta qualunque, e conseguente specificazione geometrica della curvatura Riemanniana*". Rend. Circ. Mat. Palermo, XLII p. 1, 1917. His geometrical explanation of the measure of curvature, however, is totally different from the one we shall give in section 5.

make a compass-rigid describe a small circuit, e.g. along a vanishing (quasi-)parallelogram. We already pointed out that in general it will *not* return to its starting-*position*. The difference between the two positions is such as might have been produced by an infinitesimal rotation around the starting-point. The amount of this rotation is proportional to the area of the circuit described, the orientation of the "axis" of rotation (which in higher extensions is of $(n-2)$ dimensions) being determined by the orientation of the plane of the circuit. The rotation is intimately connected with the curvature of space. When this *rotation of curvature*, as it may be called, vanishes in all points for every arbitrary circuit, then the space is Euclidean. [1])

The components of the operator by which from the data of the area included by the circuit the rotation of curvature for the compass-rigid can be derived, are RIEMANN's four-index-symbols, of the second kind.

Further — to confine ourselves to three-dimensional space — if we take the length of the axis of rotation equal to the amount of the angle of rotation, and construct a parallelepiped with this axis and the parallelogram of the circuit, we can consider the ratio of its volume to the square of the parellelogram as a measure for the curvature of space. Indeed, in the limit, for a vanishing circuit, this ratio is just the number indicated by RIEMANN as *the measure of curvature of the space with respect to the plane of the circuit considered.*

6. Now we shall proceed to give the required formulae. We take the length of a line-element as defined by

$$ds^2 = \Sigma(ab)\, g_{ab} dx^a\, dx^b\ ,$$

$dx^a\ dx^b$ representing increments of the coordinates along the line-element **dx**, $g_{ab}\ (= g_{ba})$ being regular functions of the coordinates of the starting-point, each index in the sum going through all the values from 1 to n, where n is the number of dimensions of space. The algebraical complements of g_{ab} will be denoted by g^{ab}, so that

[1]) The fundamental idea of a recent article by H. WEYL (*Gravitation und Elektrizität*, Berl. Sitz. Ber. May, 1918) may be considered the hypothesis that a small rigid (= "Vektorraum") after turning about an infinitesimal circuit of "translations" (of a somewhat more general kind) not only will have got in a changed position, but in general will have *changed its dimensions* as well. In four-dimensional space-time the linear dilatation of the (4-dimensional) rigid would be half the scalar product of the alternating electromagnetic tensor and the area included by the circuit. (Note added during the revisal of the proofs).

$$\Sigma(b) g_{ab}\, g^{nb} = \begin{cases} 1 & \text{for } n = a, \\ 0 & \text{for } n \neq a \end{cases}$$

For the sake of brevity we shall write g for the determinant formed of the g^{ab}. Further we shall avail ourselves of CHRISTOFFEL's well-known symbols:

$$\begin{bmatrix} l\,m \\ a \end{bmatrix} = \frac{1}{2}\left[\frac{\partial g_{am}}{\partial x^l} + \frac{\partial g_{la}}{\partial x^m} - \frac{\partial g_{lm}}{\partial x^a}\right], \qquad \begin{Bmatrix} l\,m \\ b \end{Bmatrix} = \Sigma(a)\, g^{ab} \begin{bmatrix} l\,m \\ a \end{bmatrix}.$$

The definition of the line-element entails the definition of the length of a vector $\mathbf{v}$, with components v^a:

$$v^2 = \Sigma(ab)\, g_{ab}\, v^a\, v^b,$$

and the definition of the angle between two vectors $\mathbf{v}$ and $\mathbf{w}$:

$$vw \cos(vw) = \Sigma(ab)\, g_{ab}\, v^a\, w^b.$$

7. Let the points of a small rigid be given by their coordinates relative to the centre: u^a, v^a, $w^a \ldots$ $(a = 1, 2, 3, n)$, these being the components of vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \ldots$ which are of the order of a vanishing quantity ε. If the centre shifts from P to a neighbouring point Q, determined by the infinitesimal increments in the coordinates dx^a (of the order Δ), then we require the new coordinates of the points of the rigid relative to Q in order to satisfy the definition and first condition of section 2: the points are to be points of a rigid, and must each cover an equal distance.

