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Astronomy. — *"Theory of Jupiter's Satellites. I: The intermediary orbit"*. By Prof. W. DE SITTER.

(Communicated in the meeting of March 29, 1919).

The following pages contain the elaboration of the theory which was outlined in my paper[1]) of 1918 March 23. Only the results will be given here; the computations will be published in detail in the Annals of the Observatory at Leiden. The present paper gives the determination of the intermediary orbit. As has been explained in the "Outlines", the motion of the satellites is thereby represented as a keplerian ellipse with the constant semi-axis a_i and excentricity e_i, the perijoves having the common retrograde motion $-\varkappa\tau$.

Instead of the time we use the independent variable τ, i.e. we count the time in units of 1.1221899034 mean solar days. The unit of mass is the mass of Jupiter, and the unit of length has been so chosen that the Gaussian constant $f = 1$. This unit of length is

$$a_0 = 0.0070854378 \text{ astronomical units.}$$

The adopted masses of the satellites are[2]):

$$m_1 = 0.00003796\ (1 + \lambda_1)$$
$$m_2 = 0.00002541\ (1 + \lambda_2)$$
$$m_3 = 0.00008201\ (1 + \lambda_3)$$
$$m_4 = 0.00004523\ (1 + \lambda_4).$$

The mass of the sun is $M = 1047.40\ (1 + \Sigma\, m_i)$, or

$$M = 1047.600$$

For the quantities J and K depending on the ellipticity, we take

$$J = 0.02186^5\ (1 + \lambda_0) \qquad K = 0.00259\ (1 + \lambda_5).$$

These occur exclusively in the combinations Jb^2 and Kb^4 respectively, of which the adopted values expressed in our units are

$$\log Jb^2 = 5.9969318 - 10$$
$$\log Kb^4 = 2.72766 - 10.$$

The mean distance of Jupiter from the sun, expressed in our unit, is

$$\log A = 2.865871.$$

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[1]) Outlines of a new theory of Jupiter's satellites, these Proceedings Vol XX, p. 1289.

[2]) See Annalen van de Sterrewacht te Leiden, Deel XII, Eerste Stuk, Appendix.

The mean anomalies of the satellites in the intermediary orbit are

$$l_i = c_i \tau$$

and the mean longitudes are

$$\lambda_i = \lambda_{00} + \pi_{i_0} + (c_i - \varkappa)\,\tau$$

where λ_{00} is the longitude of the opposition of II and III, which is taken as origin, and

$$\pi_{10} = \pi_{30} = \pi_{40} = 0, \qquad \pi_{20} = 180^\circ.$$

We have

$$\begin{aligned} c_1 &= 4 \\ c_2 &= 2 \\ c_3 &= 1 \\ c_4 &= 0.43697298 \\ \varkappa &= 0.0144839248. \end{aligned}$$

The values of c_i, $\varkappa$, M and A are considered as absolutely exact and not subject to correction. The corrections λ_i to the adopted masses and ellipticity are included explicitly in the formulas.

The perturbative function is given by the formula[1] (15). This is developed to a series of the form:

$$\begin{aligned} R = &-\tfrac{1}{2}\frac{(c_i-\varkappa)(1-\mu_i)}{1+m_i}\,\Sigma\,(C_i)^n_q\, e^n_i \cos q l_i \\ &+\tfrac{1}{2}\frac{(c_i-\varkappa)(1-\mu)}{1+m_i}\,\Sigma_j \frac{\mathrm{a}_i}{\mathrm{a}_j} m_j \Sigma\, \Pi^{n,n'}_{q,q'}(b_{ij})_p\, e^n_i\, e^{n'}_j \cos(p\lambda_j - p\lambda_i + q l_i + q' l_j) \\ &-\frac{(c_i-\varkappa)(1-\mu_i)}{1+m_i}\,\Sigma_j \frac{\mathrm{a}_i^2}{\mathrm{a}_j^2} m_j\, \Sigma\, Q^{n,n'}_{q,q'}\, e^n_i\, e^{n'}_j \cos(\lambda_j - \lambda_i + q l_i + q' l_j) \end{aligned}$$

The upper line of this formula contains the terms depending on the ellipticity of the planet and on the indetermined parameter μ_i. I will call this part the "additional" part of the perturbative function. It contains the elements of the perturbed satellite only.

