Mathematics. — *"On the number of degrees of freedom of the geodetically moving system and the enclosing euclidian space with the least possible number of dimensions"*. By Prof. J. A. SCHOUTEN. (Communicated by Prof. J. CARDINAAL).

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Suppose k to be a non-special curve in a finite part X_n of a general space of n dimensions, containing no singular points and where only one geodetic line exists between two arbitrary points. Assuming in a point O on k a system of n mutually independent directions, we can move this system *geodetically* along k.

This geodetic moving can be geometrically defined in the following way. X_n can always be placed in a euclidian space of $\frac{n(n+1)}{2}$ dimensions, without changing its linear element. There exists in this space a space Y_n developable on a euclidian space of n dimensions, tangent to X_n in k. The geodetically moving directions will now coincide at any moment with the directions moving parallel to themselves in the euclidian space Y_n. It appears analytically that the known *covariant* differential of a directed quantity e.q. a vector is a *common* differential judged from a geodetically moving system of directions. Hence if $\mathbf{v}$ is a vector stationary with respect to this system, $\mathbf{v}$ satisfies the differential-equation:

$$d\mathbf{v} = 0,$$

or in co-ordinates:

$$dv^{\lambda} + \begin{Bmatrix} \mu\nu \\ \lambda \end{Bmatrix} v^{\mu} dx^{\nu} = 0,$$

and this equation then gives the analytical definition of the notion geodetic moving. [1]) A geodetic line is characterised by the property that its linear element forms at every point the same angles with a system moving geodetically along the line.

[1]) The covariant notations in this paper are the customary ones, but the contravariant characteristic numbers of the linear element dx are written contravariant agreeing to G. HESSENBERG, but contrary to G. RICCI and T. LEVI CIVITA. For the invariant notations, the here used direct analysis, cf. "Ueber die direkte Analysis der neueren Relativitätstheorie", a paper presented to the "Koninkl. Akademie v. W." together with this note. (Verh. Vol. 12 N°. 6).

Starting from a point O a system of directions is now geodetically moved along a closed curve. On returning to O the system will generally appear to have rotated. Dependent on the choice of the curve it is generally possible to obtain in this manner $\infty^{\frac{n(n-1)}{2}}$ positions of the system. If this number is for one point and hence for every point of the area ∞^N, we call N the *number of degrees of freedom* of the geodetically moving system. Now the following theorem exists:

I. *The number of dimensions of the euclidian space, in which a given space X_n may be placed, without changing its linear element, is at most equal to the number of degrees of freedom belonging to the geodetically moving system increased with n.*

We will prove this theorem. If the number of degrees of freedom is smaller than $\frac{n(n-1)}{2}$, there will remain invariant r mutually perfectly perpendicular directions of $p_1, p_2, p_3, \ldots$ dimensions $p_1 + \ldots + p_r = n$ (by direction of two dimensions or 2-direction we mean a plane direction, etc.). The number of possibilities exactly corresponds to the number of manners, in which n can be written as the sum of whole positive numbers. We imagine the r invariant directions marked once for all in O. The system then may be brought in every point of X_n, always by geodetically moving. The invariant p_j-direction, $j = 1 \ldots, r$, will then define at every point a p_j-direction, and it is the question whether these directions will compose a system of ∞^{n-p_j} curved spaces P_j of p_j dimensions. This is a PFAFFIAN problem in a general space.

We select a definite invariant direction, say the p_j-direction, and for convenience, sake we shall write p for p_j. If we now define the p-direction belonging to this direction at every point by the simple p-vectors, ${}_p\mathbf{v} = \mathbf{v}_1 \frown \ldots \mathbf{v}_p$, which all pass into one another by geodetically moving and likewise the perfectly perpendicular $(n-p)$-direction to this, by ${}_q\mathbf{w} = \mathbf{w}_1 \frown \ldots \mathbf{w}_q$, $q = n - p$, then:

$$d\,{}_p\mathrm{v} = 0 \quad , \quad d\,{}_q\mathrm{w} = 0,$$

hence:

$$\nabla\,{}_p\mathrm{v} = 0 \quad , \quad \nabla\,{}_q\mathrm{w} = 0.$$

It is worth mentioning, that the vectors $\mathbf{v}_k$, $k = 1, \ldots, p$, do *not* pass into each other by geodetically moving and hence $d\mathbf{v}_k \neq 0$. The same holds good for $\mathbf{w}_l$, $l = 1, \ldots, q$. If now the linear element be $d\mathbf{x}$, the usual formulation of a problem as under consideration, is as follows.[1])

