Citation:

H.A. Lorentz, On Einstein's theory of gravitation part III, in:
KNAW, Proceedings, 20 I, 1918, Amsterdam, 1918, pp. 2-19

Physics. — "*On* EINSTEIN'S *Theory of gravitation*". III. By Prof. H. A. LORENTZ.

(Communicated in the meeting of April 1916.)[1]

§ 32. In the two preceding papers[2]) we have tried so far as possible to present the fundamental principles of the new gravitation theory in a simple form.

We shall now show how EINSTEIN'S differential equations for the gravitation field can be derived from HAMILTON'S principle. In this connexion we shall also have to consider the energy, the stresses, momenta and energy-currents in that field.

We shall again introduce the quantities g_{ab} formerly used and we shall also use the "inverse" system of quantities for which we shall now write g^{ab}. It is found useful to introduce besides these the quantities

$$\mathfrak{g}^{ab} = \sqrt{-g}\, g^{ab}.$$

Differential coefficients of all these variables with respect to the coordinates will be represented by the indices belonging to these latter, e.g.

$$g_{ab,p} = \frac{\partial g_{ab}}{\partial x_p} \quad , \quad g_{ab,pq} = \frac{\partial^2 g_{ab}}{\partial x_q \partial x_p}.$$

We shall use CHRISTOFFEL'S symbols

$$\begin{bmatrix} a\,b \\ c \end{bmatrix} = \tfrac{1}{2}\left(g_{ac,b} + g_{bc,a} - g_{ab,c}\right)$$

and RIEMANN'S symbol

$$(ik, lm) = \tfrac{1}{2}\left(g_{im,kl} + g_{kl,im} - g_{il,km} - g_{km,il}\right) + \\ + \Sigma(ab)\, g^{ab} \left\{ \begin{bmatrix} i\,m \\ a \end{bmatrix} \begin{bmatrix} k\,l \\ b \end{bmatrix} - \begin{bmatrix} i\,l \\ a \end{bmatrix} \begin{bmatrix} k\,m \\ b \end{bmatrix} \right\}.$$

Further we put

$$G_{im} = \Sigma\,(kl)\, g^{kl}(ik, lm) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

$$G = \Sigma\,(im)\, g^{im}\, G_{im} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

This latter quantity is a measure for the curvature of the field-figure. The principal function of the gravitation field is

[1]) Published September 1916, a revision having been found desirable.

[2]) See Proceedings Vol. XIX, p. 1341 and 1354.

$$\frac{1}{2\varkappa}\int Q dS,$$

where

$$Q = \sqrt{-g}\, G.$$

In the integral dS, the element of the field-figure, is expressed in x-units. The integration has to be extended over the domain within a certain closed surface σ; $\varkappa$ is a positive constant.

§ 33. When we pass from the system of coordinates $x_1, \ldots x_4$ to another, the value of G proves to remain unaltered, it is a scalar quantity. This may be verified by first proving that the quantities (ik, lm) form a covariant tensor of the fourth order[1]. Next, (g^{kl}) being a contravariant tensor of the second order[2], we can deduce from (40) that (G_{im}) is a covariant tensor of the same order[3]. According to (41) G is then a scalar. The same is true[4] for $Q\, d\, S$.

We remark that $g_{ba} = g_{ab}$[5] and $g_{ab,fe} = g_{ab,ef}$. We shall suppose Q to be written in such a way that its form is not altered by interchanging g_{ba} and g_{ab} or $g_{ab\,fe}$ and $g_{ab,ef}$. If originally this condition is not fulfilled it is easy to pass to a "symmetrical" form of this kind.

It is clear that Q may also be expressed in the quantities g^{ab} and their first and second derivatives and in the same way in the $\mathfrak{g}^{ab}$'s and first and second derivatives of these quantities.

If the necessary substitutions are executed with due care, these new forms of Q will also be symmetrical.

§ 34. We shall first express the quantity Q in the g_{ab}'s and their

1) This means that the transformation formulae for these quantities have the form

$$(ik,lm)' = \Sigma\, (abce)\, p_{ai} p_{bk} p_{cl} p_{em}\, (ab,ce)$$

See for the notations used here and for some others to be used later on my communication in Zittingsverslag Akad Amsterdam **23** (1915), p. 1073 (translated in Proceedings Amsterdam **19** (1916), p. 751). In referring to the equations and the articles of this paper I shall add the indication 1915.

2) Namely:

$$g'^{kl} = \Sigma\, (ab)\, \pi_{ak}\, \pi_{bl}\, g^{ab}.$$

The symbol (g^{kl}) denotes the complex of all the quantities g^{kl}.

3) Namely:

$$G'_{im} = \Sigma\, (ab)\, p_{ai}\, p_{bm}\, G_{ab}.$$

4) On account of the relation

$$\sqrt{-g'}\, dS' = \sqrt{-g}\, dS.$$

5) Similarly:

$$g^{ba} = g^{ab},\ \mathfrak{g}^{ba} = \mathfrak{g}^{ab}$$

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derivatives and we shall determine the variation it undergoes by arbitrarily chosen variations δg_{ab}, these latter being continuous functions of the coordinates. We have evidently

$$\delta Q = \Sigma(ab) \frac{\partial Q}{\partial g_{ab}} \delta g_{ab} + \Sigma(abe) \frac{\partial Q}{\partial g_{ab,e}} \delta g_{ab,e} + \Sigma\,(abef) \frac{\partial Q}{\partial g_{ab,ef}} \delta g_{ab\,ef}.$$

