Physics. — *"Contribution to the theory of adiabatic invariants."* (Preliminary communication). [1]) By G. KRUTKOW. (Communicated by Prof. H. A. LORENTZ).

(Communicated in the meeting of Dec. 29, 1918).

Introduction. Any quantity that has to be quanticized, which may be called a "quantum-quantity", must satisfy two conditions:

1. it must be a function of the integrals of the equations of motion of the system under consideration. This condition is self-evident, since the quantity must not change by the motion of the system, and has therefore never been explicitly stated;

2. it must be an adiabatic invariant, i. e. it must not change when the system is submitted to a reversible adiabatic influence. This demand was first formulated by EHRENFEST and proved by means of general statistical reasoning [2]). Assuming that the adiabatic influence may be calculated by the methods of mechanism this condition follows directly from the fact, that the quantum-quantity varies abruptly, whereas the external influence may be infinitely small; the quanticizable quantity therefore cannot vary at all, it must be an adiabatic invariant.

Calling the quantum-quantity v, the integrals of the equations of motion $c_1, c_2 \ldots .$ and the adiabatic invariants $v_1, v_2 \ldots .$ the conditions (1) and (2) are expressed by

$$v = funct\,(c_1, c_2, \ldots .) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$v = funct\,(v_1, v_2, \ldots .) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

3. There is still another condition which a quantum-quantity has to satisfy: it must have a meaning which is independent of the system of co-ordinates. This condition appears to me to embody the notion of the coherence of the degrees of freedom established by PLANCK [3]). To this condition I hope to be able to return in a later

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[1]) Address delivered in the Petrograd. Phys. Ges. in Dec. 1917 and April 1918.

[2]) P. EHRENFEST. Ann. d. Phys. 51 (1916) p. 327, Phys. Zschr. (1914) p. Acad. Amsterdam 22 (1913) p. 586. Ann. d. Phys. 36 (1911) p. 98. Verh. d. D. phys. Ges. 15 (1913) p. 451.

[3]) M. PLANCK. Ann d. Phys. 50 (1916) p. 285.

paper: on this occasion it will be left out of account and we shall only deal with condition (2).

This condition imposes on us the task to find the adiabatic invariants of a given mechanical system and to look for a general method of solving the "adiabatic" problem.[1]) A method of that kind was unknown so far; the adiabatic invariants had to be guessed at and their adiabatic invariability had to be tested a posteriori. In this way the following invariants were found:

a. the quantity V of statistical mechanics, which measures the phase-extension limited by the "energy-surface"[2]);

b. the "action" calculated for a full period of a periodical system;

$$v = \int_{\tau} 2\, T\, dt\ ^{3});$$

c. the quantum integrals of the "conditionally periodic" systems;

$$v_i = \int_{0} p_i\, dq_i = 2 \int_{a_i}^{b_i} p_i\, dq_i\ .\ ^{4})$$

In what follows I shall sketch out a general method of finding adiabatic invariants and apply it to certain special cases, viz.

α. Cyclic systems. Properly speaking these systems come under the head of conditionally periodic systems; but as the conditions are particularly simple in this case and bring out the very natural character of the method, I shall discuss them separately;

β. conditionally periodic systems;

γ. ergodic systems.

Under (β) I shall only consider the limiting case, in which there are no commensurable relations between the periodicity-moduli. To the further cases and in particular their relation to the third condition stated above I hope to return on a later occasion.

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[1]) This point was specially emphasized by EHRENFEST. Compare for instance P. EHRENFEST Phil. Mag. VI Vol. 33. p. 513 (1917).

[2]) P. HERTZ. Ann. d. Phys. 33 (1910) p. 544.

[3]) L. BOLTZMANN. Prinz. d. Mechanik II p. 181. P. EHRENFEST Ann. d. Phys. 51 (1916) p. 327 Anhang.

[4]) J. M. BURGERS. Ann. d. Phys. 52 (1917) p. 195. To the papers in the Proceedings of the Amst. Acad referred to by the author I had unfortunately no access.

THE GENERAL METHOD.

