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Physics. — "*On* EINSTEIN's *Theory of gravitation.*" IV. By Prof. H. A. LORENTZ.

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§ 53. The expressions for the stress-energy-components of the gravitation field found in the preceding paper call for some further remarks. If by δ_h^e we denote a quantity having the value 1 for $e = h$ and being 0 for $e \neq h$, those expressions can be written in the form (comp. equations (52) and (78))

$$t_h^e = \frac{1}{2\varkappa} \left\{ - \delta_h^e Q + \Sigma (ab) \frac{\partial Q}{\partial g_{ab,e}} g_{ab,h} + \Sigma (abf) \frac{\partial Q}{\partial g_{ab,fe}} g_{ab\,fh} - \right.$$
$$\left. - \Sigma (abf) \frac{\partial}{\partial x_f} \left(\frac{\partial Q}{\partial g_{ab,ef}} \right) g_{ab,h} \right\} \quad . \quad . \quad . \quad . \quad . \quad (88)$$

They contain the first and second derivatives of the quantities g_{ab}. EINSTEIN on the contrary has given values for the stress-energy-components which contain the first derivatives only and which therefore are in many respects much more fit for application.

It will now be shown how we can also find formulae without second derivatives, if we start from (88).

§ 54. For this purpose we shall consider the complex $\mathfrak{u}$ defined by

$$\mathfrak{u}_h^e = \frac{1}{2\varkappa} \left\{ \delta_h^e Q - \Sigma (abf) \frac{\partial}{\partial x_h} \left(\frac{\partial Q}{\partial g_{ab,fe}} g_{ab,f} \right) \right\} \quad . \quad . \quad . \quad (89)$$

and we shall seek its divergency.

We have

$$(div\ \mathfrak{u})_h = \Sigma (e) \frac{\partial \mathfrak{u}_h^e}{\partial x_e} = \frac{1}{2\varkappa} \left\{ \frac{\partial Q}{\partial x_h} - \Sigma (abfe) \frac{\partial^2}{\partial x_e \partial x_h} \left(\frac{\partial Q}{\partial g_{ab,fe}} g_{ab,f} \right) \right\}$$

or

$$(div\ \mathfrak{u})_h = \frac{1}{2\varkappa} \frac{\partial R}{\partial x_h} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (90)$$

if we put

$$R = Q - \Sigma (abfe) \frac{\partial}{\partial x_e} \left(\frac{\partial Q}{\partial g_{ab,fe}} g_{ab,f} \right) \quad . \quad . \quad . \quad . \quad (91)$$

Now $Q = \sqrt{-g}\, G$ can be divided into two parts, the first of which Q_1 contains differential coefficients of the quantities g_{ab} of the first order only, while the second Q_2 is a homogeneous linear function of the second derivatives of those quantities. This latter involves that, if we replace (91) by

$$R = Q_1 + Q_2 - \Sigma\,(abfe)\left(\frac{\partial Q}{\partial g_{ab,fe}}\, g_{ab,fe}\right) - \Sigma(abfe)\,\frac{\partial}{\partial x_e}\left(\frac{\partial Q}{\partial g_{ab,fe}}\right) g_{ab,f},$$

the second and the third term annul each other. Thus

$$R = Q_1 - \Sigma\,(abfe)\,\frac{\partial}{\partial x_e}\left(\frac{\partial Q}{\partial g_{ab,fe}}\right) g_{ab,f} \quad . \quad . \quad . \quad . \quad (92)$$

If now we define a complex $\mathfrak{v}$ by the equation

$$\mathfrak{v}_h{}^e = -\frac{1}{2\varkappa}\,\delta_h{}^e\,R, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (93)$$

we have

$$(div\ \mathfrak{v})_h = -\frac{1}{2\varkappa}\frac{\partial R}{\partial x_h}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (94)$$

If finally we put

$$\mathfrak{t}' = \mathfrak{t} + \mathfrak{u} + \mathfrak{v},$$

we infer from (90) and (94)

$$div\ \mathfrak{t}' = div\ \mathfrak{t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (95)$$

and from (88), (89), (93) and (92)

$$\mathfrak{t}'_h{}^h = \frac{1}{2\varkappa}\left\{ - Q_1 + \Sigma\,(ab)\,\frac{\partial Q}{\partial g_{ab,h}}\, g_{ab,h} - \Sigma\,(abf)\,\frac{\partial}{\partial x_h}\left(\frac{\partial Q}{\partial g_{ab,fh}}\right) g_{ab,f} - \right.$$

$$\left. \Sigma\,(abf)\,\frac{\partial}{\partial x_f}\left(\frac{\partial Q}{\partial g_{ab,hf}}\right) g_{ab,h} + \Sigma\,(abfe)\,\frac{\partial}{\partial x_e}\left(\frac{\partial Q}{\partial g_{ab,fe}}\right) g_{ab,f} \right\} \quad (96)$$

and for $e \neq h$

$$\mathfrak{t}'_h{}^e = \frac{1}{2\varkappa}\left\{ \Sigma\,(ab)\,\frac{\partial Q}{\partial g_{ab,e}}\, g_{ab,h} - \Sigma\,(abf)\,\frac{\partial}{\partial x_h}\left(\frac{\partial Q}{\partial g_{ab,fe}}\right) g_{ab,f} - \right.$$

$$\left. - \Sigma\,(abf)\,\frac{\partial}{\partial x_f}\left(\frac{\partial Q}{\partial g_{ab,ef}}\right) g_{ab,h} \right\} \quad . \quad . \quad . \quad . \quad . \quad (97)$$