By means of the equations

$$\delta g_{ab,ef} = \frac{\partial}{\partial x_f} \delta g_{ab,e} \text{ and } \delta g_{ab,e} = \frac{\partial}{\partial x_e} \delta g_{ab}$$

this may be decomposed into two parts

$$\delta Q = \delta_1 Q + \delta_2 Q, \qquad (42)$$

namely

$$\delta_1 Q = \Sigma\,(ab) \left\{ \frac{\partial Q}{\partial g_{ab}} - \Sigma\,(e) \frac{\partial}{\partial x_e} \frac{\partial Q}{\partial g_{ab,e}} + \Sigma(ef) \frac{\partial^2}{\partial x_e \partial x_f} \frac{\partial Q}{\partial g_{ab,ef}} \right\} \delta g_{ab}\,. \qquad (43)$$

$$\delta_2 Q = \Sigma\,(abe) \frac{\partial}{\partial x_e} \left(\frac{\partial Q}{\partial g_{ab,e}} \delta g_{ab} \right) + \Sigma\,(abef) \frac{\partial}{\partial x_f} \left(\frac{\partial Q}{\partial g_{ab,ef}} \delta g_{ab,e} \right) -$$
$$- \Sigma\,(abef) \frac{\partial}{\partial x_e} \left\{ \frac{\partial}{\partial x_f} \left(\frac{\partial Q}{\partial g_{ab,ef}} \right) \delta g_{ab} \right\} \qquad (44)$$

The last equation shows that

$$\int \delta_2 Q\, dS = 0 \qquad (45)$$

if the variations δg_{ab} and their first derivatives vanish at the boundary of the domain of integration.

§ 35. Equations of the same form may also be found if Q is expressed in one of the two other ways mentioned in § 33. If e.g. we work with the quantities $\mathfrak{g}^{ab}$ we shall find

$$(\delta Q) = (\delta_1 Q) + (\delta_2 Q),$$

where $(\delta_1 Q)$ and $(\delta_2 Q)$ are directly found from (43) and (44) by replacing g_{ab}, $g_{ab,e}$, $g_{ab,ef}$, δg_{ab} and $\delta g_{ab,e}$ etc. by $\mathfrak{g}^{ab}$, $\mathfrak{g}^{ab,e}$, etc. If the variations chosen in the two cases correspond to each other we shall have of course

$$(\delta Q) = \delta Q.$$

Moreover we can show that the equalities

$$(\delta_1 Q) = \delta_1 Q, \qquad (\delta_2 Q) = \delta_2 Q,$$

exist separately. [1])

[1]) Suppose that at the boundary of the domain of integration $\delta g_{ab} = 0$ and $\delta g_{ab,e} = 0$. Then we have also $\delta \mathfrak{g}^{ab} = 0$ and $\delta \mathfrak{g}^{ab,e} = 0$, so that

$$\int (\delta_2 Q)\, dS = 0, \qquad \int \delta_2 Q\, dS = 0$$

and from

The decomposition of δQ into two parts is therefore the same, whether we use g_{ab}, g^{ab} or $\mathfrak{g}^{ab}$.

It is further of importance that when the system of coordinates is changed, not only $\delta Q dS$ is an invariant, but that this is also the case with $\delta_1 Q dS$ and $\delta_2 Q dS$ separately. [2])

We have therefore

$$\frac{\delta_1 Q'}{\sqrt{-g'}} = \frac{\delta_1 Q}{\sqrt{-g}} \quad \ldots \ldots \ldots \quad (46)$$

§ 36. For the calculation of $\delta_1 Q$ we shall suppose Q to be expressed in the quantities $\mathfrak{g}^{ab}$ and their derivatives. Therefore (comp. (43))

$$\delta_1 Q = \Sigma\,(ab)\, M_{ab}\, \delta \mathfrak{g}^{ab}, \quad \ldots \ldots \quad (47)$$

if we put

$$M_{ab} = \frac{\partial Q}{\partial \mathfrak{g}^{ab}} - \Sigma\,(e)\frac{\partial}{\partial x_e}\frac{\partial Q}{\partial \mathfrak{g}^{ab,e}} + \Sigma(ef)\frac{\partial^2}{\partial x_e \partial x_f}\frac{\partial Q}{\partial \mathfrak{g}^{ab,ef}}$$

Now we can show that the quantities M_{ab} are exactly the quantities G_{ab} defined by (40). To this effect we may use the following considerations.

We know that $\left(\frac{1}{\sqrt{-g}}\mathfrak{g}^{ab}\right)$ is a contravariant tensor of the second

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$$\int (\delta Q)\, dS = \int \delta Q\, dS,$$

we infer

$$\int (\delta_1 Q)\, dS = \int \delta_1 Q\, dS.$$

As this must hold for every choice of the variations δg_{ab} (by which choice the variations $\delta \mathfrak{g}^{ab}$ are determined too) we must have at each point of the field-figure

$$(\delta_1 Q) = \delta_1 Q$$

[2]) This may be made clear by a reasoning similar to that used in the preceding note. We again suppose δg_{ab} and $\delta g_{ab,e}$ to be zero at the boundary of the domain of integration. Then $\delta g'_{ab}$ and $\delta g'_{ab,e}$ vanish too at the boundary, so that

$$\int \delta_2 Q' dS' = 0 \quad , \quad \int \delta_2 Q dS = 0.$$

From

$$\int \delta Q' dS' = \int \delta Q dS$$

we may therefore conclude that

$$\int \delta_1 Q' dS' = \int \delta_1 Q dS.$$

As this must hold for arbitrarily chosen variations δg_{ab} we have the equation

$$\delta_1 Q' dS' = \delta_1 Q dS.$$

order. From this we can deduce that $\left(\frac{1}{\sqrt{-g}}\delta \mathfrak{g}^{ab}\right)$ is also such a tensor.

Writing for it ε^{ab} we find according to (46) and (47) that

$$\Sigma\,(ab)\,M_{ab}\,\varepsilon^{ab}$$

is a scalar for every choice of (ε^{ab}).

This involves that (M_{ab}) is a covariant tensor of the second order and as the same is true for (G_{ab}) we must prove the equation

$$M_{ab} = G_{ab}$$

only *for one special* choice of coordinates.

§ 37. Now this choice can be made in such a way that at the point P of the field-figure $g_{11} = g_{22} = g_{33} = -1, g_{44} = +1, g_{ab} = 0$ for $a \neq b$ and that moreover all first derivatives $g_{ab,e}$ vanish. If then the values g_{ab} at a point Q near P are developed in series of ascending powers of the differences of coordinates $x_a(Q) - x_a(P)$ the terms directly following the constant ones will be of the second order. It is with these terms that we are concerned in the calculation both of M_{ab} and of G_{ab} for the point P. As in the results the coefficients of these terms occur to the first power only, it is sufficient to show that each of the above mentioned terms separately contributes the same value to M_{ab} and to G_{ab}.