Given the pn functions of $x^1, \ldots, x^n$:

$$v_k^1, \ldots, v_k^n \quad ; \quad k = 1, \ldots, p$$

(the contravariant characteristic numbers of the vectors $\mathbf{v}$), and the qn functions:

$$w_{l_1}, \ldots, w_{l_n} \quad ; \quad l = 1, \ldots, q$$

(the covariant characteristic numbers of the vectors $\mathbf{w}$), satisfying the relation:

$$\sum_{\lambda}^{1,\ldots,n} v_k^{\lambda} w_{l\lambda} = 0$$

equivalent to

$$\mathbf{v}_k \cdot \mathbf{w}_l = 0,$$

we ask, when the system of the total differential-equations

$$\left\| \begin{matrix} dx^1 & \ldots & dx^n \\ v_1^1 & \ldots & v_1^n \\ \vdots & & \\ v_p^1 & \ldots & v_p^n \end{matrix} \right\| = 0$$

equivalent to

$$d\mathbf{x} \frown \mathbf{v}_1 \ldots \frown \mathbf{v}_p = 0$$

is perfectly integrable.

If $\mathbf{r}$ and $\mathbf{s}$ are two vectors, lying in the p-vector ${}_p\mathbf{v}$, and consequently satisfying the relations:

$$\mathbf{w}_l \cdot \mathbf{r} = 0 \quad , \quad \mathbf{w}_l \cdot \mathbf{s} = 0 \quad , \quad l = 1, \ldots, q$$

which is equivalent to:

$$\sum_{\lambda}^{1,\ldots,n} w_{l\lambda} r^{\lambda} = 0 \quad , \quad \sum_{\lambda}^{1,\ldots,n} w_{l\lambda} s^{\lambda} = 0 \quad , \quad l = 1, \ldots, q,$$

but being otherwise arbitrary, the conditions of integrability are, as known:

$$\sum_{\mu,\nu}^{1,\ldots,n} \left(\frac{\partial w_{l\mu}}{\partial x^{\nu}} - \frac{\partial w_{l\nu}}{\partial x^{\mu}} \right) r^{\mu} s^{\nu} = 0 ; \quad l = 1, \ldots, q.$$

These equations are generally covariant[2]) and are equivalent to:

$$\mathbf{r} \frown \mathbf{s} \; {}^2 \nabla \frown \mathbf{w}_l = 0 ; \quad l = 1, \ldots, q$$

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[1]) Cf. e.g. E. VON WEBER, Vorlesungen über das PFAFF'sche Problem, pages 93 and f.f.

[2]) Owing to the circumstance that the expression $\frac{\partial w_{l\mu}}{\partial x^{\nu}} - \frac{\partial w_{l\nu}}{\partial x^{\mu}}$ is generally covariant.

Now it follows from this mode of notation that they may be replaced by the invariant equation:

$$\mathrm{r} \frown \mathrm{s} \overset{2}{.} \nabla \frown (\mathrm{w}_1 \overset{\frown}{\ldots} \mathrm{w}_q) = \mathrm{r} \frown \mathrm{s} \overset{2}{.} \nabla \frown {}_q\mathrm{w} = 0,$$

or, as follows from the preceding, still more simplified without making use of two auxiliary vectors:

$${}_p\mathrm{v} \overset{2}{.} \nabla \frown {}_q\mathrm{w} = 0.\ ^{1})$$

But this equation is identically true, $\nabla \frown {}_q\mathbf{w}$ being a multiple-sum of isomers of $\nabla_q\mathbf{w}$, and $\nabla_q\mathbf{w}$ being zero.

As the plane tangent-spaces of p_j dimensions, in the various points of the spaces P_j have p_j-directions, which by means of geodetic motion pass into each other and in the invariant p_j-direction in O, but never in any other direction, two spaces P_j can therefore never intersect. A geodetic line in X_n, which has a linear element in common with a definite space P_j, is apparently altogether contained in that space and in that space it is geodetic too. Hence two different spaces P_j can never be tangent to one another. Therefore we call the spaces P_j parallel ones. As any geodetic line having two points in common with a P_j space, falls completely within that space, which will be proved later on, we call a P_j space geodetic. The r obtained systems of parallel geodetic spaces $P_1, \ldots, P_r$ are at every point of X_n perfectly perpendicular to each other.

