Mathematics. — "*Observations on the expansion of a function in a series of factorials.*" II. By Dr. H. B. A. BOCKWINKEL. (Communicated by Prof. H. A. LORENTZ).

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5. We now consider another example of NIELSEN's theorem, not belonging to the cases mentioned under N°. 4 of the remarks made in the preceding paragraph. We choose

$$\varphi(t) = \frac{1}{e^{i\theta} - t}$$

where θ is a number between 0 and 2π, not equal to one of these numbers. For this function we have

$$\lambda = -\infty \quad , \quad \lambda' = 0,$$

the first of these equations resulting from the fact that $t = 1$ is an ordinary point of the function. It is further easily found that the n^{th} derivative of $\varphi(t)$ satisfies the equation

$$\frac{\varphi^{(n)}(t)(1-t)^{\delta+n-1}}{\Gamma(\delta+n)} = \frac{n\Gamma(n)}{\Gamma(\delta+n)}\left(\frac{1-t}{e^{i\theta}-t}\right)^n \times \frac{(1-t)^{\delta-1}}{e^{i\theta}-t} \quad . \quad . \quad (21)$$

The modulus of the expression $\frac{1-t}{e^{i\theta}-t}$ is given by the relation

$$\left|\frac{1-t}{e^{i\theta}-t}\right| = 1 - \frac{2t(1-\cos\theta)}{1-t+\sqrt{1-2t\cos\theta+t^2}} \quad . \quad . \quad . \quad . \quad (22)$$

and it is not very difficult to see that it increases monotonously from the value 0 to 1, if t decreases from 1 to 0.

We divide the interval (0,1) of t into two parts, $(0, \nu)$ and $(\nu, 1)$, where ν is a number given by

$$\nu = n^{\delta_1 - 1} \qquad (0 < \delta_1 < 1) \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

so that ν depends on n and approaches to zero as a limit when n becomes indefinitely large. The positive number δ_1 is at our disposal and will be fixed immediately. The maximum value of the modulus (22) then differs from unity by a quantity greater than

$$kn^{\delta_1 - 1}$$

if t lies in the second interval, k being a certain positive number, which is independent of n and t; thus we have in this interval

$$\left|\frac{1-t}{e^{i\theta}-t}\right|^n < e^{-kn^{\delta_1}}$$

so that the left-hand member of (21) for these values of t approaches *uniformly* to zero for $n=\infty$ (the factor $n\,\Gamma(n):\Gamma(\delta+n)$ is only equivalent to $n^{1-\delta}$ and does therefore not affect this statement). The integral

$$\int\limits_{\nu}^{1} \frac{n\Gamma(n)}{\Gamma(\delta+n)}\left(\frac{1-t}{e^{i\theta}-t}\right)^n \frac{(1-t)^{\delta-1}}{e^{i\theta}-t}\,dt$$

consequently has zero for its limit if n increases indefinitely, however small the value of δ may be fixed.

For the interval $(0,\nu)$ we have, independently of t and n,

$$\left|\frac{n\Gamma(n)}{\Gamma(\delta+n)}\left(\frac{1-t}{e^{i\theta}-t}\right)^n \frac{(1-t)^{\delta-1}}{e^{i\theta}-t}\right| < \frac{kn\,\Gamma(n)}{\Gamma(\delta+n)}$$
$$< kn^{1-\delta},$$

where k is again a positive number not depending on n and t.[1] Thus, considering (23), it follows

$$\left|\int\limits_{0}^{\nu} \frac{n\Gamma(n)}{\Gamma(\delta+n)}\left(\frac{1-t}{e^{i\theta}-t}\right)^n \frac{(1-t)^{\delta-1}}{e^{i\theta}-t}\,dt\right| < kn^{1-\delta}\times\nu$$
$$< kn^{-(\delta-\delta_1)}$$

We therefore need only choose δ_1 less than δ, to see that also the integral over the interval $(0,\nu)$ is zero for $n=\infty$. Thus the whole remainder (11) is zero for $n=\infty$, if only $R(x)>0$, i.e., since $\lambda=-\infty$ and $\lambda'=0$, if $R(x)>\lambda'$ and $R(x)>\lambda$. For these values of x the integral

$$\int\limits_{0}^{1}\frac{(1-t)^{x-1}}{e^{i\theta}-t}\,dt$$

can therefore be expanded into a series of factorials; and the theorem of Nielsen holds in this case.