From these considerations we may conclude that

$$\delta_1 Q = \Sigma(ab)\,G_{ab}\,\delta\mathfrak{g}^{ab} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

Expressions containing instead of $\delta\mathfrak{g}^{ab}$ either the variations δg^{ab} or δg_{ab} might be derived from this by using the relations between the different variations. Of these we shall only mention the formula

$$\delta g^{ab} = \frac{1}{\sqrt{-g}}\,\delta\mathfrak{g}^{ab} - \frac{g^{ab}}{2\sqrt{-g}}\,\Sigma\,(cd)\,g_{cd}\,\delta\mathfrak{g}^{cd} \quad . \quad . \quad . \quad (49)$$

§ 38. In connexion with what precedes we here insert a consideration the purpose of which will be evident later on. Let the infinitely small quantity ξ be an arbitrarily chosen continuous function of the coordinates and let the variations δg_{ab} be defined by the condition that at some point P the quantities g_{ab} have *after* the change the values which existed *before* the change at the point Q, to which P is shifted when x_h is diminished by ξ, while the three other coordinates are left constant. Then we have

$$\delta g_{ab} = -\,g_{ab,h}\,\xi$$

and similar formulae for the variations $\delta\mathfrak{g}^{ab}$.

If for $\delta_1 Q$ and $\delta_2 Q$ the expressions (48) and (44) are taken, the equation

$$\delta Q - \delta_2 Q = \delta_1 Q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

is an identity for every choice of the variations.

It will likewise be so in the special case considered and we shall also come to an identity if in (50) the terms with the derivatives of ξ are omitted while those with ξ itself are preserved.

When this is done δQ reduces to

$$-\frac{\partial Q}{\partial x_h}\xi$$

and, taking into consideration (44) and (48), we find after division by ξ

$$-\frac{\partial Q}{\partial x_h} + \Sigma(abe)\frac{\partial}{\partial x_e}\left(\frac{\partial Q}{\partial g_{ab,e}}g_{ab,h}\right) + \Sigma(abef)\frac{\partial}{\partial x_e}\left(\frac{\partial Q}{\partial g_{ab,fe}}g_{ab,fh}\right) -$$
$$-\Sigma(abef)\frac{\partial}{\partial x_e}\left\{\frac{\partial}{\partial x_f}\left(\frac{\partial Q}{\partial g_{ab,ef}}\right)g_{ab,h}\right\} = -\Sigma(ab)G_{ab}\mathfrak{g}^{ab,h}. \quad (51)$$

In the second term of (44) we have interchanged here the indices e and f.

If for shortness' sake we put, for $e \neq h$

$$\mathfrak{s}_h^e = \Sigma(ab)\frac{\partial Q}{\partial g_{ab,e}}g_{ab,h} + \Sigma(abf)\frac{\partial Q}{\partial g_{ab,fe}}g_{ab,fh} - \Sigma(abf)\frac{\partial}{\partial x_f}\left(\frac{\partial Q}{\partial g_{ab,ef}}\right)g_{ab,h} \quad (52)$$

and for $e = h$

$$\mathfrak{s}_h^h = -Q + \Sigma(ab)\frac{\partial Q}{\partial g_{ab,h}}g_{ab,h} + \Sigma(abf)\frac{\partial Q}{\partial g_{ab,fh}}g_{abfh} -$$
$$-\Sigma(abf)\frac{\partial}{\partial x_f}\left(\frac{\partial Q}{\partial g_{ab,hf}}\right)g_{ab,h}, \quad . \quad . \quad . \quad . \quad . \quad (53)$$

we may write

$$\Sigma(e)\frac{\partial \mathfrak{s}_h^e}{\partial x_e} = -\Sigma(ab)\,G_{ab}\,\mathfrak{g}^{ab,h} \quad . \quad . \quad . \quad . \quad . \quad (54)$$

The set of quantities $\mathfrak{s}_h^e$ will be called the *complex* $\mathfrak{s}$ and the set of the four quantities which stand on the left hand side of (54) in the cases $h = 1, 2, 3, 4$, the *divergency* of the complex.[1] It will be denoted by $div\,\mathfrak{s}$ and each of the four quantities separately by $div_h\,\mathfrak{s}$.

The equation therefore becomes

$$div_h \mathfrak{s} = -\Sigma(ab)G_{ab}\,\mathfrak{g}^{ab,h} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

[1] Einstein uses the word "divergency" in a somewhat different sense. It seemed desirable however to have a name for the left hand side of (54) and it was difficult to find a better one.

If we take other coordinates the right hand side of this equation is transformed according to a formula which can be found easily. Hence we can also write down the transformation formula for the left hand side. It is as follows

$$div'_h \mathfrak{S}' = p\Sigma(m)p_{mh} div_m \mathfrak{S} - Q\Sigma(a)p_{ah}\frac{\partial p}{\partial x_a} + 2p\Sigma(abc)p_{ah\,c}\,\mathfrak{g}^{bc}G_{ab}. \quad . \quad (56)$$

§ 39. We shall now consider a second complex $\mathfrak{S}_0$, the components of which are defined by

$$\mathfrak{S}^e{}_{0h} = - G\Sigma\,(a)\mathfrak{g}^{ae}\,g_{ah} + 2\Sigma\,(a)\mathfrak{g}^{ae}\,G_{ah} \quad . \quad . \quad . \quad (57)$$

Taking also the divergency of this complex we find that the difference

$$div'_h \mathfrak{S}'_0 - p\,\Sigma\,(m)p_{mh}\,div_m \mathfrak{S}_0$$

has just the value which we can deduce from (56) for the corresponding difference

$$div'_h\,\mathfrak{S}' - p\Sigma(m)p_{mh} div_m \mathfrak{S}$$

It is thus seen that

$$div'_h \mathfrak{S}' - div'_h \mathfrak{S}'_0 = p\Sigma(m)p_{mh}(div_m \mathfrak{S} - div_m \mathfrak{S}_0)$$

and that we have therefore

$$div\ \mathfrak{S} = div\ \mathfrak{S}_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

for all systems of coordinates as soon as this is the case for one system.

