Mathematics. — *"On Elementary Surfaces of the third order"*. (First communication). By B. P. HAALMEIJER. (Communicated by Prof. BROUWER).

(Communicated in the meeting of May 26, 1917).

Introduction. The existence of certain numbers of real straight lines on cubic surfaces is well known. In Math. Ann. 76 C. JUEL makes a clever attempt to prove the existence of straight lines on certain surfaces of the third order which are non-analytically defined and which he calls elementary surfaces. His methods however are not always convincing and some conditions he puts to his surfaces seem to be artificial and out of place. The object of this note is to introduce elementary surfaces of the third order in a natural way and to prove the existence of at least one straight line on such a surface. Our starting point is formed by the elementary curves of the third order which are extensively dealt with by JUEL in the Proc. of the R. Acad. of Denmark, 7th series, t. 11 N°. 2. Besides this we shall principally use well known theorems of the analysis situs and the theory of sets of points.

In carrying out the following researches I am indebted for many suggestions to Prof. L. E. J. BROUWER, who also has attracted my attention to this subject.

Definitions and exposition of the problem. An open JORDAN curve, which, together with the linesegment [1]) between its endpoints, forms the boundary of a convex region, is called *convex arch.* These convex arches form the building material for the elementary curves. Let a set of points be composed of a finite number of convex arches, in such a way that it forms the continuous representation of a circle. To every point of the circle is to correspond one and only one point of the set under consideration. Besides, the *tangent* (*touching line, Stutze*) is to change continuously with the corresponding point of the circle and lastly the set of points is not to contain linesegments, but may include entire lines. A closed set of points consisting of a finite or countably infinite number of these above defined sets is called *elementary curve.* Isolated points are admitted though tangents in the ordinary sense disappear.

[1]) In the following *line* will be used for *straight line.*

An elementary curve is said to be of the n^{th} order, when lines exist which have n, but no lines which have more than n points in common with the curve (unless the curve includes the entire line).

In this note we chiefly consider elementary curves of the third order. Some of the results obtained by JUEL which shall prove most useful are the following:

The possible forms of elementary curves of the third order are:

1. One connected curve of the third order without double point or cusp.
2. One connected curve of the third order with a cusp (the two branches arrive at the cusp from different sides of the tangent, cusps where the two branches meet from the same side cannot exist on curves of the third order, as a slight change in the position of the tangent would produce 4 points of intersection).
3. One connected curve of the third order with double point, (this variety can be considered as composed of a curve of the third order and one of the second [1]) having only the double point in common and each forming an angle at that point).
4. One connected curve of the third order and one of the second [1]) (that is: oval, boundary of convex region) having no points in common.
5. One connected curve of the third order and isolated point.
6. Straight line and oval [1]).
7. Straight line and isolated point.
8. Three straight lines.

As points of intersection with a line are counted:

double: ordinary point (that is: internal point of a convex arch) on the tangent, isolated point on every line, cusp on every line except the tangent and double point on every line except on either of the tangents.

triple: point of inflexion on tangent, cusp on tangent and double point on both tangents.

All other modes of intersection are counted single.

We define as *elementary surface of the third order* F^3 any set of points in the projective R_3 possessing the two following properties [2]):

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[1]) These curves of the second order of course need not have finite breadth, but can have one or two points in common with the line at infinity. (We always consider projective space).

[2]) Ultimately it may be advisable to make this definition less restricting. In order to admit conical points it will be necessary to extend the first condition and to make it possible that the surface degenerates both conditions have to be revised. The ultimate definition must be couched in such terms that no essential alterations are required for defining elementary surfaces of order higher than the third.

1. F^3 is to answer the most general definition of a twodimensional continuum [1]).

2. Every plane section of F^3 is an elementary curve of the third order.

This note is divided into two parts:

In the first part we shall prove: *The tangents to plane sections passing through an arbitrary point A of F^3, not situated on a line of F^3, form one plane,* which may be called *tangent plane* to F^3 in A Only one exceptional point is possible having the following character: It is isolated in every plane except the planes through one line, and in these it is cusp with that line as cuspidal tangent.

In the second part we begin by proving some further theorems concerning points of F^3 not situated on a line of F^3. At the end we assume that no point of a certain plane section is situated on a line of F^3. By showing that this leads to contradictory results, the existence of at least one straight line on F^3 is established.

First part. We divide the proposition as follows:

§ 1. If A is isolated in a plane α, then α is tangent plane, to F^3 in A or A is exceptional point.

§ 2. Only one exceptional point is possible.

§ 3. If A is double point in a plane α and cusp in not more than one plane, then α is tangent plane.

§ 4. If A is cusp in one and not more than one plane α, then α is tangent plane.

§ 5. If A is cusp in two different planes, then A is exceptional point.

§ 6. Through A passes at least one plane in which A is either isolated point, double point or cusp.