Again we take the example

$$\varphi(t)=\frac{1}{e^{i\theta}-t}+\frac{1}{(1-t)^{\mu}},\ \begin{pmatrix}0<\theta<2\pi\\0<\mu<1\end{pmatrix}.$$

Here $\lambda'=0$, on account of the first term, and $\lambda=\mu$, on account

[1] We shall always, in future, denote by k a finite positive number, without always meaning the *same* number by this letter. This will not cause any ambiguity, because the exact value of k is of no importance in our reasonings. For the sake of clearness, however, we shall often mention the quantities on which k does *not* depend.

of the second. If $\varphi(t)$ were equal to the second term only, the integral (1) could be expanded into a series of factorials for $R(x) > \mu$ only, and this series would be *absolutely* converging for these values of x. Thus the *whole* function may also be represented by such a series for $R(x) > \mu$, i.e. for $R(x) > \lambda$ and $R(x) > \lambda'$, but the convergence is, on account of the first term, only *conditional* for $\lambda < R(x) < \lambda' + 1$. This, again, is exactly the proposition of NIELSEN.

6. If, in the first example of the foregoing paragraph, we account for the reason of the validity of this proposition, we infer that it is a consequence of the fact that the expression

$$\left|\frac{1-t}{e^{i\theta}-t}\right|^n,$$

for a fixed value of $t > 0$, decreases with $1/n$ as the n-th power of a number less than 1, which causes that, in the integral (11), only an interval has to be considered which, in a proper manner, approaches to zero as n becomes infinite, so that the value of $R(x)$ for which expansion is possible can be depressed by unity. This suggests the idea that something of the kind might occur *as a rule*, if $\varphi(t)$ has $t = 1$ for an *ordinary* point. The truth of this presumption is proved by the following investigation.

We again divide the interval (0,1) of t into two partial intervals, with the point $t = v$ as a common end-point, which is to approach ultimately to zero as n becomes indefinitely large; and we assume, as in the preceding paragraph, for v the value (23). Consider the circle, with centre v and passing through two fixed points C and C' lying on the circumference of the circle of convergence (0,1) of $\varphi(t)$, symmetrically with regard to the axis of real quantities, and in the interior of an arc DAD' of the latter circle, which does not contain a singular point of $\varphi(t)$, D and D' being also conjugate points, whereas A is the point with the affix $t = 1$. Then, from and after some value of n the value of v will be so small that the circle with centre v does not contain any singular point of $\varphi(t)$ in its interior and on its circumference; and at all points of the latter between the radii OD and OD', including an arc EBE' of it (B being the point on that arc with argument zero), the modulus of $\varphi(t)$ will remain under a finite quantity K, independent of n and t. As regards points of the supplementary arc EFE' of circle (v), F being the point opposite to B, we may remark that $\varphi(t)$ there has a modulus no greater than

$$\bar{\varphi}(1-v''),$$

$\overline{\varphi}(t)$ means the natural majorant of $\varphi(t)$, and v'' the distance of the points D and E.