Now a direct calculation starting from (52), (53) and (57) teaches us that the terms with the highest derivatives of the quantities g_{ab}, (viz. those of the third order) are the same in $div_h\,\mathfrak{S}$ and $div_h\,\mathfrak{S}_0$. Further it is evident that in the system of coordinates introduced in § 37 these terms with the third derivatives are the only ones. This proves the general validity of equation (58) It is especially to be noticed that if $\mathfrak{S}$ and $\mathfrak{S}_0$ are determined by (52), (53) and (57) and if the function defined in § 32 is taken for G, the relation is an identity.

§ 40. We shall now derive the differential equations for the gravitation field, first for the case of an electromagnetic system. [1]) For the part of the principal function belonging to it we write

$$\int \mathrm{L}\,d\,S,$$

where L is defined by (35) (1915). From L we can derive the stresses, the momenta, the energy-current and the energy of the

[1]) This has also been done by DE DONDER, Zittingsverslag Akad. Amsterdam, **25** (1916), p. 153.

electromagnetic system; for this purpose we must use the equations (45) and (46) (1915) or in EINSTEIN's notation, which we shall follow here, [1])

$$\mathfrak{T}_c^c = -\mathrm{L} + \sum_{a=|=c} (a)\, \psi^*_{ac}\, \psi_{a'c'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (59)$$

and for $b =|= c$

$$\mathfrak{T}_c^b = \sum_{a=|=c} (a)\, \psi^*_{ab}\, \psi_{a'c'} \quad . \quad . \quad . \quad . \quad \quad . \quad . \quad . \quad (60)$$

The set of quantities $\mathfrak{T}_c{}^b$ might be called the stress-energy-complex (comp. § 38). As for a change of the system of coordinates the transformation formulae for $\mathfrak{T}$ are similar to those by which tensors are defined, we can also speak of the stress-energy-*tensor*. We have namely

$$\frac{1}{\sqrt{-g'}}\, \mathfrak{T}'^{b}_{c} = \frac{1}{\sqrt{-g}}\, \Sigma(kl)\, p_{kc}\, \pi_{lb}\, \mathfrak{T}_k^l \, .$$

§ 41. The equations for the gravitation field are now obtained (comp. §§ 13 and 14, 1915) from the condition that

$$\delta_\psi \int \mathrm{L} d\, S + \frac{1}{2\varkappa}\, \delta \int Q\, d\, S = 0 \quad . \quad . \quad . \quad . \quad . \quad (61)$$

for all variations δg_{ab} which vanish at the boundary of the field of integration together with their first derivatives. The index ψ in the first term indicates that in the variation of L the quantities ψ_{ab} must be kept constant.

If we suppose L to be expressed in the quantities g^{ab} and if (42), (45) and (48) are taken into consideration, we find from (61) that at each point of the field-figure

$$\Sigma(ab) \left(\frac{\partial \mathrm{L}}{\partial g^{ab}}\right)_\psi \delta g^{ab} + \frac{1}{2\varkappa}\, \Sigma(ab)\, G_{ab}\, \delta \mathfrak{g}^{ab} = 0 \quad . \quad . \qquad (62)$$

If now in the first term we put

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[1]) The notations ψ_{ab}, $\overline{\psi}_{ab}$ and ψ^*_{ab} (see (27), (29) and § 11, 1915), will however be preserved though they do not correspond to those of EINSTEIN. As to formulae (59) and (60) it is to be understood that if p and q are two of the numbers 1, 2, 3, 4, p' and q' denote the other two in such a way that the order $p\ q\ p'\ q'$ is obtained from 1 2 3 4 by an even number of permutations of two ciphers.

If x_1, x_2, x_3, x_4 are replaced by x, y, z, t and if for the stresses the usual notations X_x, X_y, etc., are used (so that e.g. for a surface element $d\sigma$ perpendicular to the axis of x, X_x is the first component of the force per unit of surface which the part of the system situated on the positive side of $d\sigma$ exerts on the opposite part) then $\mathfrak{T}_1{}^1 = X_x$, $\mathfrak{T}_1{}^2 = X_y$, etc. Further $-\mathfrak{T}_1{}^4$, $-\mathfrak{T}_2{}^4$, $-\mathfrak{T}_3{}^4$ are the components of the momentum per unit of volume and $\mathfrak{T}_4{}^1$, $\mathfrak{T}_4{}^2$, $\mathfrak{T}_4{}^3$ the components of the energy-current. Finally $\mathfrak{T}_4{}^4$ is the energy per unit of volume.

$$\left(\frac{\partial L}{\partial g^{ab}}\right)_{\psi} = \tfrac{1}{2}\sqrt{-g}\,T_{ab}\,, \quad \ldots \ldots \quad (63)$$

and if for δg^{ab} the value (49) is substituted, this term becomes

$$\tfrac{1}{2}\,\Sigma\,(ab)\,T_{ab}\,\delta\mathfrak{g}^{ab} - \tfrac{1}{4}\,\Sigma\,(abcd)\,g^{ab}\,g_{cd}\,T_{ab}\,\delta\mathfrak{g}^{cd},$$

or if in the latter summation a, b is interchanged with c, d and if the quantity

$$T = \Sigma\,(cd)\,g^{cd}\,T_{cd} \quad \ldots \ldots \quad (64)$$

is introduced,

$$\tfrac{1}{2}\,\Sigma\,(ab)\,(T_{ab} - \tfrac{1}{2}\,g_{ab}\,T)\,\delta\mathfrak{g}^{ab}.$$

Finally, putting equal to zero the coefficient of each $\delta\mathfrak{g}^{ab}$ we find from (62) the differential equation required

$$G_{ab} = -\varkappa\,(T_{ab} - \tfrac{1}{2}\,g_{ab}\,T) \quad \ldots \ldots \quad (65)$$