The second line is the principal part of the perturbative function. It has been written down for the case of an inner satellite perturbed by an outer one, i.e. for $j > i$. If the perturbed satellite is the outer one, i.e. if $i > j$, then the factor $\mathrm{a}_i/\mathrm{a}_j$ must be omitted, and $\Pi^{n,n'}_{q,q'}(b_{ij})_p$ must be replaced by $\Pi^{n'\,n}_{-q',-q}(b_{ji})_p$. The $\Pi^{n,n'}_{q,q'}$ are the well known operators of Newcomb and $(b_{ij})_p$ are the coefficients of Laplace in the development of a'/Δ.

[1]) These Procedings Vol XX, page 1298.

The third line contains the complementary part of the perturbative function. The coefficients $Q^{n,n'}_{q,q'}$ differ from those of the usual development by the introduction of the terms with J_j and K_j. The values of $(C_i)^n_q$ and $Q^{n,n'}_{q,q'}$ have been given in Leiden Annals XII, 1, pages 22 and 23.

For the determination of the intermediary orbit we only use the non-periodic part $[R_i]$ of the perturbative function. We add to this those terms of the secular part of the perturbative function corresponding to the action of the sun, which do not contain angular elements of the sun. These are with sufficient approximation

$$\frac{(c_i-\varkappa)(1-\mu_i)\,\mathrm{a}_i^3}{1+m_i}\,\frac{}{A^3}\,M\left[\tfrac{1}{4}-\tfrac{3}{8}\,i_\omega^2+\tfrac{3}{8}\,(\mathrm{e}_i^2+e_0^2)\right],$$

where e_0 is the excentricity and i_0 the inclination of the sun's orbit.

The perturbative function is developed in powers of e_i. For this reason I have also, instead of η_i, taken as unknown $e_i=\eta_i\sqrt{1-\frac{1}{4}\eta_i^2}$. The equations determining μ_i and e_i then become, instead of (18) and (19):

$$a_i\,\frac{\partial\,[R_i]}{\partial\,a}+\tfrac{1}{4}\varkappa\,\mathrm{e}_i^2=0$$

$$\sqrt{1-\mathrm{e}_i^2}\,\frac{\partial\,[R_i]}{\partial\,e_i}+\varkappa\,\mathrm{e}_i=0$$

If we denote by $[R'_i]$ the non-periodic terms of the principal and complementary parts of the perturbative function, these equations become

$$\frac{\mu_i}{1-\mu_i}-[J_i+\tfrac{1}{2}K_i]-(\tfrac{2}{3}J_i+\tfrac{5}{2}K_i)\mathrm{e}_i^2+\tfrac{1}{2}\,\frac{M}{1+m_i}\,\frac{\mathrm{a}_i^3}{A^3}+$$

$$+\frac{\varkappa\,\mathrm{e}_i^2}{4\,(c_i-\varkappa)(1-\mu_i)}+\frac{1}{(c_i-\varkappa)(1-\mu_i)}\,a_i\,\frac{\partial\,[R'_i]}{\partial a_i}=0\ .\ \ .\qquad(1)$$

$$A_i\,\mathrm{e}_i+2\,(c_i-\varkappa)(1-\mu_i)\,J_i\,\mathrm{e}_i^3+(1-\tfrac{1}{2}\,\mathrm{e}_i^2)\,\frac{\partial\,[R'_i]}{\partial \mathrm{e}_i}=0\ .\ \ .\qquad(2)$$

$$A_i=\varkappa+(c_i-\varkappa)(1-\mu_i)(J_i+K_i)+\tfrac{3}{4}\,\frac{(c_i-\varkappa)\,(1-\mu_i)}{1+m_i}\,M\,\frac{\mathrm{a}_i^3}{A^3}.$$

The first equation gives $\mu'_i=\dfrac{\mu_i}{1-\mu_i}$. Then we find a_i from

$$\mathrm{a}_i^3\,(c_i-\varkappa)^2=(1+m_i)(1+\mu'_i).$$

For the solution of the equations (1) and (2) we started from the approximations [1])