1. *Definition of adiabatic invariants* [1]). We consider a mechanical system of n degrees of freedom, the equations of motion of which must be written in the Hamiltonian form

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad \dot{q}_i = \frac{\partial H}{\partial p_i}, \quad (i = 1, 2, \ldots, n) \quad . \quad . \quad . \quad (3)$$

H is here a function of the p_i and q_i. It must not contain t explicitly. Moreover it is supposed to depend on certain external co-ordinates, which we shall call the parameters a_x. These parameters may either retain constant values, in this case we have the iso-parametric problem, or they may vary, which gives the rheo-parametric problem, or they may vary very slowly [2]), which is the herpo-parametric or adiabatic problem, to which we shall give special attention.

We shall make the following assumptions:

I. None of the quantities p_i or q_i increases to infinity. The q_i are confixed within fixed limits.

II. During the time in which each q_i goes to and fro many times between its extreme values, the a_x must change by an infinitely small amount of the first order. Moreover each $\dot{a}_x$ must be approximately constant. Equations (3) must remain valid during the process. It follows from these assumptions that the herpo-parametric problem, will be obtained by putting $\dot{a}_x = const.$ in the rheo-parametric problem and then taking for all the quantities the time-average in the corresponding iso-parametric problem.

In our discussion we shall confine ourselves to one parameter a. This is not an essential limitation of the problem, but it simplifies the formulae considerably.

An adiabatic invariant is a function v of the integrationconstants $c_1, c_2, \ldots$ of the iso-parametric motion and of the parameter a, the total "adiabatic" derivative of which with respect to a disappears:

$$\frac{\overline{dv}}{da} = \frac{\partial v}{\partial a} + \frac{\partial v}{\partial c_1}\frac{\overline{dc_1}}{da} + \frac{\partial v}{\partial c_2}\frac{\overline{dc_2}}{da} + \ldots \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where the horizontal line indicates the time-average.

2. *The iso-parametric problem.* In the equations of motion (3) we

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[1]) Comp. P. EHRENFEST l. c. and J. M. BURGERS l. c.

[2]) Implicitly this condition will show itself in the fact, that HAMILTON's function only contains the parameters a_x itself and not the corresponding moments.

put $a = \text{const.}$ and integrate according to JACOBI's method. If

$$H_1 = c_1, H_2 = c_2, \ldots . H_n = c_n \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

is a set of normal integrals of the equations, from which p_i may be solved, the characteristic function

$$V = \int \sum_i F_i \, dq_i, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

may be formed, where the functions $F(q_i, c_i, a)$ represent the quantities p_i deduced from equations (5), and putting

$$\frac{\partial V}{\partial c_1} = t_1 \quad \frac{\partial V}{\partial c_2} = t_2 \quad \ldots . \frac{\partial V}{\partial c_n} = t_n \quad . \quad . \quad . \quad . \quad . \quad (7)$$

these will be the additional integrals, where

$$t_1 = t + c_1^* \quad t_2 = c_2^* \quad \ldots . t_n = c_n^* \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The quantities c_i^* are the n integration-constants. The iso-parametric problem is thereby solved.

3. *The differential equations of the rheo-parametric problem.* In order to obtain these equations we shall pass from the variable quantities p_i and q_i to the variables c_i and t_i. This is a "contact-transformation". It is obtained by means of the characteristic function

$$V(q_i, c_i, a) = \int \sum_i F_i \, dq_i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6')$$

as transformation-function

$$\frac{\partial V}{\partial q_i} = p_i \qquad \frac{\partial V}{\partial c_i} = t_i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The differential equations retain the Hamiltonian form. If a remains constant, the new Hamiltonian function is equal to the transformed old one, i.e. to c_i and the following trivial result is obtained:

$$\dot{c}_1 = 0, \dot{c}_2 = 0, \ldots . \dot{c}_n = 0 ; \dot{t}_1 = 1, \dot{t}_2 = 0, \ldots . \dot{t}_n = 0$$

We now allow a to change, i.e. we put $a = \text{function } (t)$. The transformation-function V is now an implicit function of t through the intermediary of q_i, c_i *and* a:

$$\frac{\partial V}{\partial t} = \sum_i \left(\frac{\partial V}{\partial q_i} \dot{q}_i + \frac{\partial V}{\partial c_i} \dot{c}_i \right) + \frac{\partial V}{\partial a} \dot{a}$$