Formula (95) shows that the quantities $\mathfrak{t}'_h{}^e$ can be taken just as well as the expressions (88) for the stress-energy-components and we see from (96) and (97) that these new expressions contain only the first derivatives of the coefficients g_{ab}; they are homogeneous quadratic functions of these differential coefficients.

This becomes clear when we remember that Q_1 is a function of this kind and that only Q_1 contributes something to the second term of (96) and the first of (97); further that the derivatives of Q occurring in the following terms contain only the quantities g_{ab} and not their derivatives.

§ 55. Einstein's stress-energy-components have a form widely different from that of the above mentioned ones. They are

$$\mathfrak{t}^e_{(E)h} = \frac{1}{2\varkappa} \delta_h^e \Sigma (abcf) g^{ab} \Gamma_{ac}^f \Gamma_{bf}^c - \frac{1}{\varkappa} \Sigma (abc) g^{ab} \Gamma_{ac}^e \Gamma_{bh}^c,$$

where for the sake of simplicity it has been assumed that $\sqrt{-g}=1$. Further we have

$$\Gamma_{ab}^c = -\left\{ {ab \atop c} \right\} = - \Sigma (e) g^{ce} \left[{ab \atop e} \right].$$

If now our formulae (96) and (97) are likewise simplified by the assumption $\sqrt{-g}=1$ (so that Q becomes equal to G), we may expect that $\mathfrak{t}'$ will become identical with $\mathfrak{t}_{(E)}$. This is really so in the case $g_{ab} = 0$ for $a \neq b$; by which it seems very probable that the agreement will exist in general.

In the preceding paper it was shown already that the stress-energy-components t_h^e do not form a "tensor", but what was called a "complex". The same may be said of the quantities $\mathfrak{t}'^e_h$ defined by (96) and (97) and of the expressions given by EINSTEIN. If we want a stress-energy-*tensor*, there are only left the quantities $\mathfrak{t}^e_{0h}$ defined by (86) and (57), the values of which are always equal and opposite to the corresponding stress-energy-components $\mathfrak{T}_h^e$ for the matter or the electromagnetic field.

It must be noticed that the four equations

$$\Sigma (e) \frac{\partial}{\partial x_e} (\mathfrak{T}_h^e + \mathfrak{T}^e_{(g)h}) = 0$$

always express the same relations, whether we choose $\mathfrak{t}^e_{0h}$, t_h^e, $\mathfrak{t}'^e_h$ or $\mathfrak{t}^e_{(E)h}$ as stress-energy-components $\mathfrak{T}^e_{(g)h}$ of the gravitation field. If however in a definite case we want to use the equations in order to calculate how the momentum and the energy of the matter and the electromagnetic field change by the gravitational actions, it is best to use $\mathfrak{t}'^e_h$ or $\mathfrak{t}^e_{(E)h}$, just because these quantities are homogeneous quadratic functions of the derivatives $g_{ab,c}$.

Experience namely teaches us that the gravitation fields occurring in nature may be regarded as feeble, in this sense that the values of the g_{ab}'s are little different from those which might be assumed if no gravitation field existed. For these latter values, which will be called the "normal" ones, we may write in orthogonal coordinates

$$g_{11} = g_{22} = g_{33} = -1, \quad g_{44} = c^2, \quad g_{ab} = 0, \quad \text{for} \quad a \neq b. \tag{98}$$

In a first approximation, which most times will be sufficient, the deviations of the values of the g_{ab}'s from these normal ones may be taken proportional to the gravitation constant $\varkappa$. This factor also appears in the differential coefficients $g_{ab,c}$; hence, according to the character of the functions $\mathfrak{t}'^e_h$ mentioned above (and on account of the factor $\frac{1}{\varkappa}$ in (96) and (97)) these functions become proportional to $\varkappa$, so that in a feeble gravitation field they have low values.

§ 56. Because of the complicated form of equations (96) and (97), we shall confine ourselves to the calculation for some cases of $\mathfrak{t}'^{\,4}_{4}$, i.e. of the energy per unit of volume. This calculation is considerably simplified if we consider stationary fields only. Then all differential coefficients with respect to x_4 vanish, so that we have according to (96)

$$\mathfrak{t}'^{\,4}_{4} = \frac{1}{2\varkappa}\left\{- Q_1 + \Sigma\,(abfe)\,\frac{\partial}{\partial x_e}\left(\frac{\partial Q}{\partial g_{ab,fe}}\right) g_{ab,f}\right\} \quad . \quad . \quad (99)$$

We shall work out the calculation, first for a field without gravitation and secondly for the case of an attracting spherical body in which the matter is distributed symmetrically round the centre.