$\log a_1 = 9.5997740-10$	$e_1 = 0.00404164$
$\log a_2 = 9.8015496-10$	$e_2 = 0.00936330$
$\log a_3 = 0.0042524$	$e_3 = 0.00059680$
$\log a_4 = 0.2494696$	$e_4 = 0$

The coefficients of LAPLACE corresponding to these values of a_i were derived from those given by SOUILLART by the application of the corrections necessary to reduce from SOUILLART's values of the ratios of the mean distances to ours. Then with these coefficients the NEWCOMB's operators occurring in the formulas (1) and the other coefficients of these formulas were computed, and the values of μ'_i were solved and from these the values of a_i were derived. The coefficients of LAPLACE were then reduced to these new values of a_i, and a second approximation of μ'_i was derived, which differed only very little from the first. The corresponding values of a_i were considered as final, and were used as the basis for an entirely new computation of the LAPLACE coefficients. Then with these coefficients we computed the operators necessary for the equations (2). These equations (2) are not, like (1), independent of each other, but must be solved by successive approximations. The approximations, which converged very rapidly, were continued until the ninth decimal place of e_i was no longer affected. The values of e_i thus derived are the definitive ones. They were substituted in (1) instead of the original approximations, but this did not produce any change in the values of μ'_i and a_i.

The elements of the intermediary orbit are thus determined. The different terms of μ'_i are given below. The terms marked "*add*" are the second and third of the formula (1), "$\varkappa$" is the fifth and "*sun*" the fourth term. The effect of the last term is given for each perturbing satellite separately. The quantities $\varkappa$ and e_i are considered to be of the first order, the masses and J_i are of the second order. A term containing the factor me^2 is thus of the fourth order. We found [2])

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[1]) See Leiden, Annals XII, 1, pages 52 and 53.

[2]) The computations were made with two more decimals than are published here. Consequently it may happen that the sum of the printed numbers differs one unit in the last decimal place from the printed sum.

75*

μ'_1 : *add.* :	2nd order	+	·000 62824	+	·000 62826
	4th ,,	+	2		
$\varkappa$:	3rd ,,			—	3·
sun :	3rd ,,			—	8
m_2 :	2nd ,,	—	·000 00552		
	3rd ,,	+	48	—	507
	4th ,,	—	3		
m_3 :	2nd ,,			—	303
m_4 :	2nd ,,			—	27

$$\mu'_1 = + \cdot 00061977$$

μ'_2 : *add.* :	2nd order	+	·000 24791	+	·000 24794
	4th ,,	+	3		
$\varkappa$:	3rd ,,			—	32
sun :	3rd ,,			—	34
m_1 :	2nd ,,	+	·000 05599		
	3rd ,,	—	281	+	5328
	4th ,	+	10		
m_3 :	2nd ,,	—	·000 01761		
	3rd ,,	+	157	—	1609
	4th ,,	—	5		
m_4 :	2nd ,,			—	119

$$\mu'_2 = + \cdot 000\ 28328$$

μ'_3 : *add.* :	2nd order			+	·000 09739
sun :	3rd ,,			—	136
m_1 :	2nd ,,			+	4313
m_2 :	2nd ,,	+	·000 03738		
	3rd ,,	—	114	+	3628
	4th ,,	+	3		
m_4 :	2nd ,,			—	643

$$\mu'_3 = + \cdot 000\ 16901$$

μ'_4 : *add.* :	2nd order	+	·000 03148
sun :	3rd ,,	—	741
m_1 :	2nd ,,	+	3946
m_2 :	2nd ,,	+	2816
m_3 :	2nd ,,	+	11069

$$\mu'_4 = + \cdot 000\ 20238$$

From these we find

$$\begin{aligned}
log\,a_1 &= 9\;559\;77215 - 10 + \delta\,log\,a_1\\
log\,a_2 &= 9{\cdot}801\;46241 - 10 + \delta\,log\,a_2\\
log\,a_3 &= 0{\cdot}004\;26052 \qquad\quad + \delta\,log\,a_3\\
log\,a_4 &= 0{\cdot}249\;49217 \qquad\quad + \delta\,log\,a_4
\end{aligned}$$