The differential equations (3) retain their form all the time, but the new Hamiltonian function K now becomes

$$K = c_1 + \left(\frac{\partial V}{\partial a} \right) \dot{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

where the brackets are intended to indicate, that the derivative $\partial V/\partial a$ must be expressed in the variables c_i and t_i. The differential equations of the rheo-parametric problem therefore are as follows:

$$\left.\begin{aligned} \dot{c}_1 &= -\frac{\partial K}{\partial t_1} \quad \dot{c}_2 = -\frac{\partial K}{\partial t_2} \ldots \dot{c}_n = -\frac{\partial K}{\partial t_n} \\ \dot{t}_1 &= \frac{\partial K}{\partial c_1} \quad \dot{t}_2 = \frac{\partial K}{\partial c_2} \quad \ldots \dot{t}_n = \frac{\partial K}{\partial c_n} \end{aligned}\right\} \quad \ldots \quad (11)$$

4. *The herpo-parametric or adiabatic problem.*

To begin with we put $\dot{a} = \text{const}$. Substituting the value of K from (10) the equations (11) then assume the form:

$$\left.\begin{aligned} \dot{c}_i &= -\dot{a}\frac{\partial}{\partial t_i}\left(\frac{\partial V}{\partial a}\right) \qquad (i = 1, 2, \ldots u) \\ \dot{t}_1 &= \dot{a}\frac{\partial}{\partial c_i}\left(\frac{\partial V}{\partial a}\right) + 1 \qquad \dot{t}_x = \dot{a}\frac{\partial}{\partial c_x}\left(\frac{\partial V}{\partial a}\right) \quad (x = 2, 3, \ldots n) \end{aligned}\right\} \quad \ldots \quad (12)$$

or, indicating the differentiation with respect to the parameter a by means of a dash:

$$c'_i = -\frac{\partial}{\partial t_i}\left(\frac{\partial V}{\partial a}\right) \qquad t'_1 = \left(\frac{\partial V}{\partial a}\right) + \frac{1}{\dot{a}} \qquad t'_x = \frac{\partial}{\partial c_x}\left(\frac{\partial V}{\partial a}\right) \quad \ldots \quad (12')$$

We now only need to put the line which indicates the mean value on the left side and on the right actually to calculate the time-average in order to obtain the differential equations of the herpo-parametric or adiabatic problem. The integration, in which the said line on the left is omitted, gives the adiabatic invariants; indeed, the equations being

$$c'_i = f_i(c_i, t_i, a) \qquad t'_i = g_i(c_i, t_i, a)$$

and $\varphi(c_2, t_2, a)$ their integrals, the total differential $d\varphi/da$ owing to the equations must disappear, or

$$\frac{d\varphi}{da} \frac{\partial \varphi}{\partial a} + \sum_i \left(\frac{\partial \varphi}{\partial c_i} f_i + \frac{\partial \varphi}{\partial t_i} g\right) = 0$$

$$\frac{\partial \varphi}{\partial a} + \sum_i \left(\frac{\partial \varphi}{\partial c_i} c'_i + \frac{\partial \varphi}{\partial t_i} t'_i\right) = \frac{d\varphi}{da} = 0$$

but this is no other than equation (4), i.e. the equation which expresses the definition of adiabatic invariants.

In this manner the problem set in the introduction: to derive a general method of finding adiabatic invariants, has been solved. Before discussing the more general applications two special problems — classical ones for the quantumhypothesis — may be treated by our method.

5 (A). *The linear oscillator.* The parameter is here the frequency. The solution is as follows:

$$H=\tfrac{1}{2}p^2+\tfrac{1}{2}a^2q^2=c_1 \quad p=F=\sqrt{2c_1-a^2q^2} \quad V=\int dq\,F=\int dq\sqrt{2c_1-a^2q^2} \tag{13}$$

$$\frac{\partial V}{\partial a}=-\int dq\,\frac{aq^2}{F} \qquad \frac{\partial V}{\partial c_1}=\int\frac{dq}{F}=t_1 \quad . \quad . \quad . \quad . \tag{14}$$

$$c'_1=\frac{\partial}{\partial t_1}\left(\frac{\partial V}{\partial a}\right)=\frac{\partial}{\partial q}\left(\frac{\partial V}{\partial a}\right)\cdot\frac{dq}{dt_1}=aq^2=\frac{1}{a}a^2q^2 \quad . \quad . \quad . \tag{15}$$

The mean value of the right-hand side is c_1/a. Thus we obtain the well-known adiabatic invariant c_1/a.