If there is no gravitation field we may take for the quantities g_{ab} the "normal" values. For the case of orthogonal coordinates these are given by (98). When we want to use the polar coordinates introduced into § 48 we have the corresponding formulae

$$\left.\begin{aligned} g_{11} = -\frac{r^2}{1-x_1^{\,2}},\quad g_{22} = -r^2(1-x_1^{\,2}),\quad g_{33} = -1,\quad g_{44} = c^2, \\ g_{ab} = 0, \quad \text{for } a \neq b. \end{aligned}\right\} \quad (100)$$

If, using polar coordinates, we have to do with an attracting sphere and if we take its centre as origin, we may put

$$g_{11} = -\frac{u}{1-x_1^{\,2}},\quad g_{22} = -(1-x_1^{\,2})\,u,\quad g_{33} = -v,\quad g_{44} = w, \quad (101)$$

where u, v, w are functions of r. The g_{ab}'s which belong to an orthogonal system of coordinates may be expressed in the same functions.

These g_{ab}'s are

$$g_{11} = -\frac{u}{r^2} + \frac{x_1^{\,2}}{r^2}\left(\frac{u}{r^2} - v\right), \ etc.$$

$$g_{12} = \frac{x_1 x_2}{r^2}\left(\frac{u}{r^2} - v\right), \ etc.$$

$$g_{14} = g_{24} = g_{34} = 0, \qquad g_{44} = w.$$

The "etc." means that for g_{22}, g_{33} we have similar expressions as for g_{11} and for g_{23}, g_{31} similar ones as for g_{12}.

§ 57. In order to deduce the differential equations determining u, v, w we may arbitrarily use rectangular or polar coordinates; the latter however are here to be preferred. If differentiations with respect to r are indicated by accents, we have according to (40) and (101)

$$G_{11} = \frac{1}{1-x_1^2}\left(-1 + \frac{u''}{2v} - \frac{u'v'}{4v^2} + \frac{u'w'}{4vw}\right),$$

$$G_{22} = (1-x_1^2)\left(-1 + \frac{u''}{2v} - \frac{u'v'}{4v^2} + \frac{u'w'}{4vw}\right),$$

$$G_{33} = \frac{u''}{u} - \frac{u'^2}{2u^2} - \frac{u'v'}{2uv} - \frac{v'w'}{4vw} + \frac{w''}{2w} - \frac{w'^2}{4w^2},$$

$$G_{44} = -\frac{u'w'}{2uv} + \frac{v'w'}{4v^2} - \frac{w''}{2v} + \frac{w'^2}{4vw},$$

$$G_{ab} = 0, \text{ for } a \neq b.$$

So we have found the left hand sides of the field equations (65). Before considering these equations more closely we shall introduce the simplification that the g_{ab}'s are very little different from the normal values (100). For these latter we have

$$u = r^2 \quad , \quad v = 1 \quad , \quad w = c^2 \quad . \quad . \quad . \quad . \quad . \quad (102)$$

and therefore we now put

$$u = r^2(1+\lambda) \quad , \quad v = 1 + \mu \quad , \quad w = c^2(1+\nu) \quad . \quad . \quad (103)$$

The quantities λ, μ, ν, which depend on r, will be regarded as infinitely small of the first order and in the field equations we shall neglect quantities of second and higher orders.

Then we may write for G_{11} etc.

$$G_{11} = \frac{1}{1-x_1^2}(\lambda + 2r\lambda' + \tfrac{1}{2}r^2\lambda'' - \mu - \tfrac{1}{2}r\mu' + \tfrac{1}{2}r\nu'),$$

$$G_{22} = (1-x_1^2)(\lambda + 2r\lambda' + \tfrac{1}{2}r^2\lambda'' - \mu - \tfrac{1}{2}r\mu' + \tfrac{1}{2}r\nu'),$$

$$G_{33} = \frac{2}{r}\lambda' + \lambda'' - \frac{1}{r}\mu' + \tfrac{1}{2}\nu'',$$

$$G_{44} = -c^2\left(\frac{1}{r}\nu' + \tfrac{1}{2}\nu''\right).$$

On the right hand-sides of the field equations (65) we may take for g_{ab} the normal value; moreover we shall take for T_{ab} and T the values which hold for a system of incoherent material points. We may do so if we assume no other internal stresses but those caused by the mutual attractions; these stresses may be neglected in the present approximation.

As we supposed the attracting matter to be at rest we have according to (10), (16) and (15) (1915) $w_1 = w_2 = w_3 = 0, w_4 = \varrho$, $u_1 = u_2 = u_3 = 0, u_4 = c^2\varrho, P = c\varrho$.

