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**Physics.** — "*Magnetic properties of cubic lattices.*" By Prof. L. S. ORNSTEIN and Dr. F. ZERNIKE. (Communicated by Prof. H. A. LORENTZ.)

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The well-known model of EWING has been treated more in detail by different scientists. A few have taken the very unsatisfactory standpoint, that elementary magnets are distributed at random in space<sup>1)</sup>. More in accordance with reality is the supposition, from which W. PEDDIE<sup>2)</sup>, and later on also HONDA and OKUBA<sup>3)</sup> have started, that the magnetic particles are arranged in a cubic lattice. The reasonings however show two important fallacies.

In the first place they neglected the demagnetising force in a sphere; accordingly they think that dipoles cannot yield a result, which made them unnecessarily consider magnets of finite length. In the second place they considered only those rotations at the research of stability, in which the magnetic axes of all particles are moved in mutual parallelism.

As will be shown hereafter, the consequence of this unfounded limitation in the freedom of motion of the particles is that the stability becomes much greater than is in reality the case.

If we sweep this limitation, we find that the arrangement of magnetic atoms in a cubic lattice is unstable without exterior field. A body of such a structure, can therefore possess no coercitive force.

§ 1. We consider a cubic lattice with edge  $d$ . In the corners of the lattices we imagine dipoles possessing the strength  $p$ , and which can rotate freely. Be those dipoles directed all parallel to an edge of the lattice by a strong exterior field  $H$ . Now we put the question how far the exterior of the field must be weakened to reach the limit of stability. If the system without exterior field is stable the intensity  $H_g$  at which this is the case, will be negative.

The magnetic properties of the lattice considered will consequently

<sup>1)</sup> GANS. u. HERTZ, Zeitsch. für Mathematik und Physik.

<sup>2)</sup> Edinburgh Proc. 195 and 7.

<sup>3)</sup> Phys. Review, X, 1917, p. 705.

— if  $H_g$  is negative — be roughly speaking analogous with those of a ferro-magnetic body with hysteresis. If on the contrary we find a positive value for  $H_g$ , we have to do with a body without hysteresis that can only be magnetised up to saturation by a strong field  $H_g$ . With a weaker exterior field the magnetic atoms will not remain totally directed and consequently  $M$  will decrease. We shall deduce for that case the connection between the intensity  $H$  and magnetisation, in other words: the permeability.

In order to find from  $H_g$  the coercitive force  $H_c$ , we must bear in mind that the latter is defined as the negative *interior* field required to make the magnetisation change its sign. This interior field will always be found by adding the field  $H_{con}$ , which is caused by the magnetised body itself, to the exterior field  $H_g$ . The field  $H_{con}$  must be calculated on the supposition, that the body has a continuous space-magnetisation.

So we have

$$H = H_g + H_{con} \text{ and especially } H_c = -H_g - H_{con}.$$

Here  $-H_{con}$  is the so-called demagnetising force. By this definition  $H_c$  will become independent of the form of the body considered, which consequently is not the case for  $H_g$ . In the preceding paragraph we must consequently read for  $H_g$  everywhere  $H_g + H_{con}$ . In our calculation we shall always take the limit spherical. Then  $H_{con}$  is  $-\frac{1}{3}M$ .

It is easy to demonstrate that  $H_g$  becomes  $=0$  when we impose on the turning of the atoms the limitation discussed above that their axes always remain parallel. For this purpose we only have to sum up the reciprocal energy of two dipoles over the whole lattice. From considerations of symmetry we then find that this sum is zero.<sup>1)</sup>

We shall give another proof of the theorem mentioned, the principle of which can also be useful for our further calculation.