B. *Body rotating about a fixed axis.* Calling the moment of inertia (the parameter) A and the moment of momentum p, we have:

$$H=\frac{1}{2A}p^2=c_1 \quad p=F=\sqrt{2\,Ac_1} \quad V=\int dqF=q\sqrt{2Ac_1} \quad . \quad . \tag{16}$$

$$\frac{\partial V}{\partial A}=\frac{c_1q}{F} \qquad \frac{\partial V}{\partial c_1}=\frac{Aq}{F}=t_1 \qquad \left(\frac{\partial V}{\partial A}\right)=\frac{c_1}{A}t_1 \quad . \quad . \tag{17}$$

$$c'_1=-\frac{\partial}{\partial_1}\left(\frac{\partial V}{\partial A}\right)=-\frac{c_1}{A} \quad . \quad . \quad . \quad . \quad . \quad . \tag{18}$$

which gives $c_1A=\text{const.}$, hence also $p=T=\sqrt{2Ac_1}=\text{const.}$

Applications.

6. *The cyclic system.* We call *cyclic* those co-ordinates which do not occur in the expression for the Hamiltonian function (ignorable co-ordinates according to Thomson and Tait's terminology). They will be indicated by $q_x\,(x=1,2,\ldots,k)$, the remaining, non-cyclic co-ordinates by $q_\lambda\,(\lambda=k+1,\ k+2,\ldots,n)$.

Hence we have

$$H=H(q_\lambda,\,q_x,\,p_\lambda;\,a) \qquad \dot{p}_x=-\frac{\partial H}{\partial q_x}=0 \qquad p_x=c_x \quad . \quad . \tag{19}$$

The characteristic function now will be

$$V=\sum_x c_x q_x+W(q_\lambda,\,c_x,\,c_\lambda;\,a) \quad . \quad . \quad . \quad . \quad . \tag{20}$$

We shall further assume that c_n is the energy-constant; we then obtain

$$\frac{\partial V}{\partial c_x}=q_x+\frac{\partial W}{\partial c_x}=t_x \qquad \frac{\partial V}{\partial c_\lambda}=\frac{\partial W}{\partial c_\lambda}=t_\lambda \quad . \quad . \quad . \tag{20'}$$

where all t, excepting t_n, are constants and $t_n=t+\text{const.}$

From the equations for t_λ the q_λ may be derived as functions of the c_x, c_λ and t_λ. Further we have

$$\frac{\partial V}{\partial a} = \frac{\partial W}{\partial a} \quad \ldots\ldots\ldots \quad (21)$$

from which it follows, that $\frac{\partial V}{\partial a}$ is a function of the c_x, c_λ and t_γ and independent of the t_x, hence:

$$c'_x = -\frac{\partial}{\partial t_x}\left(\frac{\partial V}{\partial a}\right) = 0 \qquad c_x = \text{adiab. Invar} \quad \ldots \quad (22)$$

In the case, when $K = n$, i. e. when all the co-ordinates are cyclic, we have

$$H = H(p_i) \qquad p_i = c_i \qquad V = \sum_i c_i q_i \qquad (i = 1, 2, \ldots n) \; . \quad (23)$$

If the energy-constant c is a function of the c_i which is found by substituting the c_i in H, the new Hamiltonian function will be

$$K = C + \left(\frac{\partial V}{\partial a}\right)$$

But $\frac{\partial V}{\partial a}$ is equal to zero, hence all c_i are adiabatic invariants. As the co-ordinates corresponding to the moments c_i the old co-ordinates q_i must be taken — they are all linear functions of the time. This fact brings out the natural character of the method, hence it appears to be a very natural generalization of the method of reasoning followed in the theory of cyclic systems.

The simplest instance of a cyclic system — a body rotating about a fixed axis — was discussed above under 5.

7. *The conditionally periodic system.* As is well-known a conditionally periodic system possesses besides the energy-integral $(n-1)$ other integrals which are of the second degree with respect to the moments. They all contain the moments only as squares, not as products, thus only p_i^2 and no $p_i p_x$. Solving the p_i^2 we get

$$p_i^2 = \psi_i(q_i, c_1 \ldots c_n) \qquad p_i = \sqrt{\psi_i} \quad (c_1 \ldots c_n \text{ integration const.}) \quad (24)$$

therefore each p_i depends only on the corresponding co-ordinates q_i. If the initial value of q_i lies in between two simple successive roots a_i and b_i of the equation $\psi_i = 0$, the co-ordinate displays librational motion. We shall here consider the case in which this holds for all the co-ordinates q_i.