§ 1. *If A is isolated in a plane α then α is tangent plane or A is exceptional point.*

The first thing to be done is to construct a plane in which A is not isolated. The vicinity of A on F^3 is the (1,1) continuous representation of the vicinity of a point in a plane, hence a sequence of points $A_1, A_2, A_3 \ldots$ of F^3 can be chosen having A for sole limiting point. Let a be an arbitrary line through A in α and $\beta_1, \beta_2, \beta_3 \ldots$ the planes passing through a and $A_1, A_2, A_3 \ldots$ respectively. These planes have at least one limiting plane β passing through a also. In case A is isolated in each of the planes $\beta_1, \beta_2, \beta_3 \ldots$ it can be shown that A is not isolated in β.

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[1]) Brouwer, Math Ann. 71, p 97.

In a plane in which A is isolated the remaining points belonging to F^3 form a connected curve of the third order or a straight line. This restcurve is a closed set of points (it is the continuous representation of a circle), hence A has a finite minimum distance from it. Let this minimum distance be ε_1 in β_1, ε_2 in β_2 etc. When a point B moves along the restcurve in β_1 the distance AB changes continuously from ε_1 to ∞, in β_2 from ε_2 to ∞ etc. (when a point A is situated at distances b_1 and b_2 from two points B_1 and B_2 belonging to a connected set of points then to every distance b_3 such that $b_1 > b_3 > b_2$ corresponds at least one point B_3 of the set such that $AB_3 = b_3$).

The sequence $\varepsilon_1, \varepsilon_2, \varepsilon_3 \ldots$ has zero for limit. Let $\delta_1, \delta_2, \delta_3 \ldots$ be a decreasing sequence chosen from it and let the corresponding planes be represented by $\beta_{\delta_1}, \beta_{\delta_2}, \beta_{\delta_3} \ldots$.

In β_{δ_2} we choose a point B_2' of F^3 such that $\delta_1 > AB_2' > \delta_2$
„ β_{δ_3} „ „ „ „ B_2'' „ „ „ „ $\delta_1 > AB_2'' > \delta_2$
and „ „ B_3' „ „ „ „ $\delta_2 > AB_3' > \delta_3$
In β_{δ_4} we choose a point B_2''' „ „ „ „ $\delta_1 > AB_2''' > \delta_2$
and „ „ B_3'' „ „ „ „ $\delta_2 > AB_3'' > \delta_3$
and „ „ B_4' „ „ „ „ $\delta_3 > AB_4' > \delta_4$

etc.

$B_2', B_2'', B_2''' \ldots$ have a limiting point B_2 in β such that $\delta_1 \geqq AB_2 \geqq \delta_2$
$B_3', B_3'', B_3''' \ldots$ „ „ „ „ B_3 „ β „ „ $\delta_2 \geqq AB_3 \geqq \delta_3$
etc.

F^3 is a closed set of points, hence $B_2, B_3, B_4 \ldots$ all belong to F^3. Besides $\delta_1, \delta_2, \delta_3 \ldots$ is a decreasing sequence having zero for limit hence A is limiting point of F^3 in β.

We now proceed to construct a finite sphere round A inside of which F^3 is entirely situated on one side of the plane α (except the point A in α). A is isolated in α, hence with A as centre there exists in α a finite circle c containing no other points of F^3. Let b be the sphere with A as centre passing through c. The vicinity of A on F^3 is the (1,1) continuous representation of the vicinity of a point in a plane. Let A_1 be the point corresponding to A. The correspondence is (1,1) continuous, hence a finite circle c_1 round A_1 can be found in the plane such that all internal points of c_1 have corresponding points inside the sphere b and a sphere b' concentric with b can be found such that all internal points of b' belonging to F^3 have corresponding points inside c_1.

Inside b' F^3 lies on only one side of α for if this were not the case, a contradiction might be obtained as follows: Two points B

and C of F^3 are chosen, both internal to b' and on different sides of α. The corresponding points B_1 and C_1 are situated inside c_1 and can be joined by an open JORDAN curve not passing through A_1 and entirely internal to c_1. The set of points K corresponding to this curve is closed and connected (both these properties are invariants for (1,1) continuous transformations). K is situated entirely inside b, contains points on both sides of α but no points of α itself (A is the only internal point of c belonging to F^3). Hence K is composed of two closed sets of points, one on each side of α, but this is impossible, because K is connected.