The corrections $\delta\,log\,a_i$ have been added to take account of eventual corrections λ_i to the masses and to J and K. We also put

$$e_i = e_{i_0}(1 + \eta_i)$$

Then we have

$$\begin{aligned}
10^7\,\delta\,log\,a_1 &= +\,907\,\lambda_0 + \;\;2\,\lambda_5 + 55\,\lambda_1 - \;\;7\,\lambda_2 - 4\,\lambda_3\\
10^7\,\delta\,log\,a_2 &= +\,359\,\lambda_0 + 77\,\lambda_1 + 22\,\lambda_2 - \;22\,\lambda_3 - 2\,\lambda_4 - \eta_1 - \eta_2\\
10^7\,\delta\,log\,a_3 &= +\,140\,\lambda_0 + 62\,\lambda_1 + 52\,\lambda_2 + 119\,\lambda_3 - 9\,\lambda_4 - \eta_2\\
10^7\,\delta\,log\,a_4 &= +\;\;45\,\lambda_0 + 57\,\lambda_1 + 41\,\lambda_2 + 160\,\lambda_3 + 66\,\lambda_4
\end{aligned}$$

For the coefficients A_i of the first term of the equations (2) we find

	I	II	III	IV
$\varkappa$	0·014 48392	0·014 48392	0·014 48392	0·014 48392
add.	2 50655	49242	9599	1330
sun	50	100	201	469
A_1	0·016 99097	0·014 97735	0·014 58193	0.014 50191

The second term can be neglected for the satellites III and IV. For I and II it contributes —2 and —5 respectively to the eighth decimal place of e_i.

The third term gives

$$\begin{array}{llllrl}
10^{16}\left(1 - \frac{1}{2}e_1^2\right)\dfrac{\partial\,[R'_1]}{\partial e_1} & : m_2 & : 2^{nd}\text{order} & - & 752429 & \\
 & & 3^{rd} \;\;" & + & 43589 & \\
 & & 4^{th} \;\;" & - & 1908 & \left.\right\} - 710677\\
 & & 5^{th} \;\;" & + & 73 & \\
 & & 6^{th} \;\;" & - & 3 & \\
 & m_3 & : 3^{rd} \;\;" & + & 801 & \left.\right\} + \quad 793\\
 & & 4^{th} \;\;" & - & 7 & \\
 & m_4 & : 3^{rd} \;\;" & & & + \quad\;\; 70\\
\hline
 & & & & & - 709814
\end{array}$$

$10^{10}\left(1-\frac{1}{2}e_2^2\right)\frac{\partial\,[R'_2]}{\partial e_2}$:

m_1 :	2nd order	$-$ 312553	
	3rd „	$+$ 72951	$-$ 242876
	4th „	$-$ 3396	
	5th „	$+$ 125	
	6th „	$-$ 3	
m_3 :	2nd order	$-$ 1198671	
	3rd „	$+$ 42122	$-$ 1157462
	4th „	$-$ 930	
	5th „	$+$ 18	
m_4 :	3rd „		$+$ 368
			$-$ 1399970

$10^{10}\left(1-\frac{1}{2}e_3^2\right)\frac{\partial\,[R'_3]}{\partial e_3}$:

m_1 :	3rd order	$-$ 75	$-$ 72
	4th „	$+$ 4	
m_2 :	2nd „	$-$ 100444	
	3rd „	$+$ 13874	$-$ 86928
	4th „	$-$ 366	
	5th „	$+$ 7	
m_4 :	3rd		$+$ 77
			$-$ 86923

$10^{10}\left(1-\frac{1}{2}e_4^2\right)\frac{\partial\,[R'_4]}{\partial e_4}$:

m_1	3rd order	$+$ 5
m_2	3rd „	$+$ 51
m_3	3rd „	$-$ 71
		$-$ 15

The terms of the sixth order have not been computed directly, but have been derived by extrapolation of the series of logarithms of the terms of the second to fifth orders.