Denoting the new relative coordinates by $u^a + du^a$, $v^a + dv^a \ldots$ etc. it is easy to formulate the latter half of the condition. For the increments of the coordinates of the point designed by $\mathbf{u}$ will be $dx^a + du^a$, and the starting point of the line-element through which it runs lies beside P, at a distance defined by $\mathbf{u}$. So, if this line-element is to be equal to that from P to Q, i.e.

$$ds^2 = \Sigma(ab)\, g_{ab}\, dx^a\, dx^b,$$

we necessarily must have

$$0 = \Sigma\,(ab\,m)\,\frac{\partial g_{ab}}{\partial x^m}\, u^m\, dx^a\, dx^b + \Sigma(ab)\, 2 g_{ab}\, dx^a\, du^b \quad . \quad . \quad (1)$$

If the aggregate of points is to form a rigid, both the lengths of the relative vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \ldots.$ and the included angles must be constant, and expressions such as

$$u^2 = \Sigma(ab)\, g_{am}\, u^a\, u^m \quad , \quad uv \cos(uv) = \Sigma\,(am)\, g_{am}\, u^a\, v^m,$$

must have the same value in P and Q. This implies

$$0 = \Sigma\,(a\,m\,l)\,\frac{\partial g_{am}}{\partial x^l}\, dx^l\, u^a\, u^m + \Sigma(am)\, 2 g_{am}\, u^a\, du^m, \quad . \quad . \quad . \quad (2)$$

and

$$0 = \Sigma\,(a\,m\,l)\,\frac{\partial g_{am}}{\partial x^l}\,dx^l u^a v^m + \Sigma(am)\,g_{am}\,\{u^a\,dv^m + v^m\,du^a\} \quad . \quad (3)$$

These are the equations which must be applicable to du^a, dv^a... etc. in the translations mentioned. It is not difficult to find espressions satisfying the equations. We can add to (1) the identity

$$0 \equiv \Sigma\,(a\,l\,m)\left(\frac{\partial g_{am}}{\partial x^l} - \frac{\partial g_{lm}}{\partial x^a}\right) dx^a\,dx^l\,u^m\,,$$

and a similar identity to (2):

$$0 \equiv \Sigma\,(a\,l\,m)\left(\frac{\partial g_{al}}{\partial x^m} - \frac{\partial g_{lm}}{\partial x^a}\right) u^a\,dx^l\,u^m\,.$$

Replace at the same time the index b in the first term, right-hand side of (1) by l, and m in the second term, right-hand side of (2), by b, and we get

$$0 = \Sigma\,(a)\left[\Sigma\,(lm)\left(\frac{\partial g_{am}}{\partial x^l} + \frac{\partial g_{al}}{\partial x^m} - \frac{\partial g_{lm}}{\partial x^a}\right) dx^l u^m + \Sigma(b)\,2g_{ab}\,du^b\right] dx^a,$$

and

$$0 = \Sigma\,(a)\left[\Sigma(lm)\left(\frac{\partial g_{am}}{\partial x^l} + \frac{\partial g_{al}}{\partial x^m} - \frac{\partial g_{lm}}{\partial x^a}\right) dx^l u^m + \Sigma(b)\,2g_{ab}\,du^b\right] u^a.$$

Dividing by 2 we can reduce the equations to the form

$$0 = \Sigma\,(ab)\,g_{ab}\,dx^a\left[du^b + \Sigma(lm)\begin{Bmatrix} lm \\ b \end{Bmatrix} dx^l u^m\right], \quad . \quad . \quad . \quad (1')$$

$$0 = \Sigma\,(ab)\,g_{ab}\,u^a\left[du^b + \Sigma(lm)\begin{Bmatrix} lm \\ b \end{Bmatrix} dx^l u^m\right], \quad . \quad (2')$$

Similarly, we can put for the third equation

$$0 = \Sigma(ab)\,g_{ab}\left[u^a\,[dv^b + \Sigma\begin{Bmatrix} lm \\ b \end{Bmatrix} dx^l v^m] + v^a\,[du^b + \Sigma\begin{Bmatrix} lm \\ b \end{Bmatrix} dx^l u^m]\right] \quad (3')$$

So we can satisfy the imposed condition by taking

$$du^b = -\,\Sigma(lm)\begin{Bmatrix} lm \\ b \end{Bmatrix} dx^l u^m \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and similar expressions for dv^b, dw^b....