We further remark that the radius of the circle (v) is greater than $1-v$, say $1-v+v'$. It is evident that the numbers v' and v'' both approach to zero together with v, but that their ratios to the latter number remain finite and *different from zero.*

At a point P of the interval $(v, 1)$ we have, according to a well-known proposition

$$\left|\frac{\varphi^{(n)}(t)}{n!}\right| < \frac{M}{(1-t+v')^n},$$

if M is the greater of the numbers K and $\overline{\varphi}\,(1-v'')$. Instead of this inequality we may write

$$\left|\frac{(1-t)^{n-1}\varphi^{(n)}(t)}{n!}\right| < \left(\frac{1-t}{1-t+v'}\right)^{n-1} M v'^{-1},$$

or, since for $0 \overline{<} t \overline{<} 1$, $\frac{1-t}{1-t+v'} < \frac{1}{1+v'}$,

$$\left|\frac{(1-t)^{n-1}\varphi^{(n)}(t)}{n!}\right| < \frac{M v'^{-1}}{(1+v')^n} \quad \ldots\ldots\ldots \quad (24)$$

With regard to $\overline{\varphi}\,(1-v'')$ the following remarks may be made. If, in the equivalence-equation

$$\lim a_n \frown n^{\lambda'}$$

the quantity λ' is no less than -1, we have, according to the proposition of Cesarò, for any fixed $\delta > 0$,

$$\lim_{v''=0} (v'')^{\lambda'+1+\delta}\, \overline{\varphi}(1-v'') = 0,$$

and hence, in virtue of the remark made above on the relation between v'' and v,

$$\lim_{v=0} v^{\lambda'+1+\delta}\, \overline{\varphi}(1-v'') = 0$$

and further

$$\lim_{v=0} v^{\lambda'+1+\delta} \times M = 0$$

since, as a matter of course, the expression $K \times v^{\lambda'+1+\delta}$ has, too, zero for its limit.

Thus we may write for (24), in connection with the assumption (23) and the finite, not disappearing ratio between v and v'

$$\left|\frac{(1-t)^{n-1}\varphi^{(n)}(t)}{\Gamma(n-1)}\right| < \frac{kn^k}{(1+kn^{\delta_1-1})^n}$$

$$< kn^k e^{-n^{\delta_1}},$$

where k is again a positive number not depending on n and t. Hence, corresponding to any fixed positive quantity ε chosen arbitrarily small, there is an integral number N, such that the left-hand member of the latter inequality is less in value than ε, for every value of t in the interval $(\nu,1)$, if only $n > N$. For these values of n we have therefore

$$\left| \int_{\nu}^{1} \frac{(1-t)^{x+n-1} \varphi^{(n)}(t)}{\Gamma(x+n)} \, dt \right| < k\varepsilon \int_{\nu}^{1} (1-t)^{R(x)} \, dt$$
$$< k\varepsilon$$

if $R(x) > -1$. For any such value of x, i.e. a fortiori for $R(x) > \lambda'$, since λ' was supposed greater than -1, the part of the integral (11) taken over the interval $(\nu,1)$ has zero for its limit for $n = \infty$.

For the integration over the remaining interval $(0,\nu)$ we apply the mode of treatment of § 3 and the inequality (17). According to the latter there is, corresponding to any fixed δ and ε, chosen as small as we please, an integer N such that we have *uniformly* in the interval $(0,1)$, and hence in $(0,\nu)$,

$$\left| \frac{\varphi^{(n)}(t)\,(1-t)^{\lambda'+\delta+n+1}}{\Gamma(\lambda'+\delta+n+1)} \right| < \varepsilon, \text{ if } n > N.$$

For the interval $(0,\nu)$ it follows from this that, for $n > N$

$$\left| \frac{\varphi^{(n)}(t)\,(1-t)^{x+n-1}}{\Gamma(x+n)} \right| < k\varepsilon\, n^{1-R(x)+\lambda'+\delta}$$

thus

$$\left| \int_{0}^{\nu} \frac{\varphi^{(n)}(t)\,(1-t)^{x+n-1}}{\Gamma(x+n)} \, dt \right| < k\varepsilon\, n^{-R(x)+\lambda'+\delta+\delta_1}$$