The above results may be taken together as follows:

Through the line a passes a plane α, in which A is isolated, and a plane β, in which A is not isolated. Besides, inside a sufficiently small neighbourhood of A the surface F^3 lies entirely on one side of α, let us say below α. Hence inside that neighbourhood of A the intersection of β and F^3 lies entirely below a (always excepting the point A itself, which is situated *on* a). Considering the possible forms of elementary curves of the third order, there remain two possibilities:

1. A is ordinary point in β with a as tangent.
2. A is cusp in β.

Let A be cusp in β with b as cuspidal tangent. In no plane through b can A be isolated, because the two branches meeting at the cusp in β furnish points of F^3 on both sides of each of these planes inside every vicinity of A. But above α there is a finite neighbourhood of A containing no points of F^3, hence in every plane through b, A is either cusp or ordinary point with the tangent in α. We proceed to show that *if A is cusp in β it cannot be ordinary point in two other planes through b.*

Let $\alpha_1, \alpha_2, \alpha_3 \dots$ be a sequence of parallel planes each of which lies above all preceding ones and which have α for limiting plane. Let the points of intersection of b and $\alpha_1, \alpha_2, \alpha_3 \dots$ be respectively $B_1, B_2, B_3 \dots$ If the sequence is started high enough every plane $\alpha_1, \alpha_2 \dots$ has a point in common with each of the branches meeting at the cusp in β. Let these points be B_1' and B_1'', B_2' and B_2'', B_3' and $B_3'' \dots$ None of these points B_1', $B_1'' \dots$ can be isolated in the planes $\alpha_1, \alpha_2 \dots$ considering the branches meeting at the cusp in β furnish points on both sides of each of these planes in every vicinity of B_1', $B_1'' \dots$

A sequence of connected sets of points, each having a breadth $> p$, has for limit a connected set of points with breadth $\geqq p$. From this follows that when n increases the points B_n' and B_n''

cannot continue to be situated on odd curves in α_n, for an odd curve is never entirely internal to a finite region (in other words: always has infinite breadth), so the limiting set would be a curve in α passing through A. But if for n larger than some finite value the points B_n' and B_n'' can neither be isolated nor situated on odd curves, they must lie on even curves. which in this case must be ovals. Obviously these ovals contract when n increases and A is the sole limiting point. Let γ and δ be the planes through b in which A is supposed to be ordinary point (with the tangents in α).

Let a_n, c_n and d_n be the lines of intersection of α_n and β, γ, δ respectively. Obviously a_n intersects the oval in plane α_n at B_n' and B_n''. B_n is a point of the cuspidal tangent in β and B_n' and B_n'' are points of the branches meeting at the cusp from different sides of the tangent, so on line a_n, B_n is situated between B_n' and B_n'', hence B_n is internal point of the oval in α_n. From this follows that the lines c_n and d_n passing through B_n have each two points in common with the oval, one on either side of B_n. Let these points be C_n', C_n'' and D_n', D_n''.

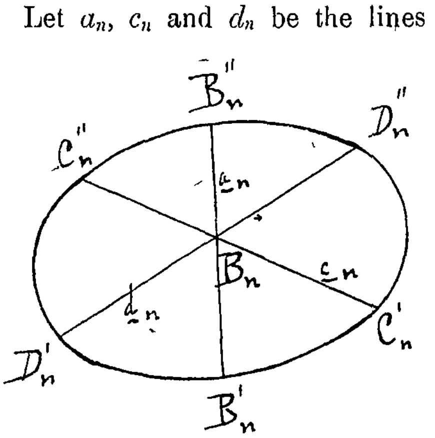


Fig. 1.

In plane β A is cusp with b as tangent, but in γ and δ A is supposed to be ordinary point with the tangents in α. From this follows that by taking n large enough the ratios $\frac{B_n B_n'}{B_n C_n'}$ and $\frac{B_n B_n'}{B_n D_n'}$ may be made as small as desired, even of the second order with respect to the distance of the planes α and α_n. Besides the angles of a_n, c_n and d_n are the same for every n, hence for n large enough, the linesegments $C_n' D_n'$ and $B_n B_n'$ will have no point in common and this result contradicts one of the fundamental properties of ovals.

The following question arises: Is it possible that A is ordinary point in γ and δ and cusp in β, but with a cuspidal tangent not coinciding with b? We shall show that the answer must be negative. The notation of points of intersection etc. is kept the same as above. In β the branches meeting at the cusp would arrive from the same side of b, but in γ and δ the branches meeting at A arrive from different sides of b. Hence for n large enough the oval in α_n would be such that on the lines c_n and d_n the point B_n is situated *between* the points of intersection with the oval, but on the line a_n both points of intersection lie on the same side of B_n. This means that B_n is at the same time internal and external to the oval, and this is impossible.

The above results may be taken together as follows: A is supposed to be isolated in plane α, and b is a line through A not situated in α. *Now, if A is ordinary point in two different planes through b, it cannot be cusp in any other plane through b.* But if A is ordinary point in a plane through b, the branches meeting at A in this plane furnish points of F^3 on both sides of every plane through b inside every vicinity of A. Hence in no plane through b can A be isolated. Besides above α there is a finite vicinity of A containing no points of F^3 so in no plane can A be double point, point of inflexion or ordinary point with tangent not situated in α. Hence when A is supposed to be isolated in α, and b is a line through A not situated in α, the final result may be formulated as follows: *If through b pass two different planes in which A is ordinary point, then in every plane through b, A is ordinary point and all the tangents are situated in α.*

Above we found that in β the point A is either:

1. Ordinary point with a as tangent.
2. Cusp.

Let the first possibility be assumed. We turn the tangent a in the plane β round the point A in both directions to the positions a' and a''. Provided these rotations be small enough the lines a' and a'' have each three different points in common with F^3 [1]). Hence in no plane through a' or a'' can A be isolated point, double point or cusp. Points of inflexion are also excluded, because one of the branches meeting at such a point would arrive from above α, but above α there is a finite neighbourhood of A containing no points of F^3. The only remaining possibility is that in every plane through a' or a'', A is ordinary point and the tangents must all be situated in α because above α there is a finite neighbourhood of A containing no points of F^3.