We choose a system of axes parallel with the edge of the lattice and take the origin in one of its points. We imagine in all the points of the lattice except in the origin, Northpoles of unity strength, and we imagine the lattice limited by a very large sphere about 0. Let  $V_0(x,y,z)$  represent the potential in a point  $x,y,z$ . The potential of dipoles with moment  $p$  in the  $x$ -direction is  $p \frac{\partial V_0}{\partial x}$ ,

the intensity in 0 is consequently  $p \frac{\partial^2 V_0}{\partial x^2}$ ,  $p \frac{\partial^2 V_0}{\partial y \partial x}$ ,  $p \frac{\partial^2 V_0}{\partial z \partial x}$  respectively in the  $x$ ,  $y$  and  $z$ -direction. The potential energy of a dipole

<sup>1)</sup> H. A. LORENTZ, Theory of Electrons. Note 55, p. 208.

with moment  $p'$  in 0 will consequently be  $pp' \frac{\partial^2 V_0}{\partial x^2}$ , when this dipole was also directed along the  $x$ -axis, whilst it amounts to  $-pp' \frac{\partial^2 V_0}{\partial y \partial x}$ , when the last dipole is directed along the axis of  $y$ .

When we place in all corners equally directed dipoles, we can dissolve these in dipoles according to the direction of the axes in components with moments  $p_x, p_y, p_z$ . And so the potential energy of the dipole in the origin is:

$$-\left\{ p_x^2 \frac{\partial^2 V_0}{\partial x^2} + p_y^2 \frac{\partial^2 V_0}{\partial y^2} + p_z^2 \frac{\partial^2 V_0}{\partial z^2} + 2p_x p_y \frac{\partial^2 V_0}{\partial x \partial y} + 2p_x p_z \frac{\partial^2 V_0}{\partial x \partial z} + 2p_z p_y \frac{\partial^2 V_0}{\partial z \partial y} \right\}.$$

On account of the symmetry the three mixed differential-quotients are zero, and we have further  $\frac{\partial^2 V_0}{\partial x^2} = \frac{\partial^2 V_0}{\partial y^2} = \frac{\partial^2 V_0}{\partial z^2}$ . Consequently these differential-quotients are also zero because  $V_0$  fulfills the equation of LAPLACE. In consequence therefore the interior energy of the lattice is zero, independent of the direction of the dipoles (provided all dipoles are parallel). So a very weak exterior field will be sufficient to let all dipoles assume the directions of this field, in other words:  $H_q$  is zero.

The same result holds good for the two other BRAVAIS cubic arrangements: the centred cubic and the plane-centred cubic lattice. The limitation used thus yields a coercitive force which is equal to one third of the magnetisation of saturation. For steel the coercitive force is at least 80 times smaller.

2. In what follows we shall want the potential  $V$  of a rectangular lattice with unequal edges  $a, b$ , and  $c$  for the case that every corner carries a pole of unity strength; this potential depends upon the form of the boundary even if we imagine it at great distance. We shall avoid the difficulties of the boundary by the following artifice. Besides the point-charges 1 in the corners we give the body a homogeneous space-charge of  $-1$  per volume  $a, b, c$ . In total the body is thus uncharged and the parts at a great distance of the particle considered have a vanishing influence. So we are able to calculate the potential  $V'$  for this case of a lattice infinitely extended in all directions. From this we shall then find  $V$  for the case of a sphere by adding the potential in a homogeneous sphere with a charge-

density  $+\frac{1}{abc}$ , which with exception of a constant is equal to  $-\frac{x^2+y^2+z^2}{6abc}$ .

We begin by calculating the potential  $U$ , which is caused by the charges lying between the planes  $z = \pm \frac{1}{2}c$ . Evidently  $U$  is a periodical function of  $x$  and  $y$  with the periods  $a$  and  $b$ . So it may be represented by a double series of FOURIER:

$$U = \sum Z_{mn} \cos \frac{2\pi m}{a} x \cos \frac{2\pi n}{b} y \quad \begin{matrix} m = 0, 1, 2, \dots \\ n = 0, 1, 2, \dots \end{matrix}$$

in which for the sake of symmetry only the cosinus appears. The coefficients  $Z_{mn}$  are functions of  $z$ , which can be determined from the equations:

$$\Delta U = \frac{1}{abc} \quad \text{for } |z| < \frac{1}{2}c \quad \Delta U = 0 \quad \text{for } |z| > \frac{1}{2}c$$

with the conditions of limit

$$U_{z=\pm\infty} = 0 \quad \left(\frac{\partial U}{\partial z}\right)_1 = \left(\frac{\partial U}{\partial z}\right)_2 \quad \text{for } z = \pm \frac{1}{2}c.$$

Now the FOURIER-series for  $z \neq 0$  may be twice differentiated, so that after substitution in these equations every term separately must fulfill the homogeneous equations, and  $Z_{00}$  the equation  $\Delta Z_{00} = \frac{1}{abc}$ .