In the notations we are now using we have further, according to (23) (1915),

$$\mathfrak{T}e_h = \frac{u_h w_e}{P},$$

so that of the stress-energy-components of the matter only one is different from zero, namely

$$\mathfrak{T}_4{}^4 = c\varrho.$$

Further (66) involves that, also of the quantities T_{ab}, only one, namely T_{44}, is not equal to zero. As we may put $\sqrt{-g} = cr^2$, we have namely

$$T_{44} = \frac{c^2}{r^2}\varrho, \quad T = \frac{1}{r^2}\varrho.$$

Finally we are led to the three differential equations

$$\lambda + 2r\lambda' + \tfrac{1}{2}r^2\lambda'' - \mu - \tfrac{1}{2}r\mu' + \tfrac{1}{2}r\nu' = -\tfrac{1}{2}\varkappa\varrho, \quad . \quad (104)$$

$$2r\lambda' + r^2\lambda'' - r\mu' + \tfrac{1}{2}r^2\nu'' = -\tfrac{1}{2}\varkappa\varrho, \quad . \quad . \quad . \quad (105)$$

$$r\nu' + \tfrac{1}{2}r^2\nu'' = \tfrac{1}{2}\varkappa\varrho \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (106)$$

It may be remarked that $\varrho dx_1 dx_2 dx_3$ represents the "mass" present in the element of volume $dx_1 dx_2 dx_3$. Because of the meaning of x_1, x_2, x_3 (§ 48) the mass in the shell between spheres with radii r and $r + dr$ is found when $\varrho dx_1 dx_2 dx_3$ is integrated with respect to x_1 between the limits -1 and $+1$ and with respect to x_2 between 0 and 2π. As ϱ depends on r only, this latter mass becomes $4\pi\varrho dr$, so that ϱ is connected with the "density" in the ordinary sense of the word, which will be called $\overline{\varrho}$, by the equation

$$\varrho = r^2\overline{\varrho}.$$

The differential equations also hold outside the sphere if ϱ is put equal to zero. We can first imagine ϱ to change gradually to 0 near the surface and then treat the abrupt change as a limiting case.

In all the preceding considerations we have tacitly supposed the second derivatives of the quantities g_{ab} to have everywhere finite values. Therefore ν and ν' will be continuous at the surface, even in the case of an abrupt change.

§ 58. Equation (106) gives

$$\nu' = \frac{\varkappa}{r^2}\int_0^r \varrho\, dr, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (107)$$

where the integration constant is determined by the consideration that for $r = 0$ all the quantities g_{ab} and their derivatives must be finite, so that for $r = 0$ the product $r^2\nu'$ must be zero. As it is natural to suppose that at an infinite distance ν vanishes, we find further

$$v = \varkappa \int_{\infty}^{r} \frac{dr}{r^2} \int_{0}^{r} \varrho dr \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (108)$$

The quantities λ and μ on the contrary are not completely determined by the differential equations. If namely equations (105) and (106) are added to (104) after having been multiplied by $-\frac{1}{2}$ and $+\frac{1}{2}$ respectively, we find

$$\lambda + r\lambda' - \mu + rv' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (109)$$

and it is clear that (104) and (105) are satisfied as soon as this is the case with this condition (109) and with (106). So we have only to attend to (108) and (109). The indefiniteness remaining in λ and μ is inevitable on account of the covariancy of the field equations. It does not give rise to any difficulties.

Equation (107) teaches us that near the centre

$$v' = \frac{1}{3} \varkappa \overline{\varrho}_0 r$$

if $\overline{\varrho}_0$ is the density at the centre, whereas from (108) we find a finite value for v itself. This confirms what has been said above about the values at the centre. We shall assume that at that point λ, μ and their derivatives have likewise finite values. Moreover we suppose (and this agrees with (109)) that λ, μ, λ' and μ' are continuous at the surface of the sphere.

If a is the radius of the sphere we find from (108) for an external point

$$v = -\frac{\varkappa}{r} \int_{0}^{a} \varrho \, dr.$$

Without contradicting (109) we may assume that at a great distance from the centre λ and μ are likewise proportional to $\frac{1}{r}$, so that λ' and μ' decrease proportionally to $\frac{1}{r^2}$.

§ 59. We can now continue the calculation of t'^4_4 (§ 56). Substituting (101) in (99) and using polar coordinates we find

$$t'^4_4 = -\frac{1}{2\varkappa} u \sqrt{\frac{w}{v}} \left(\frac{1}{2} \frac{u'^2}{u^2} + \frac{u'w'}{uw} \right),$$

whence by substituting (102) we derive for a field without gravitation

$$t'^4_4 = -\frac{c}{\varkappa}.$$

This equation shows that, working with polar coordinates, we should have to ascribe a certain negative value of the energy to a field without gravitation, in such a way (comp. § 57) that the energy in the shell between the spheres described round the origin with radii r and $r + dr$ becomes

$$-\frac{4\pi c}{\varkappa}\, dr.$$

The density of the energy in the ordinary sense of the word would be inversely proportional to r^2, so that it would become infinite at the centre.