The equation (4) is covariant. It will retain its form wathever be the choice of coordinates.

It defines the position of the points of the small rigid when, by a first approximation, they have all covered the same infinitesimal distance.

It is seen, from (1), that in developing g_{ab} into a series we have neglected terms with products $u^m\,u^n$. The squares of the distances covered therefore can differ from PQ by an amount of the order $\varepsilon^2\,\Delta^2$, so that the distances may only be taken as equal up to an amount of the order $\varepsilon^2\,\Delta$, which we shall neglect.

8. The "corrections" given by eq. (4) are of the order $\Delta\varepsilon$. In order to see if the solution defined by them is the only one, we may ask if we can satisfy the equations by "corrections" differing from du^a, such as $du^a + \delta u^a$, where δu^a is of the same order as du_a.
If these are to satisfy eq. (1), (2), (3), we must evidently have.

$$0 = \Sigma(ab)\, g_{ab}\, dx^a\, \delta u^b\ ,$$
$$0 = \Sigma(ab)\, g_{ab}\, u^a\, \delta u^b\ ,$$
$$0 = \Sigma(ab)\, g_{ab}\, u^a\, \delta v^b + g_{ab}\, v^a\, \delta u^b\ , \text{ etc.}$$

If the rigid, besides the centre, consists of p points, we shall have $2p + \frac{1}{2}p(p-1)$ equations for pn variables δu^a, $\delta v^a \ldots$ etc. For $p = 2n-3$ we have as many homogeneous linear equations as there are variables. If no set of n vectors $\mathbf{u,v,w} \ldots$ are situated together in an $(n-1)$-dimensional space, then these equations only permit a solution of the form (e.g. for $n = 3$):

$$\delta u^a = \frac{\omega}{\sqrt{g}} \Sigma\,(ij) \begin{vmatrix} g_{bi} & g_{ci} \\ g_{bj} & g_{cj} \end{vmatrix} dx^i\, u^j = \frac{\omega}{\sqrt{g}} \begin{vmatrix} dx_b & dx_c \\ u_b & u_c \end{vmatrix} \quad , \quad . \quad . \qquad (5)$$

where ω is an arbitrary number, and by b, c, a, we mean a set of three indices which form an even permutation of 1, 2, 3. We denote by dx_b and u_b with lowered index the covariant combinations:

$$dx_b = \Sigma(i)\, g_{bi}\, dx^i \quad , \quad u_b = \Sigma(j)\, g_{bj}\, u_j .$$

It can easily be ascertained that the expression given for δu^a, together with similar expressions for δv^a, $\delta w^a \ldots$ satisfy the equations. They must define the displacements in the case of an infinitesimal rotation about $\mathbf{dx}$ as an axis [1]). For all vectors $\mathbf{u,v,w} \ldots$ keep their lengths unchanged and both the angles included with $\mathbf{dx}$ and the mutual angles remain unaltered.

Since the condition imposed thus far appears not to be sufficient to define a displacement without ambiguity, we must recur to the condition of reciprocity of section 2.

Shifting the centre of the compass-rigid from P to Q the particle designed by $\mathbf{u}$ might come from a position R into the position defined by the coordinates

$$x^a_P + dx^a + u^a + du^a + \delta u^a\ ,$$

or,

$$x^a_P + dx^a + u^a - \Sigma\,(lm) \left\{ \begin{matrix} lm \\ a \end{matrix} \right\} dx^l u^m + \frac{\omega}{\sqrt{g}} \begin{vmatrix} dx_b & dx_c \\ u_b & u_c \end{vmatrix} .$$

If we now shift the centre from P to R, and we then wish to find what will be the new position for the particle from Q according

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[1]) Through an angle $\omega|dx|$.

to the same displacement-law, we only have to interchange the vectors **dx** and **u**

Now we see, the determinant changing its sign, that this position will never be the same as that reached by the former particle from R, unless $\omega = 0$.