If now $R(x) > \lambda'$, we can have chosen the numbers δ and δ_1 so small that $R(x)$ is also greater than $\lambda' + \delta + \delta_1$, and in this case we infer from the latter inequality that the integral over the interval $(0,\nu)$, too, has zero as a limit for $n = \infty$, if $R(x) > \lambda'$. *Thus the theorem of* Nielsen *has been proved, in case $t = 0$ is an ordinary point of the function $\varphi(t)$.*

If a function $\varphi(t)$ has the point $t = 1$ for its only singular point on the circumference of the circle of convergence $(0,1)$, and if, moreover, it satisfies the conditions of Hadamard, i.e. if it is continuous and "à écart fini" on that circumference, or if a certain derivative of negative order $-\omega$ has this property, then we always have

$$\omega = \lambda = \lambda' + 1,$$

and the theorem of Nielsen has ceased having anything particular.

Again it may happen that $\varphi(t)$ can be divided into the sum of two functions $\varphi_1(t)$ and $\varphi_2(t)$, the first of which is regular at $t=1$ and the second of which has the latter point as its only singularity on the circumference of the circle (0,1). If, then, the number λ' for $\varphi_1(t)$ is equal to λ_1' and that for $\varphi_2(t)$ to λ_2' and if $\lambda_1' > \lambda_2'$, so that for the whole function $\varphi(t)$ the number λ' is equal to λ_1', the integral (1) can be expanded in a conditionally converging series of factorials for

$$\lambda'_2 + 1 < R(x) < \lambda' + 1$$

if $\lambda' = \lambda_1' < \lambda_2' + 1$, and for

$$\lambda' < R(x) < \lambda' + 1$$

if $\lambda_1' > \lambda_2' + 1$. If, in this case, $\varphi_2(t)$ has the properties of HADAMARD, then $\lambda_2' + 1 = \lambda_2 = \lambda$, and the proposition of NIELSEN is valid, which, now, really has a particular meaning.

7. The following proposition is, as a corollary, included in the theorem of the preceding paragraph.

If the coefficients a_n of a function $\varphi(t)$, defined by a power-series

$$\varphi(t) = \sum_{0}^{\infty} {}_n\, a_n t^n$$

which has the circle (0.1) *as its domain of convergence, are, for $n = \infty$, equivalent to a power $n^{\lambda'}$ of n, and if the series*

$$\sum_{0}^{\infty} {}_n \frac{a_n}{n^{\lambda'+\theta}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

is divergent for $0 < \theta < 1$, the point $t = 1$ is a singularity of $\varphi(t)$.

For if $t = 1$ is an *ordinary* point of $\varphi(t)$, the series (6), which, except for the factor $\Gamma(x)$, is equal to

$$\sum_{0}^{\infty} {}_n \frac{n!\, a_n}{\Gamma(x+n+1)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

is convergent for $R(x) > \lambda'$ and the convergence of (25) can be derived from it. For we may write

$$\frac{n!\, a_n}{n^{\lambda'+\theta}} = \frac{n!\, a_n}{\Gamma(x+n+1)} \times \frac{\Gamma(x+n+1)}{\Gamma(n+1) n^{\lambda'+\theta}}$$

If we choose x such that $\lambda' < R(x) < \lambda' + \theta$, the series formed by the first factor, if n takes all values from zero to infinite, converges, as we have already seen, whereas the series, composed of the terms obtained by taking the *first finite differences*, with regard to n, of the second factor, converges *absolutely*; and it is a well-known truth that the convergence of (25) is a consequence of these two facts. The same thing would hold with regard to the series

$$\Sigma \frac{a_n}{\varphi(n)}$$

if

$$\lim_{n=\infty} \varphi(n) \smile n^{\lambda'+\theta}, \quad \text{and} \quad \lim_{n=\infty} \Delta\, \varphi(n) \smile n^{\lambda'+\theta-1}.$$

Therefore, in the statement of the above theorem such a series may be chosen as well. We further remark that λ', which was hitherto supposed to be greater than -1, may also be *less* than the latter number: the theorem of NIELSEN, in the particular case demonstrated in § 6, keeps its validity for those values of λ', though we should have to apply our reasonings to an integral of the form (8) (in a footnote of § 1) in the latter case.