The characteristic function V is now given by

$$V = \sum_i \int dq_i \sqrt{\psi_i} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

hence

$$\frac{\partial V}{\partial a} = \sum_i \int dq_i \frac{\partial \sqrt{\psi_i}}{\partial a}, \quad \frac{\partial V}{\partial c_x} = \sum_i \int dq_i \frac{\partial \sqrt{4_i}}{\partial c_x} = t_x \quad . \quad . \quad (26, 27)$$

The first group of the rheo-parametric differential equations has the following form

$$\dot{c}_x = -\frac{\partial}{\partial t_x}\left(\frac{\partial V}{\partial a}\dot{a}\right)$$

or putting $a = const$ and substituting for $\partial V/\partial a$ its value from (26)

$$\dot{c}'_x = -\frac{\partial}{\partial t_x}\left(\sum_i \int dqi \frac{\partial \sqrt{\psi_i}}{\partial a}\right) \quad . \quad . \quad . \quad . \quad . \quad (28)$$

Now it follows from (27) that the integral within the brackets depends on t_x only through the intermediary of the q_i (on c_x it depends explicitly and also through the q_i; hence

$$\dot{c}'_x = -\left(\sum_i \frac{\partial \sqrt{\psi_i}}{\partial a}\frac{\partial q_i}{\partial t_x}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (28')$$

We have now on the left to put the line indicating the mean value and on the right actually to calculate the time-average. For this we need the following propositions: the curve of the orbit fills the whole region $a_i \leq q_i \leq b_i$ $(t = 1, 2 \ldots . n)$, the filling being everywhere "dense" [1]). The time-mean of an arbitrary function f of the phase of motion of the system, taken over an interval of time τ increasing indefinitely, may be replaced by the space-mean of the function over this region [2]). In the variable quantities c_i, t_i in order to compute the space-mean we have to integrate the function f over a "period-cell" and divide by the "volume" of the cell

$$\Omega = |\omega_{ix}| \qquad \left(\omega_{ix} = \int dq_i \frac{\partial \sqrt{\psi_i}}{\partial c_x} = 2\int_{a_i}^{b_i} dq_i \frac{\partial \sqrt{\psi_i}}{\partial c_x}\right) \quad . \quad . \quad (29)$$

hence:

$$\bar{f} = lim \frac{1}{\tau}\int_\tau dt\, f = \frac{1}{\Omega}\int \ldots \int_\Omega dt_1 \ldots dt_n f \quad . \quad . \quad . \quad (30)$$

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[1]) P. Stäckel. Math. Ann., 54 (1901) p. 86. In the proof it is assumed that between the ωix (equation 29) no relations of commensurability exist.

[2]) Comp. J. M. Burgers l. c. p. 200.

Representing by Ω_{ix} the sub-determinants corresponding to the ω_{ix} the mean value of the right-hand side of (28′) after some reduction may be written in the form

$$-\frac{1}{\Omega}\sum_i \Omega_{ix}\, 2\int_{a_i}^{b_i}\frac{\partial\sqrt{\psi_i}}{\partial a}dq_i \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

or putting

$$v_i = 2\int_{a_i}^{b_i} dq_i\sqrt{\psi_i} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

and noticing that the integrand disappears at the limits of the integral, also in this form

$$-\frac{1}{\Omega}\sum_i \Omega_{ix}\frac{\partial v_i}{\partial a} \quad . \quad . \quad . \quad . \quad . \quad . \quad (31')$$

Hence we obtain the relation

$$\overline{c'_x} + \frac{1}{\Omega}\sum_i \Omega_{ix}\frac{\partial v_i}{\partial a} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

We now solve this set of equations for the derivatives $\partial v_i/\partial a$

$$\frac{\partial v_i}{\partial a} + \sum_x \omega_{ix}\,\overline{c'_x} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

Instead of ω_{ix} we may write

$$\omega_{ix} = \frac{\partial}{\partial c_x}\, 2\int_{a_i}^{b_i} dq_i\sqrt{\psi_i} = \frac{\partial v_i}{\partial c_x} \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

Hence instead of (34)

$$\frac{\partial v_i}{\partial a} + \sum_x \frac{\partial v_i}{\partial c_x}\overline{c'_x} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

The v_i are functions of a and of the c_x; the left-hand side of (36) therefore is the complete "adiabatic" derivative $\frac{\overline{dv_i}}{da}$: hence the v_i are adiabatic invariants.