This is of the same form as EINSTEIN's field equations, but to see that the formulae really correspond to each other it remains to show that the quantities T_{ab} and $\mathfrak{T}_c^{\,b}$ defined by (63), (59) and (60) are connected by EINSTEIN's formulae

$$\mathfrak{T}_c^b = \sqrt{-g}\,\Sigma(a)\,g^{ab}\,T_{ac} \quad \ldots \ldots \quad (66)$$

We must have therefore

$$2\,\Sigma(a)\,g^{ac}\left(\frac{\partial L}{\partial g^{ac}}\right)_{\psi} = -L + \sum_{a \neq c}(a)\,\psi^{*}_{ac}\,\psi_{a'c'} \quad \ldots \quad (67)$$

and for $b \neq c$

$$2\,\Sigma(a)\,g^{ab}\left(\frac{\partial L}{\partial g^{ac}}\right)_{\psi} = \sum_{a \neq c}(a)\,\psi^{*}_{ab}\,\psi_{a'c'} \quad \ldots \quad (68)$$

§ 12. This can be tested in the following way. The function L (comp. § 9, 1915) is a homogeneous quadratic function of the ψ_{ab}'s and when differentiated with respect to these variables it gives the quantities $\overline{\psi}_{ab}$. It may therefore also be regarded as a homogeneous quadratic function of the $\overline{\psi}_{ab}$. From (35), (29) and (32) [1], 1915 we find therefore

$$L = \tfrac{1}{8}\sqrt{-g}\,\Sigma\,(pqrs)\,(g^{pr}g^{qs} - g^{qr}g^{ps})\,\overline{\psi}_{pq}\,\overline{\psi}_{rs} \quad \ldots \quad (69)$$

Now we can also differentiate with respect to the g^{ab}'s, while not the ψ_{ab}'s but the quantities $\overline{\psi}_{ab}$ are kept constant, and we have e.g.

$$\left(\frac{\partial L}{\partial g^{ac}}\right)_{\psi} = -\left(\frac{\partial L}{\partial g^{ac}}\right)_{\overline{\psi}}.$$

According to (69) one part of the latter differential coefficient is

[1]) The quantities γ_{ab} in that equation are the same as those which are now denoted by g^{ab}.

obtained by differentiating the factor $\sqrt{-g}$ only and the other part by keeping this factor constant.

For the calculation of the first of these parts we can use the relation

$$\frac{\partial \log(\sqrt{-g})}{\partial g^{ac}} = -\tfrac{1}{2}\, g_{ac} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

and for the second part we find

$$\tfrac{1}{2} \sqrt{-g}\, \Sigma\, (pq)\, g^{pq}\, \overline{\psi}_{ap}\, \overline{\psi}_{cq}.$$

If (32) 1915 is used (67) and (68) finally become

$$\Sigma\, (q)\, \psi_{cq} \overline{\psi}_{cq} + \underset{a \neq c}{\Sigma}\, (a)\, \psi^{*}_{ac} \psi_{a'c'} = 2\mathrm{L},$$

$$\Sigma\, (q)\, \overline{\psi}_{cq}\, \psi_{bq} + \underset{a \neq c}{\Sigma} (a) \psi^{*}_{ab}\, \psi_{a'c'} = 0\,.$$

These equations are really fulfilled. This is evident from : $\psi_{aa} = 0$, $\overline{\psi}_{aa} = 0$, $\psi_{ba} = -\psi_{ab}$ and $\overline{\psi}_{ba} = -\overline{\psi}_{ab}$; besides, the meaning of ψ^{*}_{ab} (§ 11, 1915) and equation (35) 1915 must be taken into consideration.

§ 43. In nearly the same way we can treat the gravitation field of a system of incoherent material points; here the quantities w_a and u_a (§§ 4 and 5, 1915) play a similar part as ψ_{ab} and $\overline{\psi}_{ab}$ in what precedes. To consider a more general case we can suppose "molecular forces" to act between the material points (which we assume to be equal to each other); in such a way that in ordinary mechanics we should ascribe to the system a potential energy depending on the density only. Conforming to this we shall add to the Lagrangian function L (§ 4, 1915) a term which is some function of the density of the matter at the point P of the field-figure, such as that density is when by a transformation the matter at that point has been brought to rest. This can also be expressed as follows. Let $d\sigma$ be an infinitely small three-dimensional extension expressed in natural units, which at the point P is perpendicular to the world-line passing through that point, and $\overline{\varrho}\, d\sigma$ the number of points where $d\sigma$ intersects world-lines. The contribution of an element of the field-figure to the principal function will then be found by multiplying the magnitude of that element expressed in natural units by a function of $\overline{\varrho}$. Further calculation teaches us that the term to be added to L must have the form

$$\sqrt{-g}\, \varphi \left(\frac{P}{\sqrt{-g}} \right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (71)$$

where P is given by (15) 1915. As the Lagrangian function defined by (11) 1915 equally falls under this form and also the sum of this function and the new term, the expression (71) may be regarded as the *total* function L. The function φ may be left indeterminate. If now with this function the calculations of §§ 5 and 6, 1915 are repeated, we find the components of the stress-energy-tensor of the matter.

The equations for the gravitation field again take the form (65). T_{ab} is defined by an equation of the form (63), where on the left hand side we must differentiate while the w_a's are kept constant. Relation (66) can again be verified without difficulty.

We shall not, however, dwell upon this, as the following considerations are more general and apply e.g. also to systems of material points that are anisotropic as regards the configuration and the molecular actions.

§ 44. At any point P of the field-figure the Lagrangian function L will evidently be determined by the course and the mutual situation of the world-lines of the material points in the neighbourhood of P. This leads to the assumption that for constant g_{ab}'s the variation δL is a homogeneous linear function of the virtual displacements δx_a of the material points and of the differential coefficients

$$\frac{\partial \delta x_a}{\partial x_b},$$

these last quantities evidently determining the deformation of an infinitesimal part of the figure formed by the world-lines [1]).