By substituting $t = t' e^{i\varphi}$ in the power-series for $\varphi(t)$ we obtain the more general theorem:

If the coefficients a_n of a power-series in the letter t are equivalent to $n^{\lambda'}$ for $n = \infty$, the function $\varphi(t)$ represented by that series has, on the circumference of its circle of convergence (being the circle (0,1), *singularities at all points where the series*

$$\Sigma \frac{a_n t^n}{n^{\lambda'+\theta}} \quad (0 < \theta < 1)$$

diverges. We may add that this theorem already holds, if only the *upper limit* of the coefficients a_n is, in the sense of equation (14'), equivalent to $n^{\lambda'}$ for $n = \infty$.

Finitely we observe that the reverse of the proposition does not hold: if the series (25) converges, the point $t = 1$ need not be an ordinary point. To make this clear we need only think of the case that the coefficients a_n differ from zero only for values of n lying at a certain distance from each other; it may happen then that the series (25) converges absolutely, but the function $\varphi(t)$ has its whole circle of convergence as a singular line.

8. As already remarked, we doubt of the *general* validity of NIELSEN's theorem, though we are not in a position to furnish a case of the non-validity. It is our opinion that, if $\lambda' < \lambda < \lambda' + 1$, there will be cases in which the integral (1) cannot, for all values of $R(x) > \lambda$, be expanded into a series of factorials. On the other hand we can prove that such an expansion is not possible for any value of $R(x) < \lambda$, which is a thing not immediately evident if λ lies between λ' and $\lambda' + 1$. [1])

[1]) If $R(x) < \lambda'$, the impossibility is at once evident, since the series-terms have not zero for their limits then.

Suppose the series (25) to converge for $\theta > \theta_1$ and to diverge for $\theta < \theta_1$, and, consequently, the series (26) to converge for $R(x) > \lambda' + \theta_1$, and to diverge for $R(x) < \lambda' + \theta_1$, then the integral (1) will, at any case, not admit an expansion into a series of factorials for any value of $R(a) < \lambda' + \theta_1$. We now shall prove that for any positive δ, taken as small as we please,

$$\lim_{t=1} (1-t)^{\lambda'+\theta_1+\delta} \varphi(t) = 0$$ [1]

so that $\lambda \overline{\gtrless} \lambda' + \theta_1$; by this the required proof will have been established.

For the sake of brevity we write

$$\lambda' + \theta_1 + \delta = \alpha.$$

Consider the derivative of negative order $-\alpha$ of $\varphi(t)$, which according to the definition of RIEMANN [2], is given by

$$D^{-\alpha} \varphi(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} \varphi(u) du = t^\alpha \psi(t),$$

then $\psi(t)$ is a function regular at $t = 0$ with the same circle of convergence (0,1) as $\varphi(t)$ has; its expansion into a power-series is

$$\psi(t) = \sum_0^\infty {}_n \frac{n! a_n t^n}{\Gamma(\alpha+n+1)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

From this formula it may be derived that $\psi(t)$ remains finite for $t = 1$, in virtue of the initial hypothesis.

Conversely we have

$$\varphi(t) = D^\alpha[t^\alpha \psi(t)].$$

First, let

$$\lambda' + \theta_1 < 1.$$

Then we may choose δ so small that also $\alpha < 1$ and write

$$\left.\begin{aligned} \varphi(t) &= D^{\alpha-1} D[t^\alpha \psi(t)]\,^{3)} = D^{\alpha-1}[\alpha t^{\alpha-1}\psi(t) + t^\alpha \psi'(t)] = \\ &= \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-u)^{-\alpha} [\alpha\psi(u)u^{\alpha-1} + \psi'(u)u^\alpha]\, du \end{aligned}\right\} . \quad (28)$$

Now $\psi(u)$ is, in the range $0 \overline{\gtrless} u \overline{\gtrless} 1$, finite and thus less than a certain number g. Hence

[1]) Or for negative values of $\lambda' + \theta_1$, $\lim (1-t)^{\lambda+\theta_1+n+\delta} \varphi^{(n)}(t) = 0$, if n is such that $\lambda' + \theta_1 + n \overline{>} 0$.