Let c be an arbitrary line through A, not situated in α or β. The

1) JUEL, loc. cit. Acad. of Denmark. When points of intersection are counted as explained, an elementary curve of the third order and an arbitrary line in its plane have in common either three points or one point. Hence a tangent at an ordinary point A carries one point more of the curve. Now if this tangent be turned round A over a sufficiently small angle, A is replaced by two points of intersection A and B each counting single. But there must be still another point of intersection, as there are to be three altogether, so the line in its new position has three different points in common with the curve.

two planes passing through c and through a' and a'' respectively show ordinary points in A. Hence (using the results obtained above) every plane through c shows an ordinary point in A and all the tangents are situated in α.

But c is an arbitrary line through A only subjected to the condition not to lie in α or β, so it follows that in every plane, except α and β, A is ordinary point with tangent in α. Besides in β A was assumed to be ordinary point and the tangent was found to lie in α, hence the only remaining exception is α in which plane A is isolated and which has now been proved to answer our definition of tangent plane.

We now assume the second possibility given above:

The point A is isolated in α and cusp in β. Let b be the cuspidal tangent. In no plane through b can A be ordinary point, for if this were the case, it might be shown in the same way as above that A cannot be cusp in β. Also in no plane through b can A be isolated because b has only the point A in common with F^3. Taking into consideration that above α there is a finite vicinity of A containing no points of F^3, the only remaining possibility is that A is cusp in every plane through b. b must be cuspidal tangent in every one of these planes because b has only the point A in common with F^3. Now a cusp counts double as point of intersection on any line except the tangent, hence every line through A (=|=b) has one and only one other point in common with F^3, because in the plane through that line and b the point A is cusp with b for tangent. Thus in a plane through A which does not contain b, *every* line through A has one and only one other point in common with F^3, hence A is isolated in every plane which does not contain b. Thus it has been shown that A is exceptional point.

Before proceeding further we shall just rehearse what has been done in § 1:

A was assumed to be isolated in plane α. Then a plane β was constructed in which A was *not* isolated. From the assumed isolation in α it followed that only two things were possible, namely that A is ordinary point in β with tangent in α or that A is cusp in β. Assuming the first possibility we proved that α must be tangent plane, while the second assumption lead to the conclusion that A is exceptional point.

§ 2. *Only one exceptional point is possible.*

Suppose there could be two: A and B. In a plane through A and B there are a priori four possibilities:

1. A and B are both isolated.
2. A and B are both cusps.
3. A is isolated and B is cusp.
4. A is cusp and B is isolated.

But no elementary curve of the third order can have two isolated points, two cusps or one of each, hence the required contradiction is obtained.

§ 3. *If A is double point in a plane α and cusp in not more than one plane, then α is tangent plane.*

The points of F^3 situated in the plane α form an elementary curve of the third order K, which has a double point in A. A is the point of intersection of two convex arches K_1 and K_2. Let c be a circle round A in α, such that all points of K which are internal to c belong to $K_1 + K_2$ and besides c must be such that it has only two points: C and E in common with K_1 and only two points: D and F with K_2. All these conditions can be fulfilled by taking c small enough.

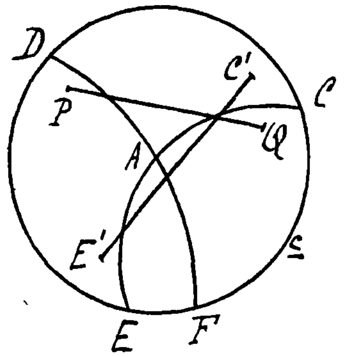


Fig. 2.

Now the branches AC, AD, AE and AF are connected by four sets of points I, II, III and IV, having no points in common, all belonging to F^3 and each of which is entirely situated on one side of α. Respecting these four sets of points, the JORDAN theorem for threedimensional space [1] leaves only two possibilities.

The first possibility comes to the following: AC and AD are connected by I, and AD and AE by II, AE and AF by III, and lastly AF and AC by IV. If the concave side of EC faces F, let us assume for a minute that III and IV are both situated above α. Now if a parallel linesegment converges from above towards $E'C'$ it would end up by having at least two points in common with both III and IV, and this is impossible. Hence III and IV cannot lie on the same side of α. If the concave side of DF faces E then II and III must also be situated on different sides of α. Hence II and IV lie on the same side of α, but then I is certainly situated on the other side, for suppose all three were on the same side then a parallel linesegment converging from that side towards PQ would

[1] BROUWER, Math. Ann. 71, p. 314.

finish up by having at least one point in common with each of II and IV, and at least two with I, and this again cannot be as no line carries four points of F^3. Hence the final result is that I and III are situated above α and II and IV below α or vice versa.