From this we shall find

$$Z_{00} = \begin{cases} 0 & |z| > \frac{1}{2}c \\ \frac{(\frac{1}{2}c - |z|)^2}{2abc} & |z| < \frac{1}{2}c \end{cases} \quad Z_{mn} = B_{mn} e^{-l|z|} \quad l = 2\pi \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$$

In order to determine still  $B_{mn}$  we can take  $z = 0$  and use the ordinary form of coefficients:

$$\frac{ab}{4} B_{mn} = \int_0^a dx \int_0^b dy U_{z=0} \cos \frac{2\pi m}{a} x \cos \frac{2\pi n}{b} y$$

in which when  $m$  or  $n$  are zero we must have  $\frac{ab}{2}$ . For  $U_{z=0}$  we have

$$U_{z=0} = \sum_i \sum_k \left( \frac{1}{4\pi r_{ik}} + C_{ik} \right)$$

in which  $r_{ik}$  is the distance to the point  $(ia, kb)$  and  $C_{ik}$  the potential of the parallelepipedum  $abc$  with that point as centre, homogeneously filled. For the sake of convergence we shall here for a moment

introduce  $r^{-1} e^{-\epsilon r}$  as law of attraction and in the result take  $\epsilon = 0$ . Then we can write

$$U_{z=0}(\epsilon) = C(\epsilon) + \sum_i \sum_k \frac{e^{-\epsilon r_{ik}}}{4\pi r_{ik}},$$

where  $C(\epsilon)$  is the potential of the infinite space homogeneously filled, and thus is a constant, only dependent on  $\epsilon$ .

If we substitute the values of  $U(\epsilon)$  in the double integral, the term  $C(\epsilon)$  will consequently yield zero. In the other term the sum and the integration may be interchanged. The various integrals may then be united into a single one over all rectangles. And so we obtain:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\cos \frac{2\pi m}{a} x \cos \frac{2\pi n}{b} y}{4\pi \sqrt{x^2 + y^2}} e^{-\epsilon \sqrt{x^2 + y^2}} dx dy.$$

By introducing pole-coordinates this integral may be reduced to

$$\frac{1}{2\pi} \int_0^{\infty} dr e^{-\epsilon r} \int_0^{\pi} \cos(lr \sin \varphi) d\varphi = \frac{1}{2} \int_0^{\infty} J_0(lr) e^{-\epsilon r} dr = \frac{1}{2\sqrt{l^2 + \epsilon^2}}$$

and thus for  $\epsilon = 0$

$$\frac{ab}{4} B_{mn} = \frac{1}{2l}.$$

The potential found can further easily be summed up for all the layers of distance  $c$  in which the lattice may be divided by planes perpendicularly to the  $z$ -axis. In a point for which  $0 < z < \frac{1}{2}c$  all layers under the point yield

$$\begin{aligned} \frac{(\frac{1}{2}c - z)^2}{2abc} + \sum B_{mn} \{e^{-lz} + e^{-l(z+c)} + e^{-l(z+2c)} + \dots\} \cos \frac{2\pi m}{a} x \cos \frac{2\pi n}{b} y = \\ = \frac{(\frac{1}{2}c - z)^2}{2abc} + \sum B_{mn} \frac{e^{-lz}}{1 - e^{-lc}} \cos \frac{2\pi m}{a} x \cos \frac{2\pi n}{b} y \end{aligned}$$