So the application of the condition of reciprocity excludes screw-movements[1]).

9. Now in the following way we can see that the condition of the body's rigidity and the equality of the covered distances together with the condition of reciprocity are sufficient to define the variety of geodesic displacements without ambiguity.

From eq. (1) and (2) we learn that the required "corrections" du^a must be proportional both to the components of the displacement **dx** and of the vector **u**. Therefore let us put

$$du^a = \Sigma\, h^a_{st}\, dx^s u^t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4')$$

Now, according to the condition of reciprocity we must apparently have

$$h^a_{st} = h^a_{ts} .$$

Substitute (4') in (3), and we get

$$0 = \Sigma\,(alm)\, \frac{\partial g_{am}}{\partial x^l}\, dx^l u^a v^m + \Sigma\,(a\,m\,s\,t)\, g_{am}\, \{h^m_{st}\, u^a\, dx^s v^t + h^a_{st}\, v^m\, dx^s u_t\}.$$

Taking other indices and putting

$$h_{a,lm} = \Sigma(b)\, g_{ab}\, h^b_{lm}\, , \qquad (h_{a,lm} = h_{a,ml}),$$

we get

$$0 = \Sigma\,(a\,l\,m)\, dx^l u^a v^m \left\{ \frac{\partial g_{am}}{\partial x^l} + h_{a,lm} + h_{m,la} \right\}.$$

In this equation we may regard the forms in brackets as unknown variables. Because of the symmetry in the indices a and m there are $\frac{1}{2}\, n^2\, (n-1)$ of them. As the equation is to hold for an arbitrary combination

[1]) Dr. Droste remarked to me that a screw-motion might be excluded in a different manner. Let PQ be part of a geodesic. In P and in Q take two infinitesimal planes perpendicular to the geodesic. Draw the geodesics perpendicular to the first plane and intersecting it in a line-element PR. These together form a "geodesic strip", which will intersect the second plane in an element QR'. PR and QR' can be called "parallel" and in the same way each pair of elements in the same geodesic strip including equal angles with the geodesic PQ.

In our chain of thought, however, geodesic lines are defined by making use of the idea of geodesic displacements (see section 10), and so we cannot avail ourselves of the above suggestion.

of three vectors **dx, u, v,** we conclude that the variables must vanish. Thus

$$0=\frac{\partial g_{am}}{\partial x^l}+h_{a,lm}+h_{m,la}.$$

Similarly

$$0=\frac{\partial g_{la}}{\partial x^m}+h_{l,ma}+h_{a,ml}$$

$$0=\frac{\partial g_{lm}}{\partial x^a}+h_{m\,al}+h_{l,am}.$$

By adding the first two, and subtracting the last, and considering that $h_{l,am}=h_{l,ma}$ etc. we find

$$h_{a,lm}=-\begin{bmatrix} lm \\ a \end{bmatrix}.$$

Now we know

$$h^b_{lm}=\Sigma\, g^{ab}\, h_{a,lm}=-\Sigma\, g^{ab}\begin{bmatrix} lm \\ a \end{bmatrix}=-\begin{Bmatrix} lm \\ b \end{Bmatrix},$$

and so from (4') we see that our values for du^b:

$$du^b=-\Sigma(lm)\begin{Bmatrix} lm \\ b \end{Bmatrix} dx^l u^m \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

constitute the only solution consistent with all conditions.

10. To explain the applications of sections 3 and 4 we proceed as follows. Suppose a vector **V**, in point P, be marked in the compass-rigid. After a geodesical displacement to Q the marked vector will have got the components

$$V^a-\Sigma\begin{Bmatrix} lm \\ a \end{Bmatrix} dx^l\, V^m.$$

Now if we have in Q a vector with components V^a+dV^a, where dV^a now represents some increment of the component V^a, then obviously the components of the *geodesic differential* are

$$dV^a+\Sigma(lm)\begin{Bmatrix} lm \\ a \end{Bmatrix} dx^l\, V^m.$$

This geodesic differential will be a vector itself, being the difference of two vectors, while dV^a are no vector-components.