[2]) See among others BOREL, Leçons sur les séries à termes positifs, p. 75.

[3]) Properly speaking it should be $D \, . \, D^{\alpha-1}$, but this operation, in the present case, is equal to $D^{\alpha-1} \, . \, D$, since the subject of the operation is *zero* for $t = 0$.

$$\left|\int_0^t \psi(u)u^{\alpha-1}(t-u)^{-\alpha}du\right| < g\int_0^t u^{\alpha-1}(t-u)^{-\alpha}\,du$$

or, substituting $u = tv$

$$\left|\int_0^1 \psi(u)u^{\alpha-1}(t-u)^{-\alpha}du\right| < g\int_0^1 v^{\alpha-1}(1-v)^{-\alpha}\,dv,$$

so that the integral in the left-hand member of this inequality remains finite for all values of t in the closed interval $(0,1)$. Further we divide the second integral on the right-hand side of (28) as follows, supposing $t > \frac{1}{2}$,

$$\int_0^t \psi'(u)u^{\alpha}(t-u)^{-\alpha}\,du = \int_0^{t-(1-t)} + \int_{t-(1-t)}^t$$

To the first of these two integrals we apply the second mean value theorem, which is allowed, because the expression $u^{\alpha}(t-u)^{-\alpha}$ increases monotonously in the interval in question. We obtain

$$\int_0^{t-(1-t)} \psi'(u)u^{\alpha}(t-u)^{-\alpha}\,da = (2t-1)^{\alpha}(1-t)^{-\alpha}\,[\psi(2t-1)-\psi(\beta)]$$

where β is a number in the interval $(0, 2t-1)$. This part of the integral, as $\varphi(t)$ remains within finite limits, is therefore for $t=1$ at most equivalent to $(1-t)^{-\alpha}$. In order to infer the same thing with regard to the second integral, we make use of the fact that

$$\lim_{t=1}(1-t)^n\,\psi^{(n)}(t) = 0, \quad (n = 1, 2, \ldots) \quad . \quad . \quad . \quad . \quad (29)$$

We shall prove this at once; it should not be thought that it is a consequence of the proposition mentioned in a footnote of § 1: it follows solely from the convergence of the series (27) for $t=1$.

If we assume, for a moment, the formula to be true, we have for the whole interval $0 \lesssim \alpha \lesssim 1$, if K is a certain positive number, not depending on u,

$$\psi'(u) < \frac{K}{1-u},$$

and so, in the interval of integration $2t-1 \lesssim u \lesssim t$

$$\psi'(u) < \frac{K}{1-t},$$

from which it follows

$$\left|\int_{2t-1}^{t}\psi'(u)u^{\alpha}(t-u)^{-\alpha}\,du\right| < \frac{K}{1-t}\int_{2t-1}^{t}(t-u)^{-\alpha}\,du = K(1-t)^{-\alpha},$$

so that this integral, too, is for $t=1$ at most of order $(1-t)^{-\alpha}$. The same holds therefore for the function $\varphi(t)$, and since α may be supposed arbitrarily little greater than $\lambda' + \theta_1$, we have certainly for every $\delta > 0$

$$\lim_{t=1}\,(1-t)^{\lambda'+\theta_1+\delta}\,\varphi(t) = 0$$

and thus, as we proposed to show

$$\lambda \overline{<} \lambda' + \theta_1.$$

Secondly, let $\lambda' + \theta_1$ lie between the integers $p-1$ and p; we may choose δ so small that the same holds for $\lambda' + \theta_1 + \delta = \alpha$. We write