and all planes above it

$$\sum B_{mn} \frac{e^{-l(c-z)}}{1 - e^{-lc}} \cos \frac{2\pi m}{a} x \cos \frac{2\pi n}{b} y$$

so

$$V' = \frac{(\frac{1}{2}c - z)^2}{2abc} + \sum \frac{2}{abl} \frac{e^{-lz} + e^{-l(c-z)}}{1 - e^{-lc}} \cos \frac{2\pi m}{a} x \cos \frac{2\pi n}{b} y. \quad (1)$$

in which the sign  $\sum'$  means, that we must take half of the terms for which  $m = 0$  or  $n = 0$ , whilst there is no term for  $m = n = 0$ .

For the spherically limited lattice without space-charge we ultimately find

$$V = -\frac{x^2 + y^2 + z^2}{6abc} - \frac{(\frac{1}{2}c - z)^2}{2abc} + S \quad (2)$$

where  $S$  represents the series  $\Sigma'$  of (1).

Formula (1) evidently holds good for  $0 < z < c$ .

From the potential  $V$  determined in this way we can find as above the potential energy of a dipole with the components  $p'_x, p'_y, p'_z$ , when the latter is placed in a point  $(x, y, z)$  of the field caused by dipoles  $p_x, p_y, p_z$  in the corners of the lattice.

The expressions

$$-\left( p_x p'_x \frac{\partial^2 V}{\partial x^2} + \dots + (p_x p'_y + p_y p'_x) \frac{\partial^2 V}{\partial x \partial y} + \dots \right)$$

will represent this energy.

From (2) follows for the derivatives of second order occurring in this expression

$$\begin{aligned} \frac{\partial^2 V}{\partial x^2} &= \frac{\partial^2 S}{\partial x^2} - \frac{1}{3abc}, & \frac{\partial^2 V}{\partial y^2} &= \frac{\partial^2 S}{\partial y^2} - \frac{1}{3abc}, \\ \frac{\partial^2 V}{\partial z^2} &= \frac{\partial^2 S}{\partial z^2} + \frac{2}{3abc}, & \frac{\partial^2 V}{\partial y \partial z} &= \frac{\partial^2 S}{\partial y \partial z} \quad \dots \quad (3) \end{aligned}$$

3. In order to examine generally the stability of the system described in (2), we must study the behaviour of the quadric form, representing the potential energy as function of the variables determining the direction of all dipoles. The difficulty of this problem does not lie so much in the great number of variables, as in the impossibility to form a single series, in which the variables relating to neighbouring magnets, follow each other closely.

This difficulty does not present itself in the case when there are dipoles placed on one line at mutually equal distances. There the stability may easily be examined in the well-known fashion with the help of a determinant. We shall mention a few of the results, as they may guide us in the case that has our attention. If all magnets are directed by a field parallel to the line, the system is still stable if the field is abolished. If we apply a slowly increasing field, contrary to the magnetisation, there may be indicated a definite group of small deviations of the dipoles, for which the system first becomes unstable. These displacements are such that the magnets lie in one plane and alternately will make angles  $+\varphi$  and  $-\varphi$  with the direction of the field. The coercitive force found for this displacement of the magnets is only one third of the force found from the supposition, that all magnets turn parallelly.

For the case of the cubic lattice the analogous general method is impracticable for the reason mentioned above, but it is clear that also there we must find the combination of deviations, which most easily leads to an unstable position of the magnets. This combination must serve in calculating the coercitive-force, and it will yield for this quantity a smaller value than all other virtual displacements. Led by the analogy of the above mentioned simple case we shall examine those combinations of displacements, in which the dipoles of the lattice are distributed over two equal groups, which show an opposite displacement. Further it will be favourable in order to get unstability if the magnets with opposite deviations are placed as alternately as possible.