If the line-element PQ itself is drawn in the compass-rigid as a vector with components dx^a, and displaced geodesically, then in Q the arrow will have got components

$$dx^a-\Sigma(lm)\begin{Bmatrix} lm \\ a \end{Bmatrix} dx^l\, dx^m.$$

This arrow we have called the *geodesic prolongation* of the element. It is easily seen that this entails for the geodesic the equation

$$0 = d^2 x^a + \Sigma(lm) \begin{Bmatrix} lm \\ a \end{Bmatrix} dx^l dx^m .$$

This (covariant) equation coïncides with what we get from the familiar definition of a geodesic as the shortest line between two points.

11. We shall now displace geodesically a particle P', which, with relation to P, is defined by a vector $\mathbf{u}$, to a point S' near T, by shifting the compass rigid in two steps from P to T, along PQ and QT. Then, a second time, we displace the particle to S'' near T, taking the steps along PK and KT, the quadrilateral $PQTK$ being a (quasi-) parallelogram with sides $\mathbf{dx}$ (PQ and KT) and $\boldsymbol{\delta x}$ (PK and QT).

After the displacement along PQ the coordinates of the particle considered relative to Q have become

$$u^a - \Sigma(lm) \begin{Bmatrix} lm \\ a \end{Bmatrix} dx^l u^m .$$

At the second step we must be careful to take the values of CRISTOFFEL's symbols at point Q, so that after the displacements along PQ and QT the coordinates relative to T are

$$u^a - \Sigma \begin{Bmatrix} lm \\ a \end{Bmatrix} dx^l u^m - \Sigma \begin{Bmatrix} lm \\ a \end{Bmatrix} \delta x^l \left[u^m - \Sigma \begin{Bmatrix} pq \\ m \end{Bmatrix} dx^p u^q \right] - \Sigma \frac{\partial}{\partial x^p} \begin{Bmatrix} lm \\ a \end{Bmatrix} dx^p \delta x^l u^m .$$

If the displacements had been performed along PK and KT, the coordinates relative to T would have been

$$u^a - \Sigma \begin{Bmatrix} lm \\ a \end{Bmatrix} \delta x^l u^m - \Sigma \begin{Bmatrix} lm \\ a \end{Bmatrix} dx^l \left[u^m - \Sigma \begin{Bmatrix} pq \\ m \end{Bmatrix} \delta x^p u^q \right] - \Sigma \frac{\partial}{\partial x^p} \begin{Bmatrix} lm \\ a \end{Bmatrix} \delta x^p dx^l u^m .$$

Taking the difference we find

$$\zeta^a = \Sigma(lmp) \left[\frac{\partial}{\partial x^p} \begin{Bmatrix} lm \\ a \end{Bmatrix} - \Sigma(n) \begin{Bmatrix} l\,n \\ a \end{Bmatrix} \begin{Bmatrix} pm \\ n \end{Bmatrix} \right] (dx^l \delta x^p - dx^p \delta x^l)\, u^m ;$$

or

$$\zeta^a = \tfrac{1}{2}\, \Sigma(lmp) \left[\frac{\partial}{\partial x^p} \begin{Bmatrix} lm \\ a \end{Bmatrix} - \frac{\partial}{\partial x^l} \begin{Bmatrix} pm \\ a \end{Bmatrix} + \Sigma \begin{Bmatrix} pn \\ a \end{Bmatrix} \begin{Bmatrix} lm \\ n \end{Bmatrix} - \Sigma \begin{Bmatrix} ln \\ a \end{Bmatrix} \begin{Bmatrix} pm \\ n \end{Bmatrix} \right] \times$$

$$\times\, (dx^l \delta x^p - dx^p \delta x^l)\, u^m \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (6)$$

The first factor is seen to be a RIEMANN's four-index-symbol, of the second kind. Availing ourselves of his notation we can put

$$\zeta^a = \tfrac{1}{2}\, \Sigma(lmp) \begin{Bmatrix} ma, lp \end{Bmatrix} (dx^l \delta x^p - dx^p \delta x^l) u^m \quad . \; . \; . \; . \quad (6)$$

The ζ^a $(a = 1, 2, \ldots n)$ are the components of the displacement-vectors which would become manifest after a geodesical displacement of the compass-rigid about the circuit $TKPQT$. The displacement cannot be anything else but a rotation, the lengths and angles remaining the same. We see how RIEMANN's symbols determine the rotation in terms of the components of the area of the circuit. This rotation is characteristic of the curvature of space.