The calculation becomes most simple, if we put

$$\mathrm{L} = \sqrt{-g}\,H \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (72)$$

and for constant g_{ab}'s

$$\delta H = \Sigma (a)\, U_a\, \delta x_a + \Sigma (ab)\, V_a^b \frac{\partial \delta x_a}{\partial x_b} \quad . \quad . \quad . \quad . \quad (73)$$

Considerations corresponding exactly to those mentioned in §§ 4—6, 1915, now lead to the equations of motion and to the following expressions for the components of the stress-energy-tensor

$$\mathfrak{T}_c^c = -\mathrm{L} - \sqrt{-g}\, V_c^c \quad . \quad . \quad . \quad . \quad . \quad . \quad (74)$$

and for $b \neq c$

$$\mathfrak{T}_c^b = -\sqrt{-g}\, V_c^b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (75)$$

The differential equations again take the form (65) if the quantities T_{ab} are defined by

[1]) In the cases considered in § 43, δL can indeed be represented in this way.

$$\left(\frac{\partial L}{\partial g^{ab}}\right)_x = \tfrac{1}{2}\sqrt{-g}\, T_{ab};$$

in the differentiation on the left hand side the coordinates of the material points are kept constant. To show that T_{ab} and $\mathfrak{T}_c^b$ satisfy equation (66) we must now show that

$$-L - \sqrt{-g}\, V^c_c = 2\,\Sigma\,(a)\, g^{ac}\left(\frac{\partial L}{\partial g^{ac}}\right)_x$$

and for $b \neq c$

$$-\sqrt{-g}\, V^b_c = 2\,\Sigma\,(a) g^{ab}\left(\frac{\partial L}{\partial g^{ac}}\right)_x$$

If here the value (72) is substituted for L and if (70) is taken into account, these equations say that for all values of b and c we must have

$$2\,\Sigma\,(a)\, g^{ab}\left(\frac{\partial H}{\partial g^{ac}}\right)_x + V^b_c = 0 \quad . \; . \; . \; . \; . \quad (76)$$

Now this relation immediately follows from a condition, to which L must be subjected at any rate, viz. that LdS is a scalar quantity. This involves that in a definite case we must find for H always the same value whatever be the choice of coordinates.

§ 45. Let us suppose that instead of only one coordinate x_c a new one x_c' has been introduced, which differs infinitely little from x_c, with the restriction that if

$$x'_c = x_c + \xi_c$$

the term ξ_c depends on the coordinate x_b only and is zero at the point in question of the field-figure. The quantities g^{ab} then take other values and in the new system of coordinates the world-lines of the material points will have a slightly changed course.

By each of these circumstances separately H would change, but all together must leave it unaltered. As to the first change we remark that, according to the transformation formula for g^{ab}, the variation δg^{ab} vanishes when the two indices are different from c, while

$$\delta g^{cc} = 2 g^{cb}\,\frac{\partial \xi_c}{\partial x_b}$$

and for $a \neq c$

$$\delta g^{ac} = \delta g^{ca} = g^{ab}\,\frac{\partial \xi_c}{\partial x_b}.$$

The change of H due to these variations is

$$2\,\frac{\partial \xi_c}{\partial x_b}\,\Sigma\,(a)\, g^{ab}\left(\frac{\partial H}{\partial g^{ac}}\right)_x.$$

Further, in the new system of coordinates the figure formed by the world-lines differs from that figure in the old system by the variation $\delta x_c = \xi_c$ which is a function of x_b only. Therefore according to (73) the second variation of H is

$$V_c^b \frac{\partial \xi_c}{\partial x_b}.$$

By putting equal to zero the sum of this expression and the preceding one we obtain (76).

§ 46. We have thus deduced for some cases the equations of the gravitation field from the variation theorem. Probably this can also be done for thermodynamic systems, if the Lagrangian function is properly chosen in connexion with the thermodynamic functions, entropy and free energy. But as soon as we are concerned with irreversible phenomena, when e.g. the energy-current consists in a conduction of heat, the variation principle cannot be applied. We shall then be obliged to take EINSTEIN's field-equations as our point of departure, unless, considering the motions of the individual atoms or molecules, we succeed in treating these by means of the generalized principle of HAMILTON.

§ 47. Finally we shall consider the stresses, the energy etc. which belong to the gravitation field itself. The results will be the same for all the systems treated above, but we shall confine ourselves to the case of §§ 44 and 45. We suppose certain external forces K_a to act on the material points, though we shall see that strictly speaking this is not allowed.

For any displacements δx_a of the matter and variations of the gravitation field we first have the equation which summarizes what we found above

$$\delta L + \frac{1}{2\varkappa}\delta Q + \Sigma(a) K_a \delta x_a = \sqrt{-g}\,\Sigma(a) U_a \delta x_a +$$
$$+ \Sigma(ab)\frac{\partial}{\partial x_b}(\sqrt{-g}\,V_a^b\,\delta x_a) - \Sigma(ab)\frac{\partial}{\partial x_b}(\sqrt{-g}\,V_a^b)\,\delta x_a +$$
$$+ \Sigma(ab)\left(\frac{\partial L}{\partial g^{ab}}\right)_x \delta g^{ab} + \frac{1}{2\varkappa}\delta_1 Q + \frac{1}{2\varkappa}\delta_2 Q + \Sigma(a) K_a \delta x_a.$$

In virtue of the equations of motion of the matter, the terms with δx_a cancel each other on the right hand side and similarly, on account of the equations of the gravitation field, the terms with δg^{ab} and $\delta_1 Q$. Thus we can write [1])

[1]) To make the notation agree with that of § 38 b has been replaced by e.

$$\Sigma(a) K_a \delta x_a = -\delta L + \Sigma(ae) \frac{\partial}{\partial x_e} (\sqrt{-g}\, V_a^e\, \delta x_a) - \frac{1}{2\varkappa} (\delta Q - \delta_2 Q). \quad (77)$$