We can obtain a division into two groups by starting from a plane through three arbitrarily chosen points of the lattice and then using the system of parallel planes which contains all points. The dipoles lying in such planes can be assigned in a systematic way to each of the groups. The most obvious method is to count the planes alternately to the first and to the second group. Let the chosen planes divide the three edges of the elementary cube, respectively in  $l$ ,  $m$ , and  $n$  parts. The dipoles on the  $x$ -axis will belong alternately to the two groups, when  $l$  is odd, but all to the same group if  $l$  is even. From this it follows that in principle there are possible only three divisions into groups i. e. dipoles along three, along two or only along one axis belonging alternately to different groups. These divisions can be obtained by starting respectively from the octahedron, the rhomb-dodecahedron or the cube-plane.

In the same way we can examine the distribution of the points of the central cubic lattice, by paying attention to the question whether the dipoles lying on three of the cube-diagonals belong or do not belong to different groups. Here the two last cases appear to be identical. Consequently there are only two possibilities, which belong respectively to the octaeder and the rhomb.-dodecahedron plane. When we consider the distributions of the plane-centred cubic lattice we can take three diagonals in the sides which meet in one corner. Then the first and the third case are identical and belong to the octahedron-plane, the second case belongs to the cube-plane.

With each of the lattices mentioned we meet with a way of the deviations that will yield no sharper criterion for the stability than the deviation in parallel of all dipoles. These are the distributions that belong to the octahedron-plane. For it is evident that for each half separately the equivalency of the three directions of axes still exists. Analogous to what has been discussed sub 1 it

holds good not only for each part separately, but also for the parts mutually, that the energy is zero for every position of the dipoles. The coercitive-force thus becomes again a third of the magnetisation of saturation, for other divisions into two groups a much smaller, even a negative value being found.

There are still many other divisions into groups conceivable, which perhaps may be of interest when another direction of the exterior field is chosen. So e. g. the division into three groups. In the case exclusively treated here where the field is parallel to the edge of the cube, they appeared to yield a greater value for  $H_e$  than that calculated below.

4. We shall take the  $y$ -axis in the direction of the exterior field, the  $x$ - and  $z$ -axes along the two other edges. For an arbitrary division into two halves the dipoles of which are directed parallel to the  $xy$ -plane, and form angles  $+\varphi$  and  $-\varphi$  with the  $y$ -axis, we can indicate the energy as follows. Every dipole may be decomposed into a  $y$ -component  $p \cos \varphi$  and an  $x$ -component  $p \sin \varphi$  for the one and  $-p \sin \varphi$  for the other group. The  $y$ -components form a complete cubic lattice and their mutual energy is consequently zero. In consequence of the exterior field  $H_e$  each dipole has an energy  $p H_e \cos \varphi$  and so the dipoles together an energy of  $\frac{p}{d^3} H_e \cos \varphi$  per unity of volume. Also the magnetical energy of the  $x$ -dipoles and the  $y$ -dipoles is zero on account of the cubic arrangement of these latter. The mutual energy of all  $x$ -components thus remains to be calculated. In order to determine this we imagine the  $x$ -components of the dipoles of the second group inversed in sign.

Then all are directed in the same way and their mutual energy is zero. If now we inverse the dipoles again then only the mutual energy of the two groups becomes different in sign. The energy sought for of all dipoles together is thus equal to twice the mutual energy of the two groups. We shall now calculate this with the help of (2).

Call the potential caused by unit-poles placed in the first half  $V$ , then the energy of a magnet with moment  $-p \sin \varphi$  in the field of the first group the dipoles of which have a moment  $p \sin \varphi$  is according to what preceded

$$p^2 \sin^2 \varphi \frac{\partial^2 V}{\partial x^2}$$

or per unity of volume

$$\frac{p^2}{2d^3} \sin^2 \varphi \frac{\partial^2 V}{\partial x^2}.$$

The total energy per volume-unity is thus for the system

$$-\frac{p}{d^3} H_e \cos \varphi + \frac{p^2}{d^3} \frac{\partial^2 V}{\partial x^2} \sin^2 \varphi \quad (4)$$

The second derivative with respect to  $\varphi$  of this expression is for  $\varphi = 0$

$$\frac{p}{d^3} H_e + \frac{2p^2}{d^3} \frac{\partial^2 V}{\partial x^2}$$

The energy is a minimum and the equilibrium stable as long as it is positive.