12. It now remains to prove the statement of section 5 as to the interpretation of RIEMANN's measure of curvature.

The measure of curvature with respect to the plane of **dx** and **δx** is defined to be[1])

$$K = \frac{\Sigma(a\,lp\,mq)\,g_{aq}\,\{ma,\,lp\}\,(dx^l\,\delta x^p - dx^p\delta x^l)(dx^m\,\delta x^q - dx^q\delta x^m)}{\Sigma(lp\,mq)(g_{lm}g_{pq} - g_{lq}\,g_{pm})(dx^l\,\delta x^p - dx^p\delta x^l)(dx^m\,\delta x^q - dx^q\delta x^m)}.$$

The denominator is four times the square of the area of the parallelogram formed by **dx** and **δx**. For by changing indices without changing the sum we get four times

$$\Sigma\,(lp\,mq)\begin{vmatrix} g_{lm} & g_{lq} \\ g_{pm} & g_{pq}\end{vmatrix} dx^l\,dx^m\,\delta x^p\delta x^q.$$

Writing d for the length of **dx**, and δ for the length of **δx**, we find for the denominator

$$4\begin{vmatrix} d^2 & d\delta\cos(d\delta) \\ d\delta\cos(d\delta) & \delta^2\end{vmatrix},$$

this being four times the square of the area of the parallelogram.

We shall discuss the numerator for the case of three-dimensional space and show that it represents four times the volume of the parallelepiped formed by the axis of rotation and the parallelogram.

Proceeding with some caution, the analogon in more-dimensional cases is readily found in the same way. We will put for the numerator

$$2\,\Sigma\,(a\,m\,q)\,g_{aq}\,R^a_m\,(dx^m\,\delta x^q - dx^q\delta x^m) \quad . \quad . \quad . \quad . \quad . \quad (7)$$

denoting by $R^a{}_m$ the coefficients of the rotation of curvature (6):

$$\zeta^a = \Sigma(m)\,R^a_m\,u^m.$$

or,

$$\zeta^c = \Sigma(j)\,R^c_j\,w^j.$$

How are the numbers R^c_j related to the components of the axis of rotation? If we suppose the components of the latter equal to l^i, then the rotation is represented, as will be presently shown, by

$$\zeta^c = \frac{1}{\sqrt{g}}\,\Sigma(ij)\begin{vmatrix} g_{ai} & g_{bi} \\ g_{aj} & g_{bj}\end{vmatrix} l^i w^j \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

[1]) Cf. for example BIANCHI, Lect. on Diff. Geometry, section 319.

Here we mean by c that index, which with a and b forms an even permutation of 1 2 3. By pqr we shall denote a similar set:

$$abc\,(=)\,pqr\,(=)\,123.$$

We already saw that displacements of this kind constitute a rotation. To inquire whether the angular amount of the rotation is equal to the length of $\mathbf{l}$, we must observe the displacement of the end of a vector $\mathbf{u}$ which is perpendicular to $\mathbf{l}$, so that

$$\Sigma\,(ab)\,g_{ab}\,u^a\,l^b = 0\,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

This displacement ought to be $|\mathbf{u}|$ multiplied by $|\mathbf{l}|$. Let us calculate $|\zeta|^2$:

$$\zeta^2 = \Sigma(cr)\,g_{cr}\zeta^c\,\zeta^r,$$

$$= \frac{1}{g}\,(rij\,v\,w) \begin{vmatrix} g_{ai} & g_{bi} & g_{ci} \\ g_{aj} & g_{bj} & g_{cj} \\ g_{ar} & g_{br} & g_{cr} \end{vmatrix} l^i\,u^j\,. \begin{vmatrix} g_{pv} & g_{qv} \\ g_{pw} & g_{qw} \end{vmatrix} l^p u^w.$$