Let us now suppose that only the coordinate x_h undergoes an infinitely small change, which has the same value at all points of the field-figure. Let at the same time the system of values g_{ab} be shifted everywhere in the direction of x_h over the distance δx_h. The left hand side of the equation then becomes $K_h \delta x_h$ and we have on the right hand side

$$\delta L = -\frac{\partial L}{\partial x_h} \delta x_h, \quad \delta Q = -\frac{\partial Q}{\partial x_h} \delta x_h.$$

After dividing the equation by δx_h we may thus, according to (74) and (75), write

$$-\Sigma(e) \frac{\partial \mathfrak{T}_h^e}{\partial x_e} = -div_h \mathfrak{T}.$$

By the same division we obtain from $\delta Q - \delta_2 Q$ the expression occurring on the left hand side of (51), which we have represented by

$$\Sigma(e) \frac{\partial \mathfrak{s}_h^e}{\partial x_e} = div_h \mathfrak{s},$$

where the complex $\mathfrak{s}$ is defined by (52) and (53). If therefore we introduce a new complex $\mathfrak{t}$ which differs from $\mathfrak{s}$ only by the factor $\frac{1}{2\varkappa}$, so that

$$\mathfrak{t}_h^e = \frac{1}{2\varkappa} \mathfrak{s}_h^e, \quad \ldots \ldots \ldots \ldots \quad (78)$$

we find

$$K_h = -div_h \mathfrak{T} - div_h \mathfrak{t}. \quad \ldots \ldots \ldots \quad (79)$$

The form of this equation leads us to consider $\mathfrak{t}$ as the stress-energy-complex of the gravitation field, just as $\mathfrak{T}$ is the stress-energy-tensor for the matter. We need not further explain that for the case $K_h = 0$ the four equations contained in (79) express the conservation of momentum and of energy for the total system, matter and gravitation field taken together.

§ 48. To learn something about the nature of the stress-energy-complex $\mathfrak{t}$ we shall consider the stationary gravitation field caused by a quantity of matter without motion and distributed symmetrically around a point O. In this problem it is convenient to introduce for the three space coordinates x_1, x_2, x_3, (x_4 will represent the time) "polar" coordinates. By x_3 we shall therefore denote a quantity r

which is a measure for the "distance" to the centre. As to x_1 and x_2, we shall put $x_1 = \cos\vartheta, x_2 = \varphi$, after first having introduced polar coordinates ϑ, φ (in such a way that the rectangular coordinates are $r\cos\vartheta, r\sin\vartheta\cos\varphi, r\sin\vartheta\sin\varphi$). It can be proved that, because of the symmetry about the centre, $g_{ab} = 0$ for $a =|= b$, while we may put for the quantities g_{aa}

$$g_{11} = -\frac{u}{1-x_1^2}, \qquad g_{22} = -u\,(1-x_1^2), \qquad g_{33} = -v, \qquad g_{44} = w, \tag{80}$$

where u, v, w are certain functions of r. Differentiations of these functions will be represented by accents. We now find that of the complex $\mathfrak{t}$ only the components $\mathfrak{t}_1^1$, $\mathfrak{t}_3^3$ and $\mathfrak{t}_4^4$ are different from zero. The expressions found for them may be further simplified by properly choosing r. If the distance to the centre O is measured by the time the light requires to be propagated from O to the point in question, we have $w = v$. One then finds

$$\left.\begin{aligned}
\mathfrak{t}_1^1 &= \frac{1}{2\varkappa}\left(-\frac{u'^2}{2u} + 2u'' - \frac{uv'^2}{v^2} + \frac{uv''}{v}\right),\\
\mathfrak{t}_3^3 &= \frac{1}{2\varkappa}\left(-2v + \frac{u'^2}{2u} + \frac{u'v'}{v}\right),\\
\mathfrak{t}_4^4 &= \frac{1}{2\varkappa}\left(-2v - \frac{u'^2}{2u} + 2u'' + \frac{uv''}{v}\right).
\end{aligned}\right\} \quad . \quad . \quad . \quad . \tag{81}$$

§ 49. We must assume that in the gravitation fields really existing the quantities g_{ab} have values differing very little from those which belong to a field without gravitation. In this latter we should have

$$u = r^2, \qquad v = w = 1,$$

and thus we put now

$$u = r^2\,(1 + \mu), \qquad v = w = 1 + \nu,$$

where the quantities μ and ν which depend on r are infinitely small, say of the first order, and their derivatives too. Neglecting quantities of the second order we find from (81)

$$\mathfrak{t}_1^1 = \frac{1}{2\varkappa}\,(2 + 2\mu + 6r\mu' + 2r^2\mu'' + r^2\nu''),$$

$$\mathfrak{t}_3^3 = \frac{1}{\varkappa}\,(\mu - \nu + r\mu' + r\nu'),$$

$$\mathfrak{t}_4^4 = \frac{1}{2\varkappa}\,(2\mu - 2\nu + 6r\mu' + 2r^2\mu'' + r^2\nu'').$$

For our degree of approximation we may suppose that of the quantities T_{ab} only T_{44} differs from 0. If we put

$$T_{44} = \varrho\,, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (82)$$

a quantity which depends on r and which we shall assume to be zero outside a certain sphere, we find from the field equations

$$\mu = \varkappa \left\{ -\frac{2}{r}\int_0^r \frac{dr}{r}\int_0^r r^2\varrho dr - \frac{1}{r}\int_0^r r^2\varrho dr + \int_\infty^r r\varrho dr \right\},$$

$$\nu = \varkappa \left\{ -\frac{1}{r}\int_0^r r^2\varrho dr + \int_\infty^r r\varrho dr \right\}.$$

We thus obtain

$$\mathfrak{t}_1{}^1 = \frac{1}{\varkappa} + \int_\infty^r r\varrho dr - \frac{1}{r}\int_0^r r^2\varrho dr - \tfrac{1}{2} r^2\varrho, \quad . \quad . \quad . \quad . \qquad (83)$$

$$\mathfrak{t}_3{}^3 = 0\,, \quad \mathfrak{t}_4{}^4 = -\tfrac{1}{2} r^2\varrho. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (84)$$