A representative case of the *second possibility* is the following: AC and AE are connected by I above α, AE and AF above or below α by II, AF and AD below α by III and lastly AD and AC above or below α by IV. If IV be situated below α we choose in α a point A' near A and a point D' near D, such that the line-segment $A'D'$ intersects the arch AD at a point near A and at another point near D. Now a parallel linesegment converging from below towards $A'D'$ would end up by carrying at least two points of III and two of IV: a contradiction [1]). Hence the second possibility left by the JORDAN theorem is excluded and we need only consider the first. In the following it will be assumed that I and III are situated above, and II and IV below α.

Obviously the set of points $\mathrm{I} + AC + AD$ is the (1,1) continuous representation of a plane region and part of its boundary. Besides, inside a finite neighbourhood of the point corresponding to A, this region has the character of a JORDAN region, because the arches AC and AD are JORDAN curves, and the same holds for the (1,1) continuous representations. The same things can be said of $\mathrm{II} + AD + AE$, $\mathrm{III} + AE + AF$ and $\mathrm{IV} + AF + AC$.

Lastly we remark that inside a finite neighbourhood of A all points of F^3, not situated in α belong to $\mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}$.

Let b be a line in α through A such that the branches FA and EA arrive at A from different sides of this line. Then the branches CA and DA will do the same. Let β be a plane through b ($\neq \alpha$). AC and AD are joined above α by I. $\mathrm{I} + AC + AD$ is the continuous (1,1) representation of a plane region and part of its boundary. Let I_1 correspond to I, A_1C_1 to AC, and A_1D_1 to AD. Inside a finite neighbourhood of A_1 the region I_1 has the character of a JORDAN region.

We shall now have to use a property of JORDAN regions called the "Unbewalltheit". [2]) For two dimensions it may be formulated as follows: Let J be a closed JORDAN curve, I the internal and E the external region. Two points Q and R of J can always be joined by an open JORDAN curve belonging entirely to I and by an open

[1]) By using this last reasoning the *first possibility* might have been dealt with in a more simple fashion.

[2]) BROUWER, Math. Ann. 71, p. 321.
SCHOENFLIES, Mengenlehre 2, chapter 5.

JORDAN curve belonging entirely to E. Let P be a third point of J and c an arbitrary circle round P. Now the "Unbewalltheit" says that if Q and R are chosen close enough to P, the joining curves may be kept entirely inside c.

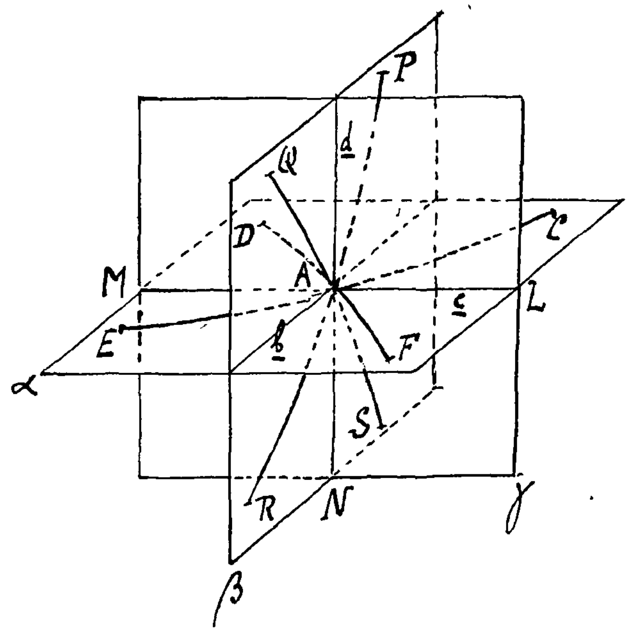
Applying this to our case it follows that points of A_1C_1 and A_1D_1 can be joined by open JORDAN curves entirely belonging to I_1, inside any vicinity of A_1. Hence in the *continuous* (1,1) representation AC and AD can be joined by open JORDAN curves entirely situated on I inside any vicinity of A. Now every one of these curves has at least one point in common with β, because AC and AD lie on different sides of that plane, hence in plane β the point A is limiting point of I, and in the same way can be proved that A is limiting point of III in β. But I and III have no points in common, hence in β one branch departs from A on I and another on III. I and III are both situated above α so in β two branches depart from A above α.

