Mathematics. — "*On the arising of a precession-motion owing to the non-euclidian linear element of the space in the vicinity of the sun*". By Prof. J. A. SCHOUTEN. (Communicated by Prof. LORENTZ).

(Communicated in the meeting of June 29, 1918).

If k be an curve in an n dimensional space X_n of arbitrary form, there will be in the euclidian space of $\frac{n(n+1)}{2}$ dimensions, into which X_n can always be placed without changing its linear element, a euclidian space Y_n, i.e. a space Y_n developable on a plane space tangent to X_n along k. If in the euclidian space Y_n a system of n mutual $\perp$ directions be moved with its origin along k parallel to itself, we find that these directions in X_n define a "geodesically moving system" [1]. If two arbitrary spaces are tangent to each other in a curve k, it follows from this definition, that a system geodesically moving along k for one space, will geodesically move for the other space too. A volume-element covered with mass can move in X_n as a solid body, but for some infinitesimals of a higher order. If a suchlike element always remains at rest with regard to a geodesically co-moving system of directions we will call it *compassbody*. Hence the compassbody mechanically realizes the geodesically moving system.

If k be a closed curve, the initial position will as a rule not coincide with the final position, if X_n is non-euclidian. Thus the position of the compassbody is changed with every rotation. Now according to the investigations of K. SCHWARZSCHILD [2] the space in the vicinity of the sun is not euclidian, but very slightly curved. The linear element is of the form

$$ds^2 = dR^2 + R^2\, d\theta^2 + R^2 \sin^2 \theta\, d\varphi^2 \quad . \quad . \quad . \quad . \qquad (1)$$

[1]) Cf. for a more detailed exposition of the geodesically moving system: "Die direkte Analysis der neueren Relativitätstheorie". Verh. of the Kon. Akad. v. Wet. Vol. 12. No. 6 and "On the number of degrees of freedom of the geodesically moving system and the enclosing euclidian space with the least possible number of dimensions". Proc. of the Kon. Akad. v. Wet. May 25, 1918.

[2]) Ueber das Gravitationsfeld eines Massenpunktes nach der Einstein'schen Theorie, Berl. Sitzungsber. 1916, p. 189—196.

L. FLAMM [1]) has indicated how a space with that linear element can be realized. The parabola situated in the xz plane:

$$z^2 = 4a(x-a) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

has the linear element:

$$ds = \frac{dx}{\sqrt{1-\frac{a}{x}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If this parabola is rotated about the z axis, in the xyz space, there arises a rotation-surface with the linear element:

$$ds^2 = \frac{dR^2}{1-\frac{a}{R}} + R^2\, d\varphi^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

φ being the rotation-angle, measured from the yz plane (fig. 1).

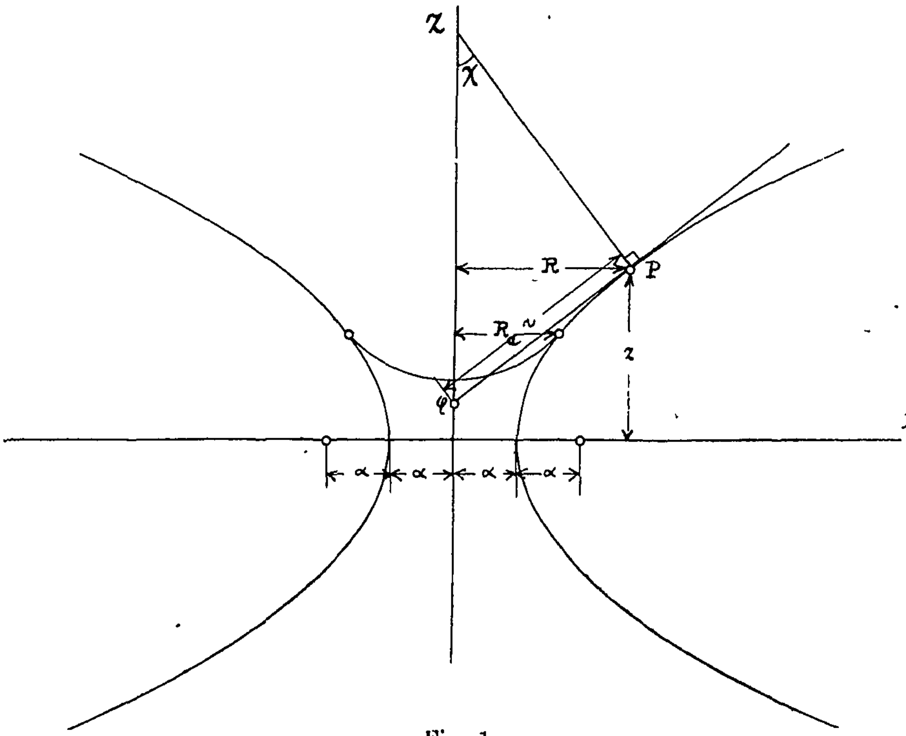

Fig. 1.