We shall first contemplate the case $r = 2$, $p_1 = p$, $p_2 = q$. The parameter-spaces of $n-1$ dimensions of the primitive variables $x^1, \ldots, x^p$ are placed thus that each of them contains ∞^{n-p-1} spaces P, those of $x^{p+1}, \ldots, x^n$ likewise with regard to the spaces Q. At every point we place the mutually perfectly perpendicular p- resp. q-vectors ${}_p\mathbf{v}$ and ${}_q\mathbf{w}$. The measure-vectors $\mathbf{e}'_\varkappa$, $\varkappa = 1, \ldots, p$ are then situated in ${}_p\mathbf{v}$ and for the measure-vectors $\mathbf{e}'_\mu$, $\mu = p+1, \ldots, n$ the same holds good with regard to ${}_q\mathbf{w}$. Because $\mathbf{e}'_\varkappa \perp \mathbf{e}'_\mu$ we have

$$g_{\varkappa\mu} = \mathbf{e}'_\varkappa \cdot \mathbf{e}'_\mu = 0 \quad ; \quad \begin{array}{l} \varkappa = 1, \ldots, p \\ \mu = p+1, \ldots, n \end{array}$$

hence the quadratic form ds^2 may be written thus:

$$ds^2 = \sum_{\varkappa,\lambda}^{1,\ldots,p} g_{\varkappa\lambda}\, dx^\varkappa\, dx^\lambda + \sum_{\mu,\nu}^{p+1,\ldots,n} g_{\mu\nu}\, dx^\mu\, dx^\nu$$

Now may be demonstrated, that $g_{\varkappa\lambda}$ is independent of $x^{p+1}, \ldots, x^n$, and likewise $g_{\mu\nu}$ is independent of $x^1, \ldots, x^p$. It is always possible to choose a scalar k as function of $x^1, \ldots, x^n$, so that:

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1) This equation can also be obtained very easily by means of the direct analysis used here. Another form of the same equation is:

$${}_q\mathrm{w} \overset{1}{.} (\nabla \overset{1}{.} {}_p\mathrm{v}) = 0.$$

$$ {}_p\mathrm{v} = \mathbf{k}\,\mathbf{e}_1' \ldots\frown\ldots \mathbf{e}_p $$

Consequently:

$$ \nabla k\, \mathbf{e}_1' \ldots \mathbf{e}_p' = (\nabla k)(\mathbf{e}_1' \ldots \mathbf{e}_p') + k \sum_{\varkappa}^{1,\ldots,p} \nabla (\mathbf{a}\,.\,\mathbf{e}_\varkappa)(\mathbf{e}_1' \ldots \mathbf{e}_{\varkappa-1}'\; \mathbf{a}\; \mathbf{e}_{\varkappa+1}' \ldots \mathbf{e}_p) = 0. $$

By complete transvection with:

$$ \mathbf{e}_p \ldots \mathbf{e}_{\varkappa+1}\; \mathbf{e}_\mu\; \mathbf{e}_{\varkappa-1} \ldots \mathbf{e}_1\; \mathbf{e}_\nu' $$

all the terms except the $(\varkappa+1)$-th give zero, hence:

$$ \left(\frac{\partial}{\partial x^\varkappa} a_\nu\right) a_\mu = 0 \qquad \begin{array}{l} \varkappa = 1, \ldots, p \\ \mu, \nu = p+1, \ldots, n. \end{array} $$

Now:

$$ g_{\mu\nu} = a_\mu\, a_\nu\,, $$

and

$$ \left(\frac{\partial}{\partial x_\varkappa} a_\nu\right) a_\mu = a_{\varkappa\nu}\, a_\mu = \begin{bmatrix} \varkappa\nu \\ \mu \end{bmatrix} = a_{\mu\varkappa}\, a_\nu, $$

thus

$$ \frac{\partial g_{\mu\nu}}{\partial a^\varkappa} = a_{\mu\varkappa}\, a_\nu + a_\mu\, a_{\nu\varkappa} = 0. $$

Hence the linear element in the Q spaces is independent of $x^1, \ldots, x^p$; the corresponding property of the P spaces relative to $x^{p+1}, \ldots, x^n$ is similarly demonstrated.

This property can also be expressed in the following manner.