The above invariants have been obtained by submitting to the series of operations prescribed by our method the first group of our rheo-parametic equations, those for c'_x. We shall now show, that we need not proceed and that we need not consider the second group of equations, those for t'_x, at all, supposing our object to be to find the condition mentioned in the introduction under (2) which every quantum-quantity of the conditionally periodic system has to

satisfy. We may briefly formulate the condition mentioned under (1) by saying, that each quantum-quantity v must retain a constant value along the "orbit" of our system; it is a function of those integrals of the iso-parametic system which do not contain the time t explicitly, i.e. of $c_1, \ldots . c_n, t_2, \ldots . t_n$. The time-mean of v is therefore v itself. We may then replace this time-mean by the space-mean for the cell Ω; this being a function of the c_x and a, v is a function of the c_x and independent of $t_2, \ldots . t_n$. Now we have found n adiabatic invariants, functions of c and a; the remaining ones, which have not been computed, all contain the t_2, hence we do not need these for our present purpose. The conditions (1) and (2) for a conditionally periodic system without commensurate relations between the w_{ix} therefore assume the form

$$\left.\begin{aligned} v &= funct\,(c_1, \ldots c_n\,;\, a) \\ v &= funct\,(v_1, \ldots v_n) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

where the v_x are given by equation (32). We know, that the quantum-theory chooses as quantum-quantities the v_x themselves [1]).

§ 8. *The ergodic system.* So far we have assumed that the iso-parametric problem is actually solved. Now we shall only suppose, that the energy-integral

$$H\,(p_i, q_i, a) = c_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

is given and in addition introduce the "ergodic" hypothesis that the system passes through every point of the "energy-surface" $H = c_2$ [2]). The time-mean F of a phase-function f is then given by

$$\left.\begin{aligned} \bar{f} = \frac{\int \ldots \int dp_2 \ldots dp_n\, dq_1 \ldots dq_n \frac{1}{\dot{q}_1} f}{\int \ldots \int dp_2 \ldots dp_n\, dq_1 \ldots dq_n \frac{1}{\dot{q}_1}}, \text{ [3]} \end{aligned}\right\} \quad . \quad . \quad . \quad (39)$$

the integrals being taken over the energy-surface $H = c_1$.

As a very natural specialisation of our general method we now take as transformation-function V the quantity

$$V = \int F dq_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

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[1]) K. SCHWARZSCHILD. Sitzungsber. Berlin 1916. p. 550. P. EPSTEIN. Ann. d. Phys. 50 (1916) p. 489; 51 (1916) p. 168. A. SOMMERFELD. Ann. d. Phys. 51 (1916) p. 1.

[2]) Cf P. and T. EHRENFEST. Enc. d. math. Wiss. IV 32. § 10.

[3]) L. BOLTZMANN. Gastheorie II p. 88. and seq.

F being the expression for p_1 which is obtained by solving $H = c_1$

$$p_1 = F(p_2 \ldots, p_n, q_1, \ldots, q_n, c_1; a) \quad \ldots \quad (41)$$

When this expression is substituted in $H = c_1$ the result will be an identity. By differentiating this with respect to $c_1, p_2, \ldots, p_n, q_1, \ldots, q_n$, we find

$$\left.\begin{aligned} \frac{\partial H}{\partial F} &= \frac{1}{\dfrac{\partial F}{\partial c_1}}, \quad \frac{\partial H}{\partial q_i} = -\frac{\dfrac{\partial F}{\partial q_i}}{\dfrac{\partial F}{\partial c_1}}, \quad \frac{\partial H}{\partial p_x} = -\frac{\dfrac{\partial F}{\partial p_x}}{\dfrac{\partial F}{\partial c_1}} \end{aligned}\right\} \quad \ldots \quad (42)$$

from which the Hamiltonian equations are easily derived as follows

$$\frac{\partial p_\lambda}{\partial p_1} = \frac{\partial F}{\partial q_x} \qquad \frac{\partial q_x}{\partial q_1} = -\frac{\partial F}{\partial p_x} \qquad \frac{dt}{dq_1} = \frac{\partial F}{\partial c_1} \quad \ldots \quad (42')$$