In β two branches arrive at A from the same side of b. Considering the possible forms of elementary curves of the third order, there are a priori three possibilities:

1. A is double point in β.
2. A is cusp in β.
3. A is ordinary point in β with b for tangent.

1. Suppose A is double point in β. Two branches AP and AQ arrive in A from above b, hence two more AS and AR arrive from below b (three from one side and one from the other is impossible because b has, besides A, another point in common with F^3). We proceed to show that the branches AR and AS are at first both situated on II or both on IV. Suppose AR and AS were situated respectively on II and IV. Then AR and AS could not be connected below α, because II and IV have no points in common. But AS would be connected via AC and AF with AP and AQ and AR would be connected via AD and AE with AP and AQ. From this follows that AR and AS would only be connected via AP and AQ. This however leads to a contradiction, because the four branches meeting at A in β must be connected in an analogous way as those in α hence AR and AS are joined by a set of points situated entirely on one side of β. Thus it has been shown that AR and AS are situated either both on II or both on IV, let us assume on II.

The vicinity of A on F^3 is the (1,1) continuous representation of the vicinity of a point in a plane. Let A correspond to A_1, AE to A_1E_1, AD to A_1D_1 II, to II_1, AR to A_1R_1 and AS to A_1S_1. Inside a finite neighbourhood of A_1 the region II_1 is divided by the open JORDAN curves A_1R_1 and A_1S_1 in three regions having no points in common. In the vicinity of A_1 all these regions have the character



Ffg. 3.

of JORDAN regions. We consider the two outside regions, namely those connecting respectively A_1E_1 with A_1R_1 and A_1S_1 with A_1D_1 [1]. The (1,1) continuous representations of these regions of F^3 connect respectively AE with AR and AS with AD. That this connection exists inside any neighbourhood of A, again follows from the "Unbewalltheit". Hence in any plane through b (fig. 3) such that AE and AR are situated on different sides, at least two branches arrive at A from below α. But we also know that in each of these planes two branches arrive in A from above α (one on I and the other on III), hence the following result has been obtained: When the plane β is turned round b (fig. 3) in such a way that the lower half moves to the left, then in every position as far as α the point A remains double point.

Let c be a line in α through A, passing between the branches AE and AD, and let d be a line in β through A, separating the branches AR and AS. The plane through c and d is denoted by γ (fig. 3). In γ two branches arrive in A from below α, one on II and the other on IV. The branch situated on II arrives in A from the right

[1]) A priori it would be possible that A_1E_1 is connected with A_1S_1 and A_1R_1 with A_1D_1, but when we consider the representations on F^3, this leads to contradiction with the JORDAN theorem for threedimensional space

hand side of AN, because the component region of II which forms the direct connection between AR and AS, is situated on the right hand side of β. This branch on II cannot have AL for tangent because in that case the branch on IV would also have AL for tangent and cusps where both branches arrive from the same side of the tangent, are excluded. Hence the branch in γ on II forms at A finite angles with both AM and AL.

The line c has, besides A, another point in common with F^3, and for this reason can never be tangent at a double point. Hence the branch in γ situated on IV cannot have AL as tangent, so it must arrive in A under a finite angle with AL, and it follows that if the plane α be turned round line b in such a way that the right hand side moves downwards (fig. 3), the point A will at first remain double point. The above results may be taken together as follows: *α cannot be limiting plane of planes through b in which A is not double point.* But by reversing α and β in our reasonings, the same can be said of plane β. Hence: If α be turned round b in either direction, A at first remains double point. In neither direction can there be a *last* plane in which A is double point, so either there is a *first* in which A is *not* double point, or all planes through b show a double point in A.

In a *first* plane in which A is not double point, there still arrive two branches in A from above α (one on I and the other on III) hence in such a plane A would be either ordinary point with b as tangent or cusp. The case that A is cusp shall be dealt with sub 2. So at present only two assumptions need be made, namely that there is a *first* plane in which A is not double point, but ordinary point with b for tangent, or that all planes through b show a double point in A. We shall successively show that both these assumptions lead to contradictions.

Let δ be first plane in which A is ordinary point with b for tangent and $\delta_1, \delta_2, \delta_3 \ldots$ a sequence of converging planes (all passing through b) in which A is double point. In δ a finite neighbourhood of A exists containing no points of F^3 on one side of the tangent b, in this case *below* b. Considering F^3 is a closed set, this is only possible when in the converging planes the loop of the curve (that is the part of the second order) ends up by being situated in the semiplane of δ_n which converges towards the lower semiplane of δ. Besides these loops must contract towards A and nothing but A. Hence for $n >$ some finite number the branches in δ_n belonging to the part of the third order depart from A above b. At first the *concave* side of these branches faces b. Both branches have infinite breadth, hence each has an infinite limiting branch. In the limiting plane σ the branches departing from A at first face b with their *convex* side (b is tangent at an ordinary point). But a sequence of finite concave branches cannot have a convex limiting branch hence a contradiction is obtained. The possibility might be put forward that on the converging branches points of inflexion may converge towards A, but a curve of the third order with double point has only *one* point of inflexion [1]), hence it may be assumed that only on either the left or the right hand branch points of inflexion converge towards A and the contradiction remains with regard to the other branch.