$$\alpha = p - 1 + \alpha' \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (30)$$

so that

$$0 < \alpha' < 1 \quad . \; . \; . \; . \; . \; . \; . \; . \quad (31)$$

In this case we have the following reduction

$$\varphi(t) = D^{\alpha'-1}\,D^{p}\,[t^{\alpha}\psi(t)]\ ^{1)}$$

$$= D^{\alpha'-1}\left[\Gamma(\alpha'+p)\sum_{0}^{p}{}_{m}\,\frac{p_m\,t^{\alpha'+m-1}\,\psi^{(m)}(t)}{\Gamma(\alpha'+m)}\right]$$

Owing to (31) we may, as in the former case, using here the inequality (29) for $n = m$, prove that the expression

$$D^{\alpha'-1}\,[t^{\alpha+m-1}\,\psi^{(m)}(t)]$$

is at most equivalent to $(1-t)^{-(\alpha'+m-1)}$ and thus $\varphi(t)$, as m is no greater than p, is of an order no higher than that of $(1-t)^{-(\alpha'+p-1)}$, that is, according to (30), of the order $(1-t)^{-\alpha}$. Thus the required result is obtained completely.

9. We now give a proof of the proposition used in the preceding paragraph. It may be stated as follows:

If the expansion in a power-series of a function $\varphi(t)$ converges at the point $t=1$ of the circle of convergence (0,1), we have for all positive integral values of n

$$\lim_{t=1}\,(1-t)^{n}\,\varphi^{(n)}(t) = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \quad (32)$$

This proposition, of course, ceases to have a particular meaning,

[1]) Properly speaking it should be $D^{p}D^{\alpha'-1}$, but this comes to the same thing as $D^{\alpha'-1}D^{p}$, because t -($p-1$)-times the subject of operation is zero for $t=0$

if $t=1$ is not a singular point of $\varphi(t)$, but if it is, the proposition is not a matter of course.

Since, if the coefficients of the power-series in question are complex, the two series formed separately by means of the real and of the imaginary parts of those coefficients must both converge, we may without loss of generality suppose the coefficients to be real quantities. We then consider, together with the function

$$\varphi(t) = a_0 + a_1 t + \ldots + a_n t^n + \ldots$$

the function

$$f(t) = \frac{\varphi(t)}{1-t} = \sum_0^{\infty} {}_n \, s_n t^n \quad \ldots \ldots \quad (33)$$

where

$$s_n = \sum_0^{n} {}_p \, a_p,$$

Since s_n, as n becomes indefinitely large, approaches to a definite limit s, the series (33) behaves, so far as regards its terms for large values of n, as the power-series of the function

$$\frac{s}{1-t},$$

and according to the reasoning of Cesàro we have not only

$$\lim_{t=1} \left[f(t) : \frac{1}{1-t} \right] = s,$$

but also

$$\lim_{t=1} \left[f^{(n)}(t) : \frac{n!}{(1-t)^{n+1}} \right] = s. \quad \ldots \ldots \quad (34)$$

Further, from n-fold differentiation of the identity

$$\varphi(t) = (1-t) f(t)$$

we obtain the new one

$$\frac{(1-t)^n \varphi^{(n)}(t)}{n!} = \frac{(1-t)^{n+1} f^{(n)}(t)}{n!} - \frac{(1-t)^n f^{(n-1)}(t)}{(n-1)!}.$$

The limit of the right-hand side of the latter equation for $t=1$, is, by (34), equal to zero for all positive integral n-values, and the required formula (32) has thus been proved.

By substituting $t = t' e^{i\varphi}$ we obtain: *If the expansion in a power-series of a function $\varphi(t)$ converges at the point $t = e^{i\varphi}$ of its circle of convergence (0,1), then, for all positive integral values of n and for real values of t', we have*

$$\lim_{t'=1} (1-t')^n \varphi^{(n)}(t' e^{i\varphi}) = 0$$