II. *If in a general space X_n is placed a system of ∞^{n-p} parallel geodetical spaces of p dimensions P, having perfectly perpendicular to it a system of ∞^p similar spaces Q of $n-p$ dimensions, a figure in a definite P-space will be congruently projected by the Q-spaces on all the other P-spaces.*

For $p=1$ this is the well-known property that the distance of two definite Q-spaces measured along the P-lines is constant. So we can here introduce in this case for primitive variable x^1 the curve length measured along these lines from a definite space, the spaces remaining parameter-spaces. Hence the linear element may then be expressed in the following way:

$$ ds^2 = dx^{1\,2} + \sum_{\mu,\nu}^{2,\ldots,n} g_{\mu\nu}\, dx^\mu\, dx^\nu $$

in which the $g_{\mu\nu}$ do not depend on x^1. [1]).

As, however, a quadratic differential form in $n-1$ variables can

[1]) This formula has already been derived by T. Levi Civita. Nozione di parallelismo in una varietà qualunque e consequente specificazione geometrica della curvatura Riemanniana. Rend. di Pal. 42 (17).

always be written as a sum of $\frac{n(n-1)}{2}$ quadrates of complete differentials, ds^2 can be reduced to a sum of $\frac{n(n-1)}{2} + 1$ similar quadrates. Hence the space X_n can be placed in this case in a euclidian space of

$$\frac{n(n-1)}{2} + 1 = \frac{(n-1)(n-2)}{2} + n$$

dimensions. As the number of degrees of freedom of the geodetic moving system amounts exactly to

$$\frac{(n-1)(n-2)}{2},$$

the required proof has been furnished.

If we now return to the case $r=2$, $p_1 = p$, $q_2 = q$, the number of degrees of freedom is $\frac{p(p-1)}{2} + \frac{q(q-1)}{2}$. The quadratic form breaks up into two forms, which may be written as a sum of $\frac{q(q+1)}{2}$ resp. $\frac{p(p+1)}{2}$ quadrates. Therefore the space X_n can be placed within a euclidian space of

$$\frac{p(p+1)}{2} + \frac{q(q+1)}{2} = \frac{p(p-1)}{2} + \frac{q(q-1)}{2} + n$$

dimensions, and here again the required proof has been furnished.

The case $q > 2$ may be reduced to the preceding one. For this purpose the differential form is divided into two. The spaces of one of the systems, say P_2, then again contain themselves at least two perfectly perpendicular systems of parallel geodetic spaces. Then the second part of the differential form is once more divided etc.

If owing to the existence of the P_j-system the division of the differential form is:

$$ds^2 = \mathbf{a} \overset{2}{\vdots} d\mathbf{x}^{2} = \mathbf{a}_p \overset{2}{\vdots} d\mathbf{x}^{2} + \mathbf{a}_q \overset{2}{\vdots} d\mathbf{x}^{2},$$

in which $\mathbf{a}_p$ and $\mathbf{a}_q$ are the ideal radices of the two parts of the fundamentaltensor $\mathbf{a}^2$, the differential equation of a geodetic line will be:

$$d\frac{d\mathbf{x}}{ds} = d\left(\mathbf{a}\cdot\frac{d\mathbf{x}}{ds}\right)\mathbf{a} = d\left(\mathbf{a}_p\cdot\frac{d\mathbf{x}}{ds}\right)\mathbf{a}_p + d\left(\mathbf{a}_q\cdot\frac{d\mathbf{x}}{ds}\right)\mathbf{a}_q = 0.$$

$\mathbf{a}_p$ compossing itself only of the measure vectors $\mathbf{e}'_1, \ldots, \mathbf{e}'_p$ and $\mathbf{a}_q$ only of $\mathbf{e}'_{p+1}, \ldots, \mathbf{e}'_n$, we have:

$$d\left(\mathbf{a}_p \cdot \frac{d\mathbf{x}}{ds}\right)\mathbf{a}_p = 0 \quad , \quad d\left(\mathbf{a}_q \cdot \frac{d\mathbf{x}}{ds}\right)\mathbf{a}_q = 0,$$

from which the property is inferred:

III. *In the proposition made in formulating II the projection of a geodetic line by means of Q spaces on a P space, or vice versa, is, as far as existent, a geodetic one itself.*

If two points A and B are situated in a P space P_1, the projections of these points on all spaces coincide. Hence the projections on a Q space of the line AB geodetic in X_n passes twice through the same point, being at the same time geodetic in Q, which is only possible when that projection has degenerated into a point. But then the geodetic line AB must be situated altogether in a P space, e.g. in the present case in P_1.

Hence any geodetic line, having two points in common with a P-space, is entirely contained in that space.