It is hardly necessary to remark that, using rectangular coordinates we find a value zero for the same case of a field without gravitation. The normal values of g_{ab} are then constants and their derivatives vanish.

§ 60. Using rectangular coordinates we shall now indicate the form of $\mathfrak{t}'^4_4$ for the field of a spherical body, with the approximation specified in § 57. Thus we put

$$\left.\begin{aligned} g_{11} &= -(1+\lambda) + \frac{x_1^2}{r^2}(\lambda-\mu), \ etc. \\ g_{12} &= \frac{x_1 x_2}{r^2}(\lambda-\mu), \ etc. \\ g_{14} &= g_{24} = g_{34} = 0 \quad , \quad g_{44} = c^2(1+\nu). \end{aligned}\right\} \quad . \; . \; . \quad (110)$$

By (109) and (110) we find[1])

[1]) Of the laborious calculation it may be remarked here only that it is convenient to write the values (110) in the form

$$g_{11} = -1 + \alpha + \frac{\partial^2\beta}{\partial x_1^2}, \ etc.$$

$$g_{12} = \frac{\partial^2\beta}{\partial x_1 \partial x_2}, \ etc.$$

where α and β are infinitesimal functions of r. We then find

$$\mathfrak{t}'^4_4 = \frac{c}{2\varkappa}\left\{-\tfrac{1}{2}\Sigma(a)\left(\frac{\partial\alpha}{\partial x_a}\right)^2 + \Sigma(a)\frac{\partial\nu}{\partial x_a}\frac{\partial\alpha}{\partial x_a} + \right.$$
$$\left. + \tfrac{1}{4}\Sigma(a\,i\,k)\left[\frac{\partial^3\beta}{\partial x_a \partial x_i^2}\frac{\partial^3\beta}{\partial x_a \partial x_k^2} - \left(\frac{\partial^3\beta}{\partial x_a \partial x_i \partial x_k}\right)^2\right]\right\}$$
$$(a, i, k = 1, 2, 3)$$

which reduces to (111) if the relations between α, β and λ, μ, viz.

$$\alpha + \frac{1}{r}\beta' = -\lambda \quad , \quad -\frac{1}{r}\beta' + \beta'' = \lambda - \mu$$

and the equality $\alpha' = \nu'$ involved in (109) are taken into consideration.

$$\mathfrak{t}'^{4}_{4} = \frac{c}{4\varkappa}\left\{ v'^{2} + \frac{1}{r}(\lambda-\mu)\left[\frac{1}{r}(\lambda-\mu) + 2(\lambda'-\mu')\right]\right\} \quad . \quad . \quad (111)$$

Thus we see (comp. § 58) that at a distance from the attracting sphere $\mathfrak{t}'^{4}_{4}$ decreases proportionally to $\frac{1}{r^4}$. Further it is to be noticed that on account of the indefiniteness pointed out in § 58, there remains some uncertainty as to the distribution of the energy over the space, but that nevertheless the total energy of the gravitation field

$$E = 4\pi \int_0^\infty \mathfrak{t}'^{4}_{4}\, r^2\, dr$$

has a definite value.

Indeed, by the integration the last term of (111) vanishes. After multiplication by r^2 this term becomes namely

$$(\lambda-\mu)^2 + 2r(\lambda-\mu)(\lambda'-\mu') = \frac{d}{dr}\left[r(\lambda-\mu)^2\right].$$

The integral of this expression is 0 because (comp. §§ 57 and 58) $r(\lambda-\mu)^2$ is continuous at the surface of the sphere and vanishes both for $r=0$ and for $r=\infty$.

We have thus

$$E = \frac{\pi c}{\varkappa}\int_0^\infty v'^2 r^2 dr, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (112)$$

where the value (107) can be substituted for v'. If e.g. the density $\bar{\varrho}$ is everywhere the same all over the sphere, we have at an internal point

$$v' = \tfrac{1}{3}\varkappa\bar{\varrho}\, r$$

and at an external point

$$v' = \tfrac{1}{3}\varkappa\bar{\varrho}\,\frac{a^3}{r^2}.$$

From this we find

$$E = \tfrac{2}{15}\pi c \varkappa \bar{\varrho}^2 a^5 .$$

§ 61. The general equation (99) found for $\mathfrak{t}'^{4}_{4}$ can be transformed in a simple way. We have namely

$$\Sigma(abfe)\frac{\partial}{\partial x_e}\left(\frac{\partial Q}{\partial g_{ab,fe}}\right) g_{ab,f} = \Sigma(abfe)\frac{\partial}{\partial x_e}\left(\frac{\partial Q}{\partial g_{ab,fe}} g_{ab,f}\right) -$$
$$- \Sigma(abfe)\frac{\partial Q}{\partial g_{ab,fe}} g_{ab,fe}$$

and we may write $-Q_2$ (§ 54) for the last term. Hence

$$t'^{4}_{4} = \frac{1}{2\varkappa}\left\{ -Q + \Sigma\,(abfe)\frac{\partial}{\partial x_e}\left(\frac{\partial Q}{\partial g_{ab,fe}}\,g_{ab,f}\right)\right\}, \quad . \quad . \quad (113)$$

where we must give the values 1, 2, 3 to e and f.