§ 50. If first we leave aside the first term of $\mathfrak{t}_1{}^1$, which would also exist if no attracting matter were present, it is remarkable that the gravitation constant $\varkappa$ does not occur in the stress $\mathfrak{t}_1{}^1$, nor in the energy $\mathfrak{t}_4{}^4$, the same would have been found if we had used other coordinates. This constitutes an important difference between EINSTEIN's theory and other theories in which attracting or repulsing forces are reduced to "field actions". The pulsating spheres of BJERKNES e.g. are subjected to forces which, for a given motion, are proportional to the density of the fluid in which they are imbedded; and the changes of pressure and the energy in that fluid are likewise proportional to this density. In this case we shall therefore ascribe to the stress-energy-complex values proportional to the intensity of the actions which we want to explain. In EINSTEIN's theory such a proportionality does not exist. The value of $\mathfrak{t}_4{}^4$ is of the same order of magnitude as $\mathfrak{T}_4{}^4$ in the matter. To our degree of approximation we find namely from (82) $\mathfrak{T}_4{}^4 = r^2\varrho$.

§ 51. If we had not worked with polar coordinates but with rectangular coordinates we should have had to put for the field without gravitation $g_{11} = g_{22} = g_{33} = -1$, $g_{44} = 1$, $g_{ab} = 0$ for $a \neq b$. Then we should have found zero for all the components of the complex. In the system of coordinates used above we found for the field without gravitation $\mathfrak{t}_1{}^1 = \frac{1}{\varkappa}$; this is due to the complex $\mathfrak{t}$ being no tensor. If it were, the quantities $\mathfrak{t}_a{}^b$ would be zero in every system of coordinates if they had that value in one system.

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It is also remarkable that in real cases the first term in (83) can be much larger than the following ones. If we consider e. g. a point P outside the attracting sphere, we can prove that the ratio of the first term to the third is of the same order as the ratio of the square of the velocity of light to the square of the velocity with which a material point can describe a circular orbit passing through P.

The following must also be noticed. In the system of polar coordinates used above there will exist in the field without gravitation the stress $\mathfrak{t}_1{}^1 = \frac{1}{\varkappa}$. If a stress of this magnitude were produced by means of actions which ~~give rise to a~~ stress-energy-*tensor*, the passage to rectangular coordinates would give us a stress which becomes infinite at the point O. In those coordinates we should namely have

$$\mathfrak{t}'_1{}^1 = \frac{\sin^2\vartheta}{r^2} \cdot \frac{1}{\varkappa}$$

§ 52. Evidently it would be more satisfactory if we could ascribe a stress-energy-*tensor* to the gravitation field. Now this can really be done. Indeed, the quantities $\mathfrak{s}^e_{0h}$ determined by (57) form a tensor and according to (58), (79) may be replaced by

$$K_h = - div_h\, \mathfrak{T} - div_h\, \mathfrak{t}_0\,, \quad . \quad . \quad . \quad . \quad . \quad . \quad (85)$$

if $\mathfrak{t}_0$ is defined by a relation similar to (78), viz.

$$\overset{e}{\mathfrak{t}}_{0h} = \frac{1}{2\varkappa}\, \overset{e}{\mathfrak{s}}_{0h} \,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (86)$$

Equation (85) shows that, just as well as $\mathfrak{t}^e_h$, we may consider the quantities $\mathfrak{t}^e_{0h}$ as the stresses etc. in the gravitation field. This way of interpretation is very simple. With a view to (41) we can namely derive from the equations for the gravitation field (65)

$$G = \varkappa T$$

and

$$T_{ab} = - \frac{1}{\varkappa}\, (G_{ab} - \tfrac{1}{2}\, g_{ab} G).$$

Further we find from (66)

$$\overset{e}{\mathfrak{T}}_h = \frac{1}{2\varkappa}\, G \Sigma(a) \mathfrak{g}^{ae} g_{ah} - \frac{1}{\varkappa}\, \Sigma(a) \mathfrak{g}^{ae} G_{ah}$$

and from (57) and (86)

$$\overset{e}{\mathfrak{t}}_{0h} = - \overset{e}{\mathfrak{T}}_h \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (87)$$

At every point of the field-figure the components of the stress-energy-tensor of the gravitation field would therefore be equal to the corresponding quantities for the matter or the electro-magnetic system with the opposite sign. It is obvious that by this the condition of the conservation of momentum and energy for the *whole* system would be immediately fulfilled. It was in fact this circumstance that made me think of the tensor $\mathfrak{t}_0 = -\mathfrak{T}$. The way in which $\mathfrak{s}_0$ was introduced in §§ 38 and 39 has only been chosen in order to lay stress on (58) being an identity, so that equation (85) is but another form of (79).

At first sight the relations (87) and the conception to which they have led, may look somewhat startling. According to it we should have to imagine that behind the directly observable world with its stresses, energy etc. there is hidden the gravitation field with stresses, energy etc. that are everywhere equal and opposite to the former; evidently this is in agreement with the interchange of momentum and energy which accompanies the action of gravitation. On the way of a light-beam e.g. there would be everywhere in the gravitation field an energy current equal and opposite to the one existing in the beam. If we remember that this hidden energy-current can be fully described mathematically by the quantities g_{ab} and that only the interchange just mentioned makes it perceptible to us, this mode of viewing the phenomena does not seem unacceptable. At all events we are forcibly led to it if we want to preserve the advantage of a stress-energy-*tensor* also for the gravitation field. It can namely be shown that a tensor which is transformed in the same way as the tensor $\mathfrak{t}_0$ defined by (57) and (86) and which in every system of coordinates has the same divergency as the latter, must coincide with $\mathfrak{t}_0$.

Finally we may remark that (78), (86), (58), (87) give

$$div\ \mathfrak{t} = div\ \mathfrak{t}_0 = -div\ \mathfrak{T},$$

so that we have, both from (79) and from (85), $K_h = 0$.

The question is this, that, so long as the gravitation field is considered as given, we may introduce "external" forces, but that in the equations for the gravitation field itself we must also take into consideration the stress-energy-tensor of the system by which those forces are exerted.

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