We now proceed to show that not all planes through b can have a double points in A. Again AE and AD are supposed to be joined by II below α and AC and AF by IV below α (fig. 3). AR and AS are situated on II. We found that if α be turned round b in such a way that the right hand side moves downwards, then at first A remains double point and the branches arriving in A from below remain situated on IV. In the same way as AC and AF are connected by IV below α, the branches AR and AS are connected by a component region of II on the right hand side of β. Taking in consideration this analogy it is obvious that if β be turned round b in such a way that the lower half moves to the right, then at first A not only remains double point, but the branches meeting at A from below α are still situated on II. This may be expressed as follows: *There cannot be a last plane in which the branches are situated on II, and the same can be said of IV.*

Let us now consider the set of semiplanes through b and situated below α. If every plane through b has a double point in A, then in each of these semiplanes two branches would arrive in A from below α. It was found that if these branches are both situated on II, then the same holds for the branches in all semiplanes situated more to the left. In the same way if both branches lie on IV this is also the case in all semiplanes more to the right. Besides the set of semiplanes with branches on II cannot have a last element on the right side and those with branches on IV cannot have a last element on the left side. But all semiplanes have two branches below α, hence the two kinds of semiplanes with branches on II and IV respectively must be separated by a semiplane with one branch on II and one on IV, and this is impossible according to page 111. Thus the assumption that *all* planes through b have double points in A leads to a contradiction.

[1]) Juel loc. cit. Acad. of Denmark § 5.

2. We now come to the second possibility given on page 111, namely that A is cusp in β. Again α denotes the plane in which A is double point and b the line of intersection of α and β. In the proposition of § 3 it was assumed that A is cusp in not more than one plane. Hence if c is a line in α $(\neq b)$ the point A can never be cusp in any plane through c. Provided c does not coincide with one of the tangents in α either, the reasoning given sub 1 shows that A cannot be double point in any plane through c (except in α). Considering the possibilities given on page 111 it follows that A must be ordinary point in every plane through c (except α), with c for tangent.

Let AF be the cuspidal tangent in β (fig. 4). The line c in α we choose in the same angle of the tangents in A, in which the line

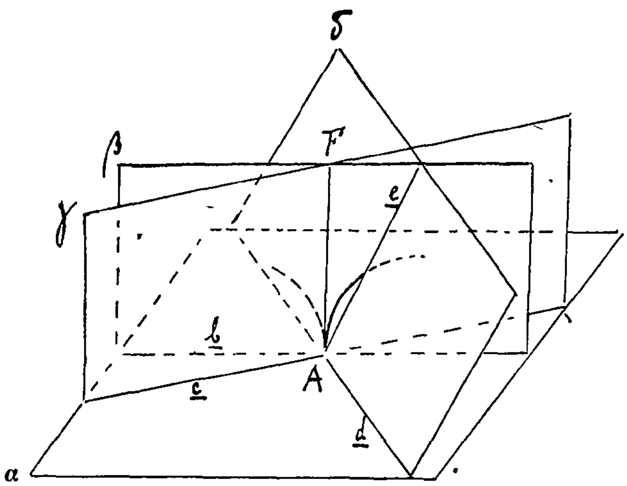


Fig. 4.

b is situated. Besides we choose in α a line d through A, not being tangent in A and in β a line e, not coinciding with AF or b. The plane through d and e is denoted by δ, that through c and AF by γ.

The branches meeting at the cusp A in β arrive from above α (one on I and the other on III). We consider a sequence of planes $\gamma: \gamma_1, \gamma_2, \gamma_3 \ldots$ turning round AF and converging towards β. In each of these planes A is ordinary point with tangent $(c_1, c_2, c_3 \ldots)$ situated in α. The branches meeting at A in each of these planes arrive from above α (one on I and the other on III), because the branches in β arrive from above and none of the lines $c_1, c_2, c_3 \ldots$ is separated from b by a tangent in A.

Each of the lines $c_1, c_2, c_3 \ldots$ has, except A, another point in common with F^3. The distance from A to these points cannot tend towards zero, because if the second point of F^3 on b is added, they form

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a closed set of points to which A does not belong, none of the lines $c_1, c_2, c_3 \ldots . b$ being tangent.

Let $e_1, e_2, e_3 \ldots$ be the lines of intersection of δ and $\gamma_1, \gamma_2, \gamma_3 \ldots$ respectively. These lines $e_1, e_2, e_3 \ldots$ converge towards e.

In γ_1 a branch departs from A between c_1 and e_1, in γ_2 between c_2 and e_2 etc. The distance from A at which these branches can cross $c_1, c_2, c_3 \ldots$ cannot tend towards zero, hence to make it possible that in plane β no branch departs from A between b and e it is unavoidable that the branches in the converging planes cross $e_1, e_2, e_3 \ldots$ in points converging towards A. This means that in plane δ the line e would be tangent in A. But considering d does not coincide with b or either of the tangents in α, the plane δ through d must show an ordinary point in A with d for tangent. Thus a contradiction has been obtained.