For the limiting case we have

$$H_q = -2p \frac{\partial^2 V}{\partial x^2}$$

And consequently the coercitive force becomes

$$H_c = \frac{p}{3d^3} + 2p \frac{\partial^2 V}{\partial x^2} = 2p \frac{\partial^2 S}{\partial x^2} \quad (5)$$

the last according to (3), where  $abc = 2d^3$ .

When this formula yields a negative value a positive field stronger than  $-H_c$  is necessary to make the position  $\varphi = 0$  stable.

For a weaker field we find the equilibrium-positive from the first derivative of (3)

$$H \sin \varphi + 2p \frac{\partial^2 V}{\partial x^2} \sin \varphi \cos \varphi = 0$$

or

$$\cos \varphi = -H/2p \frac{\partial^2 V}{\partial x^2}$$

The magnetisation is here  $I = \frac{p \cos \varphi}{d^3} = -\frac{1}{2d^3 \frac{\partial^2 V}{\partial x^2}} H = \beta$ . The

magnetic field within the sphere is  $U - \frac{1}{3} I = U(1 - \frac{1}{3}\beta)$  and the inductive  $U + I = U(1 + \beta)$ . The constant permeability of the matter is consequently

$$\mu = \frac{1 + \beta}{1 - \frac{1}{3}\beta} = \frac{\frac{d^3 H_c}{p} + \frac{2}{3}}{\frac{d^3 H_c}{p} - \frac{2}{3}}$$

For the divisions into two groups, which we have discussed sub 3 we can always calculate  $\frac{\partial^2 V}{\partial x^2}$  according to the series found in 2 and the relation (2), where in some cases we must turn the  $x$  and  $z$  axis

over an angle of  $45^\circ$  or interchange the  $x$ ,  $y$  and  $z$  axis. The series always show strong convergence; for the following numbers, the calculation of 8 terms was only necessary in one case.

The table given below gives the values calculated for  $H_c$  in this way. For the cases in which the lattice is unstable, the permeability is indicated in the unstable direction. In the third to the fifth column the values of  $a$ ,  $b$  and  $c$  used in the calculation are mentioned.

Lattice	Varying planes	$\frac{a}{d}$	$\frac{b}{d}$	$\frac{z}{d} = \frac{c}{2d}$	$\frac{H_c}{M}$	$\mu$
Cubic	111				$\frac{1}{3}$ <b>+1.104</b>	
	010	1	1	1	-0.0521	14
					-0.0521	14
	011	1	$\sqrt{2}$	$\frac{1}{2}\sqrt{2}$	+0.546 +0.546	
					-0.0925	3.2
Centred cubic	111				$\frac{1}{3}$ +0.440	
	011	1	$\sqrt{2}$	$\frac{1}{2}\sqrt{2}$	+0.440 +0.1209	
Face centred cubic	111				$\frac{1}{3}$ +0.668	
	001	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}$	+0.1610 +0.1610	

We must remark, that for the three cases always occurring with equal values of  $a$ ,  $b$  and  $c$  only one calculation was necessary. For we can interpret them as belonging to one and the same division in two groups, but with the exterior field successively in the three directions of the axes. According to (5) the values of  $H_c$  belonging to it will taken together be equal to  $\frac{p}{a^3} = M$ . Moreover on account of symmetry only two are always equal to each other, so that only one must be calculated. The smallest value of  $H_c$  for each lattice is printed in bold type, the others have importance mainly for the

calculation. So the centred and the plane-centred show coercitive force, and even in a degree much too great for steel e.g.

In all our considerations we have left unconsidered the heat-movement. The magnetic properties found here are apparently always represented by magnetisation-curves consisting of straight lines. These broken straight lines will no doubt be rounded off by the heat-movement, and consequently resemble more those under observation. Another cause for the rounding off must be looked for in the fact that the real materials are aggregates of crystals lying at random in all directions. For the present we draw the attention for the effect of this cause to the well-known theories of PIERRE WEISS.

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