The gravitation energy lying within a closed surface consists therefore of two parts, the first of which is

$$E_1 = -\frac{1}{2\varkappa}\int Q\,dx_1\,dx_2\,dx_3 \quad . \quad . \quad . \quad . \quad . \quad (114)$$

while the second can be represented by surface integrals. If namely q_1, q_2, q_3 are the direction constants of the normal drawn outward

$$E_2 = \frac{1}{2\varkappa}\,\Sigma\,(abfe)\int \frac{\partial Q}{\partial g_{ab,fe}}\,g_{ab,f}\,q_e\,d\sigma \quad . \quad . \quad . \quad . \quad (115)$$

In the case of the infinitely feeble gravitation field represented by λ, μ, ν (§ 57) both expressions E_1 and E_2 contain quantities of the first order, but it can easily be verified that these cancel each other in the sum, so that, as we knew already, the total energy is of the second order.

From $Q = \sqrt{-g}\,G$ and the equations of § 32 we find namely

$$\frac{\partial Q}{\partial g_{ab,fe}} = \tfrac{1}{2}\sqrt{-g}\,(2g^{ab}\,g^{fe} - g^{bf}\,g^{ae} - g^{af}\,g^{be}), \quad . \quad . \quad . \quad (116)$$

so that we can write

$$E_2 = \frac{1}{4\varkappa}\int \sqrt{-g}\,\Sigma\,(abfe)\,(2g^{ab}\,g^{fe} - g^{bf}\,g^{ae} - g^{af}\,g^{be})\,g_{ab,f}\,q_e\,d\sigma.$$

The factor $g_{ab,f}$ is of the first order. Thus, if we confine ourselves to that order, we may take for all the other quantities these normal values. Many of these are zero and we find

$$E_2 = -\frac{c}{2\varkappa}\,\Sigma(ae)\int g^{aa}\,(g_{aa,e} - g_{ae,a})\,q_e\,d\sigma. \quad . \quad . \quad . \quad (117)$$

Here we must take $a = 1, 2, 3, 4$; $e = 1, 2, 3$, while we remark that for $a = e$ the expression between brackets vanishes. For $a = 4$ the integral becomes $\int \frac{\partial \nu}{\partial x_e}\,q_e\,d\sigma$, which after summation with respect to e gives

$$\int \frac{\partial \nu}{\partial n}\,d\sigma, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (118)$$

n representing the normal to the surface. If a and e differ from each other, while neither of them is equal to 4, we can deduce from (110) and (109)

$$g_{aa,e} - g_{ae,a} = \frac{\partial \nu}{\partial x_e}.$$

Each value of e occurring twice, i.e. combined with the two values different from e which a can take, we have in addition to (118)

$$-2\int \frac{\partial v}{\partial n}\, d\sigma,$$

so that (117) becomes

$$E_2 = \frac{c}{2\varkappa}\int \frac{\partial v}{\partial n}\, d\sigma.$$

As now outside the sphere

$$v = -\frac{\varkappa}{r}\int_0^a \varrho\, dr$$

we have for every closed surface that does not surround the sphere $E_2 = 0$, but for every surface that does

$$E_2 = 2\pi c\int_0^a \varrho\, dr \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (119)$$

As to E_1 we remark that substituting (65) in (41) and taking into consideration (64) we find,

$$G = \varkappa T \quad , \quad Q = \varkappa \sqrt{-g}\, T \quad . \quad . \quad . \quad . \quad (120)$$

From this we conclude that E_1 is zero if there is no matter inside the surface σ. In order to determine E_1 in the opposite case, we remember that G is independent of the choice of coordinates. To calculate this quantity we may therefore use the value of T indicated in § 56, which is sufficient to calculate E_1 as far as the terms of the first order. We have therefore

$$G = \frac{\varkappa}{r^2}\varrho$$

and if, using further on rectangular coordinates, we take for $\sqrt{-g}$ the normal value c,

$$Q = \frac{c\varkappa}{r^2}\varrho\,.$$

From this we find by substitution in (114) for the case of the closed surface σ surrounding the sphere

$$E_1 = -2\pi c\int_0^a \varrho dr.$$

This equation together with (119) shows that in (113) when integrated over the whole space the terms of the first order really cancel each other. In order to calculate those of the second order and thus to derive the result (112) from (113), we should have to determine the quantity T (comp. 120)), accurately to the order $\varkappa$. The surface integrals in (115) too would have to be considered more closely. We shall not however dwell upon this.