A fan of directions in the centre of the sun determines a system of ∞^1 geodesic lines, together forming a diametral surface. Such a diametral surface can as regards the parts beyond the surface of the sun be developed on the rotation-surface (4) without changing its linear element. R may approximately be equalized with the

[1]) Beiträge zur Einstein'schen Gravitationstheorie 17 (16) 448—454.

naturally measured distance relative to the centre of the sun, while $\alpha = 2.945.10^{15}$ cm. The circle described by a definite point P:

$$\left.\begin{array}{l} x^2 + y^2 = R^2 \\ z = 2\sqrt{\alpha(R-\alpha)} \end{array}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (5)$$

consequently represents a circle in the diametral surface, having the same centre as the sun. If the sun be looked upon as a globe, filled with an incompressible liquid, a diametral surface within the sun will have the same linear element as the globe-surface, which touches the described rotation-surface in a parallel-circle with a radius R_α. This radius R_α too may approximately be equalized with the astronomically measured radius of the sun. If the described rotation-surface (4) is rotated in the four-dimensional $xyzu$ space around the yz plane, there arises a curved three-dimensional space with the linear element (1) when θ is the angle of rotation, measured from the yzu space.

We shall now investigate the motion of a compassbody moving in the circle (5) around the sun. For this purpose it is sufficient to find a space, tangent to (1) in (5), and in which the geodesic motion can conveniently be indicated. We now make the tangent line PQ rotate with the parabola. That tangent line describes a cone with the linear element:

$$ds^2 = \frac{dR^2}{\cos^2\chi} + R^2 d\varphi^2 \quad . \; . \; . \; . \; . \; . \; . \; . \quad (6)$$

in which equation χ only depends on the definitely selected point P, and therefore is a constant. With the second rotation there arises from this cone a space with a linear element:

$$ds^2 = \frac{dR^2}{\cos^2\chi} + R^2 d\theta^2 + R^2 \sin^2\theta \, d\varphi^2 \quad . \; . \; . \; . \; . \quad (7)$$

in which χ is once more a constant. The linear element of a euclidian space may be expressed (in polar-coordinates) R', φ', θ':

$$ds^2 = dR'^2 + R'^2 \, d\theta'^2 + R'^2 \sin^2\theta' \, d\varphi'^2 \quad . \; . \; . \; . \quad (8)$$

and by the substitution:

$$\left.\begin{array}{l} R = R' \cos\chi \\ \varphi = \dfrac{\varphi'}{\cos\chi} \\ \theta = \dfrac{\theta'}{\cos\chi} \end{array}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (9)$$

(7) passes into:

$$ds^2 = dR'^2 + R'^2 \, d\theta'^2 + R'^2 \sin^2 \frac{\theta'}{\cos\chi} \, d\varphi'^2 \quad . \; . \; . \quad (10)$$

On the curve (5) we have $\cos\theta = 0$. With the exception of quantities of the order χ^4 the tangent space (7) behaves along the curve (5) as the euclidian space (8). Hence we need only trace the movement of the geodesically moving system in (8) along the curve corresponding to (5). As the coordinates φ and θ have obtained a factor $\cos\chi$ according to (9), (8) can be realized by the part of the euclidian space xyu that remains, when a rotation-cone having for axis the y-axis and a top-angle of $2\pi\,(1-\cos\chi)$ is taken away. (fig. 2).

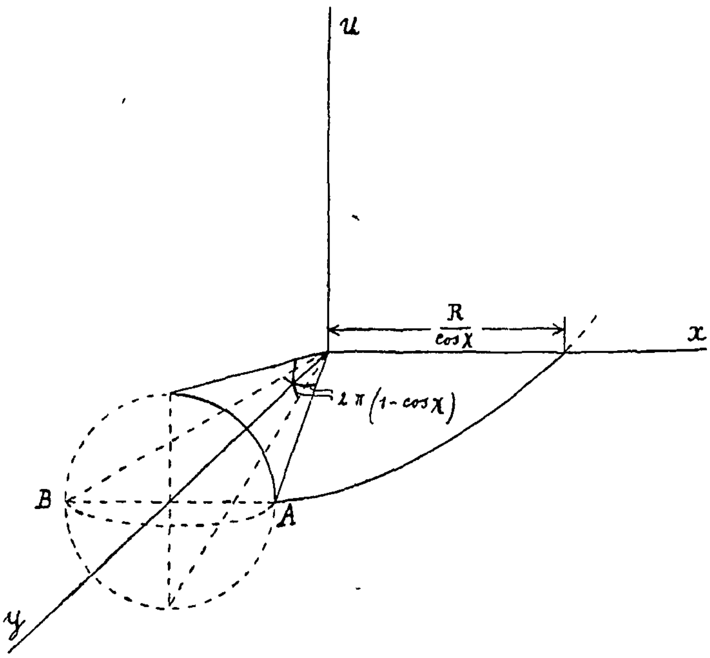

Fig. 2.

The part of the circle:

$$\left.\begin{aligned} x^2 + y^2 &= \frac{R^2}{\cos^2\chi} \\ u &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (11)$$

situated within this space viz. the part extending in fig. 2 from A via a point in the negative part of the y-axis to B, will then correspond with the entire curve (5).