We find thus:

$$e_1 = 0{\cdot}004\ 17757\ (1+\eta_1)$$
$$e_2 = 0{\cdot}009\ 34720\ (1+\eta_2)$$
$$e_3 = 0{\cdot}000\ 59610\ (1+\eta_3)$$
$$e_4 = 0{\cdot}000\ 00010\ (1+\eta_4)$$

The values of η_i are

$\eta_1 = -\ \cdot 14356\ \lambda_0 - \cdot 00049\ \lambda_5 - \cdot 00683\ \lambda_1 + \cdot 98651\ \lambda_2 - \cdot 03338\ \lambda_3 -$
$- \cdot 00009\ \lambda_4 + \cdot 021\ \lambda_0^2 + \cdot 001\ \lambda_0\lambda_1 - \cdot 140\ \lambda_0\lambda_2 + \cdot 006\ \lambda_0\lambda_3 -$
$- \cdot 006\ \lambda_1\lambda_2 + \cdot 001\ \lambda_1\lambda_3 - \cdot 013\ \lambda_2^2 + \cdot 034\ \lambda_2\lambda_3 + \cdot 02\ \lambda_0^2\lambda_2$

$\eta_2 = -\ \cdot 02951\ \lambda_0 + \cdot 16326\ \lambda_1 - \cdot 01272\ \lambda_2 + \cdot 77826\ \lambda_3 - \cdot 00024\ \lambda_4 +$
$+ \cdot 001\ \lambda_0^2 - \cdot 002\ \lambda_0\lambda_1 + \cdot 002\lambda_0\lambda_2 - \cdot 024\lambda_0\lambda_3 - \cdot 005\lambda_1^2 + \cdot 010\ \lambda_1\lambda_2$
$+ \cdot 030\ \lambda_1\lambda_3 - \cdot 001\ \lambda_2\lambda_3 - \cdot 018\ \lambda_3^2$

$\eta_3 = -\ \cdot 0023\ \lambda_0 - \cdot 0215\ \lambda_1 + \cdot 9878\ \lambda_2 - \cdot 1068\ \lambda_3 - \cdot 0008\ \lambda_4 - \cdot 02\ \lambda_1\lambda_2 -$
$- \cdot 01\ \lambda_2^2$

$\eta_4 = +\ 0{\cdot}1\ \lambda_0 - 1{\cdot}0\ \lambda_1 + 1{\cdot}0\ \lambda_2 + 1{\cdot}6\ \lambda_3 - \lambda_1\lambda_2 + 2\ \lambda_2\lambda_3$

The coordinates in the intermediary orbit are given by the formulas

$$r_i = a_i\ \varrho_i,$$
$$w_i = \lambda_{00} + \pi_{i_0} + (c_i - \varkappa)\ \tau + E_i$$
$$= \lambda_{i_0} + n_i t + E_i,$$

where ϱ_i and E_i are the ordinary elliptic values of the radius-vector and the equation of the centre. We have, in astronomical units:

$$\log a_1 = 7{\cdot}450\ \ 13884 - 10$$
$$\log a_2 = 7{\cdot}651\ \ 82910 - 10$$
$$\log a_3 = 7{\cdot}854\ \ 62721 - 10$$
$$\log a_4 = 8{\cdot}100\ \ 30886 - 10$$

From the above values of e_i we find·

$$\varrho_1 = 1.000\ 00873 - \cdot 004\ 17755\cos 4\tau - 873\cos 8\tau - 3\cos 12\tau$$
$$E_1 = +\ 0^\circ{\cdot}478714\sin 4\tau + 1250\sin 8\tau + 5\sin 12\tau$$

$$\varrho_2 = 1{\cdot}000\ 04368 - \cdot 009\ 34684\cos 2\tau - 4368\cos 4\tau - 36\cos 6\tau$$
$$E_2 = +\ 1^\circ.071098\sin 2\tau + 6258\sin 4\tau + 50\sin 6\tau$$

$$\varrho_3 = 1{\cdot}000\ 00018 - 59610\cos\tau - 18\cos 2\tau$$
$$E_3 = +\ 0^\circ{\cdot}068308\sin\tau + 25\sin 2\tau$$

$$\varrho_4 = 1 - \cdot 000\ 00010\cos c_4\tau$$
$$E_4 = +\ 0^\circ.000012\sin c_4\tau$$

The inequalities in longitude are expressed in degrees. The angle τ is counted from the opposition of II and III on 1899 June 28, $11^h 47^m 35^s$ Greenwich mean time. (See "Outlines", these Proceedings Vol. XX. p. 1299).