§ 62. From the expression for $t'^{\,4}_{4}$ given in (113) and the value

$$E = E_1 + E_2$$

derived from it, it can be inferred that, though t' is no tensor, we yet may change a good deal in the system of coordinates in which the pheno-~~mena are described, without altering the value of the total energy~~. Let us suppose e.g. that x_4 is left unchanged but that, instead of the rectangular coordinates x_1, x_2, x_3 hitherto used, other quantities x'_1, x'_2, x'_3 are introduced, which are some continuous function of x_1, x_2, x_3, with the restriction that $x'_1 = x_1$, $x'_2 = x_2$, $x'_3 = x_3$ outside a certain closed surface surrounding the attracting matter at a sufficient distance. If we use these new coordinates, we shall have to introduce other quantities g'_{ab} instead of g_{ab}. As however outside the closed surface the quantities g_{ab} and their derivatives do not change, the value of E_2 will approach the same limit as when we used the coordinates x_1, x_2, x_3, if the surface σ for which it is calculated expands indefinitely. The value which we find for E_1 after the transformation of coordinates will also be the same as before. Indeed, if $d\tau$ is an element of volume expressed in x_1, x_2, x_3-units and $d\tau'$ the same element expressed in x'_1, x'_2, x'_3-units, while Q' represents the new value of Q, we have

$$Q d\tau = Q' d\tau'.$$

It is clear that the total energy will also remain unchanged if x'_1, x'_2, x'_3 differ from x_1, x_2, x_3 at all points, provided only that these differences decrease so rapidly with increasing distance from the attracting body, that they have no influence on the limit of the expression (115).

The result which we have now found admits of another interpretation. In the mode of description which we first followed (using x_1, x_2, x_3), ϱ [1]) and g_{ab} are certain functions of x_1, x_2, x_3; in the new one ϱ', g'_{ab} are certain other functions of x'_1, x'_2, x'_3. If now, without leaving the system of coordinates x_1, x_2, x_3, we ascribe to the density and to the gravitation potentials values which depend on x_1, x_2, x_3 in the same way as ϱ', g'_{ab} depended on x'_1, x'_2, x'_3 just now, we shall obtain a new system (consisting of the attracting body and the gravitation field) which is different from the original system

[1]) By ϱ we mean here what was denoted by $\overline{\varrho}$ in § 56.

because other functions of the coordinates occur in it, but which nevertheless no observation will be able to discern from it, the indefiniteness which is a necessary consequence of the covariancy of the field equations, again presenting itself.

What has been said shows that the total gravitation energy in this new system will have the same value as in the original one, as has been found already in § 60 with the restrictions then introduced.

§ 63. If t' were a tensor, we should have for all substitutions the transformation formulae given at the end of § 40. In reality this is not the case now, but from (96) and (97) we can still deduce that those formulae hold for *linear* substitutions. They may likewise be applied to the stress-energy-components of the matter or of an electromagnetic system. Hence, if $\mathfrak{T}_a^b$ represents the total stress-energy-components, i. e. quantities in which the corresponding components for the gravitation field, the matter and the electromagnetic field are taken together, we have for any linear transformation

$$\frac{1}{\sqrt{-g'}}\,\mathfrak{T}'^{\,b}_c = \frac{1}{\sqrt{-g}}\,\Sigma\,(kl)\,p_{kc}\,\pi_{lb}\,\mathfrak{T}_k^l \quad . \quad . \quad . \quad . \qquad (121)$$

We shall apply this to the case of a relativity transformation, which can be represented by the equations

$$x'_1 = ax_1 + bcx_4, \quad x'_2 = x_2, \quad x'_3 = x_3, \quad x'_4 = ax_4 + \frac{b}{c}\,x_1, \qquad (122)$$

with the relation

$$a^2 - b^2 = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (123)$$

In doing so we shall assume that the system, when described in the rectangular coordinates x_1, x_2, x_3 and with respect to the time x_4, is in a stationary state and at rest.

Then we derive from (97) [1])

[1]) We have $g_{14} = g_{24} = g_{34} = 0$, while all the other quantities g_{ab} are independent of x_4. Thus we can say that the quantities g_{ab} and $g_{ab,c}$ are equal to zero when among their indices the number 4 occurs an odd number of times. The same may be said of g^{ab}, $g^{ab,c}$, $\frac{\partial Q}{\partial g_{ab,cd}}$ (according to (116)), $\frac{\partial}{\partial x_k}\left(\frac{\partial Q}{\partial g_{ab,cd}}\right)$ and also of products of two or more of such quantities. As in the last two terms of (97) the indices a, b and f occur twice, these terms will vanish when only one of the indices e and h has the value 4.

As to the first term of (97) we remark that, according to the formulae of § 32, each of the indices a, b and e occurs only once in the differential coefficient of Q with respect to $g_{ab,e}$, while other indices are repeated. As to the number of

$$t'_{1}{}^{4} = t'^{4}{}_{2} = t'^{4}{}_{3} = 0\,; \quad t'_{4}{}^{1} = t'_{4}{}^{2} = t'_{4}{}^{3} = 0,$$

which means that in the system (x_1, x_2, x_3, x_4) there are neither momenta nor energy currents in the gravitation field.