This result may also be obtained by replacing the cone (6) by its unfolded mantle, laid down in the yx plane symmetrically to the y axis, being a sector with an angle $2\pi\cos\chi$. The curve (5) will then coincide with (11). During the revolution about the yz

plane the circle-sector describes the space-section described. By this method it does not at once become obvious that now the obtained euclidian space may indeed replace the originally existent non-euclidian tangent-space along (5). The motion of the compass-body can now be easily traced. In the euclidian space xyu the compassbody always moves parallel to itself. If a constant direction in A in this body has the direction of the radius, then that direction in B forms with the radius an angle $2\pi(1 - \cos\chi)$ situated in a plane // at the xy-plane. Now $\frac{a}{R}$ is very small, hence:

$$\cos\chi = 1 - \frac{a}{2R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

so that the total deviation δ in one revolution amounts to:

$$\delta = \pi \frac{a}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

A compassbody moving around the sun, as its central point, in a circle with a radius equal to the average distance from the earth to the sun will show according to this formula after one revolution a deviation of 0.013″. If the radius is equal to the average distance from Mercury to the sun, the deviation amounts to 0.0328″. If the radius is equal to the radius of the sun, the deviation amounts to 2.73″. If, from another cause, the compass-body already has a revolution around an axis, which is oblique relatively to the plane of the orbit, there will set in, merely on account of the deviation described, a precession-motion, which in the first of the above-mentioned cases would result in a complete revolution of the equinoxes after ± 100.000.000 years. It is noteworthy that the effect described is of the same order as the deviation of a ray, passing the sun at a distance R from the central point. According to Einstein this deviation indeed amounts approximately to $\frac{2a}{R}$.

Whether the deviation computed of the precession motion for the earth will indeed set in, depends on the question to what extent and to what approximation a mass of the quantity and the composition of the earth has the properties of a compassbody. In order to answer this question it is necessary to make definite suppositions as regards the physical qualities of the earth, in particular the mutual attraction of her parts, and starting from these suppositions, to integrate the four-dimensional dynamical equations of motion.

ADDENDUM.

Prof. DE SITTER, to whom I have communicated the above, writes:

A precession of 0".013 per annum of course comes well within the reach of observations, since the observed value of the precessional constant is trustworthy to about 0".0010. The point is therefore with what accuracy the theoretical value can be computed. Now the lunisolar precession (the planetary precession can be taken as completely known) is given by a formula of the form

$$p_1 = (P + Q\mu) . H . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where P and Q are known numbers and μ is the mass of the moon (expressed in that of earth + moon as unit) and $H = \frac{C-A}{C}$ depends on the moments of inertia of the earth. The uncertainty of μ causes in p_1 an uncertainty of about $^1/_{2000}$ of its amount, thus, if H was exactly known, p_1 would be uncertain to the extent of $\pm$ 0",025 or twice the new precession. A better determination of μ may be expected from the opposition of EROS in 1931 [1]). However the uncertainty of H is of much greater importance. In 1915 [2]) I have with the aid of the hypothesis of isostasy, derived the ellipticity ε of the earth from H, this latter being determined from p_1 by (1). To invert this order it would be necessary, in order to get a p.e. of $\pm$ 0".005 in p_1, to know ε to about $^1/_{20000}$ of its amount. The direct determinations of ε at present do not go beyond about $^1/_{300}$. To increase this accuracy seventyfold would in my opinion be beyond the forces of geodetical science, at least in the near future.

We can determine H with greater accuracy from the constant of nutation, which is given by

$$N = R . \mu . H . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where R is again a known number. From (1) and (2) we derive

$$p_1 = \left(S + \frac{T}{\mu}\right) . N \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where S and T are also practically exactly known. The uncertainty of the multiplier owing to the uncertainty of μ is now about $^1/_{4000}$ or $^1/_{5000}$, and it will probably be reduced to $^1/_{10000}$ by the new determination of μ in 1931.

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1) The figure of the earth and some related astronomical constants. The Observatory Aug. 1915, page 322.

2) On Isostasy, the moments of inertia, and the compression of the earth. These Proceedings April 1915, Vol XVII, p. 1291.

The result of NEWCOMB's discussion in 1891 of all available determinations of the constant of nutation has determined its value to about $^{1}/_{11000}$ of its amount. To get a p.e. of $\pm$ 0″.005 in p this accuracy must be increased nine- or tenfold. This certainly is no easy task, but it would be preposterous to say that it exceeded the forces of astronomy. It will of course require very refined and prolonged observations and discussions.

I may be allowed to remark that it still remains to be investigated whether the new precession is the only effect of the new gravitational theory, and the equations (1) and (2) are not affected, i.e. whether EINSTEIN's theory gives exactly the same equations for the motion of the axis of rotation of the earth with reference to the geodetically transported system of coordinates, as are found in NEWTON's theory relatively to a "fixed" system. This cannot be asserted without a special investigation, which so far as I know has not been undertaken, and it might even happen that the precession of the geodetically transported system of coordinates was exactly cancelled by a small change in the precession of the earth relatively to that system.