The summation with respect to c has been effected by writing in full the first determinant. If we want to sum up with respect to r, we notice that the determinant vanishes but for one special value of r, which is different both from i and j. If $i = p$, $j = q$, then the determinant becomes $+g$, if $i = q$, $j = p$, then we get $-g$. In both cases we may write

$$\zeta^2 = \Sigma\,(pq\,vw) \begin{vmatrix} g_{pv}\,g_{qv} \\ g_{pw}\,g_{qw} \end{vmatrix} l^p\,l^v\,u^q\,u^w,$$

and, by (9):

$$\zeta^2 = u^2\,l^2,$$

So the correctness of formula (8) has been shown.

But then we are justified in putting

$$R^c_j = \frac{1}{\sqrt{g}}\,\Sigma(i) \begin{vmatrix} g_{ai} & g_{bi} \\ g_{aj} & g_{bj} \end{vmatrix} l^i,$$

and we can subsequently show that (7) represents four times the parallelepiped mentioned. We write (7) with a slight alteration of indices and we get:

$$2\,\Sigma\,(c\,j\,k)\,g_{ck}\,R^c_j\,\,(dx^j\,\delta x^k - dx^k\delta x^j) =$$

$$= \frac{2}{\sqrt{g}}\,\Sigma(ij\,k) \begin{vmatrix} g_{ai} & g_{bi} & g_{ci} \\ g_{aj} & g_{bj} & g_{cj} \\ g_{ak} & g_{bk} & g_{ck} \end{vmatrix} l^i\,(dx^j\,\delta x^k - dx^k\delta x^j).$$

Now if j and k assume all values, a set j, k furnishes just as much as a set k, j, the determinant taking the value $+g$ or $-g$ according to the combination i, j, k being an even or odd permutation of 1 2 3, and vanishing for other combinations. So we get for the numerator

$$4\sqrt{g}\begin{vmatrix} l^i & l^j & l^k \\ dx^i & dx^j & dx^k \\ \delta x^i & \delta x^j & \delta x^k \end{vmatrix}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

this being four times the volume of the parallelepiped formed by **l**, **dx** and δ**x**.

This sufficiently explains section 5. We may remark that formula (8) for the displacements of rotation implies a convention as to the direction in which the axis of rotation has to be drawn. The axis of rotation must be orientated in a manner to ensure that the direction of ζ is correlated to the directions of **l** and **u**, i.e. a parallelepiped constructed from **l**, **u** and ζ, in the order thus specified, must have a positive volume:

$$\sqrt{g}\begin{vmatrix} l^a & l^b & l^c \\ u^a & u^b & u^c \\ \zeta^a & \zeta^b & \zeta^c \end{vmatrix} = \textit{positive.}$$

This amounts to the same relation which exists between the directions of the positive axes of X, Y, Z.

One sees from (10) that the measure of curvature will be positive if the direction of the axis of the rotation of curvature bears the above-mentioned correlation to the directions of **dx** and δ**x**.

Similarly, in four dimensions, if the axis of a rotation in a special case be a parallelogram on the vectors **l** and **m**, then the rotation is given by

$$\zeta^d = \frac{1}{\sqrt{g}}\,\Sigma\,(i\,j\,k)\begin{vmatrix} g_{ai} & g_{bi} & g_{ci} \\ g_{aj} & g_{bj} & g_{cj} \\ g_{ak} & g_{bk} & g_{ck} \end{vmatrix}\; l^i\, m^j\, u^k\,.$$

where $a\,b\,c\,d$ (=) 1 2 3 4, and the direction of ζ is correlated to the directions of **l**, **m** and **u**, i.e.

$$\sqrt{g}\begin{vmatrix} l^a & l^b & l^c & l^d \\ m^a & m^b & m^c & m^d \\ u^a & u^b & u^c & u^d \\ \zeta^a & \zeta^b & \zeta^c & \zeta^d \end{vmatrix} = \textit{positive}$$

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