We may assume the same for the matter, so that we have for the total stress-energy-components in the system (x_1, x_2, x_3, x_4)

$$\mathfrak{T}_{1}{}^{4} = \mathfrak{T}_{2}{}^{4} = \mathfrak{T}_{3}{}^{4} = 0\,; \quad \mathfrak{T}_{4}{}^{1} = \mathfrak{T}_{4}{}^{2} = \mathfrak{T}_{4}{}^{3} = 0.$$

Let us now consider especially the components $\mathfrak{T}'_{1}{}^{4}$, $\mathfrak{T}'_{4}{}^{1}$ and $\mathfrak{T}'_{4}{}^{4}$ in the system (x'_1, x'_2, x'_3, x'_4). For these we find from (121) and (122)

$$\mathfrak{T}'_{1}{}^{4} = \frac{ab}{c}\,\mathfrak{T}_{1}{}^{1} - \frac{ab}{c}\,\mathfrak{T}_{4}{}^{4}\,; \quad \mathfrak{T}'_{4}{}^{1} = -abc\,\mathfrak{T}_{1}{}^{1} + abc\,\mathfrak{T}_{4}{}^{4} \quad . \quad (124)$$

$$\mathfrak{T}'_{4}{}^{4} = -b^2\,\mathfrak{T}_{1}{}^{1} + a^2\,\mathfrak{T}_{4}{}^{4} \quad . \quad . \quad . \quad . \quad . \quad . \quad (125)$$

It is thus seen in the first place that between the momentum in the direction of x_1 $(-\mathfrak{T}'_{1}{}^{4})$ and the energy-current in that direction $(\mathfrak{T}'_{4}{}^{1})$ there exists the relation

$$\mathfrak{T}'_{4}{}^{1} = -c^2\,\mathfrak{T}'_{1}{}^{4}$$

well known from the theory of relativity.

Further we have for the total energy in the system (x'_1, x'_2, x'_3, x'_4)

$$E' = \int \mathfrak{T}'_{4}{}^{4}\, dx'_1\, dx'_2\, dx'_3,$$

where the integration has to be performed for a definite value of the time x'_4. On account of (122) we may write for this

$$E' = \frac{1}{a}\int \mathfrak{T}'_{4}{}^{4}\, dx_1\, dx_2\, dx_3,$$

where we have to keep in view a definite value of the time x_4.

If the value (125) is substituted here and if we take into consideration that, the state being stationary in the system (x_1, x_2, x_3, x_4),

$$\int \mathfrak{T}_{1}{}^{1}\, dx_1\, dx_2\, dx_3 = 0$$

we have

$$E' = aE,$$

if E is the energy ascribed to the system in the coordinates (x_1, x_2, x_3, x_4). By integration of the first of the expressions (124) we find in the same way for the total momentum in the direction of x_1

$$G' = \frac{b}{c}\,E.$$

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times which e, h and the other indices occur we can therefore say the same of the first term of (97) as of the other terms. The first term also is therefore zero, if no more than one of the two indices e and h has the value 4.

That $t'_{4}{}^{e}$ vanishes for $e \neq 4$ is seen immediately.

§ 64. Equations (122) show that in the coordinates (x'_1, x'_2, x'_3, x'_4) the system has a velocity of translation $\frac{bc}{a}$ in the direction of x'_1. If this velocity is denoted by v, we have according to (123)

$$a = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}.$$

If therefore we put

$$M = \frac{E}{c^2},$$

we find

$$E' = \frac{Mc^2}{\sqrt{1-\frac{v^2}{c^2}}}, \quad G' = \frac{Mv}{\sqrt{1-\frac{v^2}{c^2}}} \quad . \quad . \quad . \quad . \quad (126)$$

When the system moves as a whole we may therefore ascribe to it an energy and a momentum which depend on the velocity of translation in the way known from the theory of relativity. The quantity M, to which the energy of the gravitation field also contributes a certain part, may be called the "mass" of the system. From what has been said in § 62 it follows that within certain limits it depends on the way in which the system and the gravitation field are described.

It must be remarked however that, if for the gravitation field we had chosen the stress-energy-tensor $\mathfrak{t}_0$ (§ 52), the *total* energy of the system even when in motion would be zero. The same would be true of the *total* momentum and we should have to put $M = 0$.

At first sight it may seem strange that we may arbitrarily ascribe to the moving system the momentum determined by (126) or a momentum 0; one might be inclined to think that, when a definite system of coordinates has been chosen, the momentum must have a definite value, which might be determined by an experiment in which the system is brought to rest by "external" forces. We must remember however (comp. § 52) that in the theory of gravitation we may introduce no "external" forces without considering also the material system S' in which they originate. This system S' together with the system S with which we were originally concerned, will form an entity, in which there is a gravitation field, part of which is due to S' (and a part also to the simultaneous existence of S and S'), There is no doubt that we may apply the above considerations to the *total* system (S, S') without being led into contradiction with any observation.