Let us now form the derivatives of the transformation-function V with respect to all the variables which it contains:

$$\frac{\partial V}{\partial q_1} = F \quad \frac{\partial V}{\partial q_x} = \int \frac{\partial F}{\partial q_x} dq_1 = p_x \quad \frac{\partial V}{\partial p_i} = \int \frac{\partial F}{\partial p_x} dq_1 = q_x \quad \frac{\partial V}{\partial c_1} = \int \frac{\partial F}{\partial c_1} dq_1 = t_1 \quad (43)$$

Evidently V forms the transition from the variables $p_1, \ldots, p_n, q_1, \ldots, q_n$ to the variables $p_2, \ldots, p_n, q_2, \ldots, q_n, c_1, t_1$. Of all the rheoparametric differential equations we only need the equation for c' here, viz.

$$c'_1 = -\frac{\partial}{\partial t_1}\left(\frac{\partial V}{\partial a}\right) = -\frac{\partial}{\partial t_1}\left(\int \frac{\partial F}{\partial a} dq_1\right) \quad \ldots \quad (44)$$

The integral inside the brackets only depends on t through q_1, hence

$$c'_1 = -\left(\frac{\partial F}{\partial a} \cdot q_1\right) \quad \ldots \quad (44')$$

We now form the mean value according to (39):

$$\left.\overline{c'_1} = \frac{-\int \ldots \int dp_2 \ldots dp_n \, dq_1 \ldots dq_n \dfrac{\partial F}{\partial a}}{\int \ldots \int dp_2 \ldots dp_n \, dq_1 \ldots dq_n \dfrac{\partial F}{\partial c_1}}\right\} \quad \ldots \quad (45)$$

where in the denominator $1/\dot{q}_1$ has been replaced by $\dfrac{\partial F}{\partial c_1}$ according to the last equation of the set (42'). It is easily seen, that the numerator and denominator are the partial derivatives with respect to a and c, respectively of a function V of the form

$$V = \int \ldots \int dp_1 \ldots dp_n \, dq_1 \ldots dq_n, \quad \ldots \quad (46)$$

the integration extending over the region enclosed by the energy-surface $H = c_1$. We thus have

$$\frac{\partial V}{\partial a} + \frac{\partial V}{\partial c_1} \frac{dc_1}{da} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

hence V is an adiabatic invariant. It can also easily be shown that this quantity has a meaning which is independent of the system of co-ordinates used; it therefore also satisfies the condition mentioned in the introduction under (3). The same is true for the quantity called v in § 3, b.

It remains to be seen under what conditions the quantities v_1 defined by equation (32) also satisfy this requirement. It may be expected that this enquiry will teach us how to quanticize systems which are "degenerated" in different ways. It also seems very probable, that this question will be decided on the lines indicated by PLANCK [1]) and SCHWARZSCHILD [2]). For instance, as regards the movement of a top on which no external forces are acting, of the three adiabatic invariants: the moment of momentum, its projection on the axes of the figure and its projection on ζ-axes of a fixed system of coordinates of arbitrary orientation (all three multiplied by 2π) only the first two may be quanticized. The "elementary region" thus will be not h^3 but $h^3(2n_1 + 1)$, where n_1 is the quantum-number corresponding to the moment of momentum. On this ground exception may be taken to EPSTEIN's calculation of the specific heat of hydrogen [3]). To all these problems — problems relating to the adaptation of the quantum-hypothesis to different cases — I hope to return soon.

The method above developed is independent of this question, it is the solution of a purely mechanical problem. It seems advisable to try and apply it to systems which cannot be integrated by a separation of the variables in HAMILTON-JACOBI's partial differential equation, e.g. to the POINSOT-motion. About this question also I hope to be able to make a communication shortly.

Petrograd, September 1918. *Physical Laboratory of the University.*

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[1]) M. PLANCK. l. c.

[2]) K. SCHWARZSCHILD. Sitzungsber. Berlin 1916. p. 550.

[3]) P. S. EPSTEIN. Ber. d. D. Phys. Ges. 1916 p. 398. Compare especially (10) on p. 401. Objections may also be made to the quanticizing proposed on p. 407, seeing that the quantum-quantities in that case are not adiabatic invariants,