It has been shown successively that the a priori possibilities 1 and 2 given on page 111 lead to contradictory results. Hence only the third possibility remains, namely that A is ordinary point in β with b for tangent. But b is an arbitrary line in α through A, only subjected to the condition not to coincide with either of the tangents in A, and β is an arbitrary plane through b, only supposed not to coincide with α, hence the results obtained so far may be expressed as follows: *In every plane through A which does not coincide with α and does not contain a tangent in α, the point A is ordinary point with tangent situated in α.*

Thus to complete the proof that α is tangent plane, it only remains to consider the sections of F^3 in planes through a tangent at A in α.

In α the point A is point of intersection of two convex arches, parts of which are indicated by QS and PR in fig. 5. Let a $(= DC)$ be tangent at A to PR, and let β be an arbitrary plane through a $(\neq \alpha)$. We assume the senses of curvature of the convex arches to be as indicated in fig. 5.

In β we choose a line AB $(\neq a)$ and we consider a sequence of planes $\beta_1, \beta_2, \beta_3 \ldots .$ all passing through AB and converging towards β, in such a way that the back part converges towards β from the right hand side (see fig. 5). The line of intersection of α and β_n is denoted by AC_n (a_n).

Let the part of F^3 connecting AP and AS be situated above α (the other case is treated in a strictly analogous way). In every plane β_n a branch departs from A above α in the direction AC_n. These branches have a limiting set in β belonging to the closed set F^3. Applying the same reasoning given above to show that A cannot be cusp in any plane, it can be shown that this limiting branch

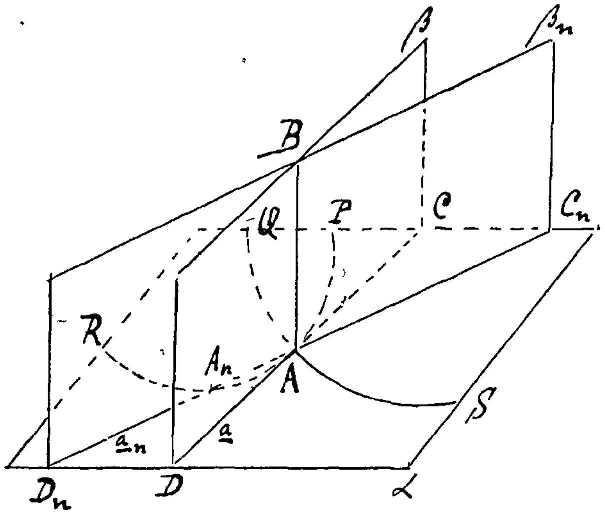

Fig. 5.

in β' departs from A above α in the direction AC. If this branch formed at A a finite angle with AC, then every line inside this angle would be tangent at A in every plane except β, and this is obviously at variance with the results already obtained. For the complete demonstration it is necessary to know that the linesegment AC_n cannot have points in common with F^3, converging towards A. Now this is obvious if we remember that when the lines a_n converge towards a, the point A_n on AR converges towards A, and that the point A only counts double on a.

The possibility might be put forward that the branches in the converging planes β_n have only A as limiting set in β. Then however, it is unavoidable that the converging planes show ovals, contracting from above towards A. Now all these ovals would cross AB, hence A would be limiting point of F^3 on AB and the entire line AB would belong to F^3, a possibility excluded at the outset.

Between the branches AP and AS the surface F^3 was assumed to be situated above α, hence the part of F^3 connecting AS and AR lies below α. Now if the planes $\beta_1, \beta_2, \beta_3 \ldots$ converge towards β from the other side and if we consider the front halves of these planes (fig 5), it may be shown in exactly the same way that in β a branch departs from A below α in the direction AD.

Taking these results together, it is found that A is point of inflexion in β with a for tangent.

Before passing on to § 4 we shall prove the following theorem: *It is impossible that a point of intersection A counts double on a* *line b in some planes through b and single in other planes through b.* To prove this it obviously is sufficient to show that *when a point of intersection A counts double on a line b in a sequence of planes* $\delta_1, \delta_2 \ldots$ *through b, converging towards a limiting plane* δ, *then A also counts double on b in* δ.

Let us imagine two parallel planes, also parallel to b, situated close to b and on different sides of that line. The lines of intersection with $\delta_1, \delta_2 \ldots\ldots \delta$ are respectively denoted by $b_1' \, b_2' \ldots\ldots b'$ and $b_1'' \, b_2'' \ldots\ldots b''$. Now if the above proposition were false, then in at least one of the two planes, for instance the first, there would be every time *two* points of intersection with b'_n, converging together towards *one* point of intersection with b'. This would remain the same when the plane. parallel to itself, moves towards b. But then it is unavoidable that two branches departing from A in δ_n which keep finite breadth, converge towards one single finite branch departing from A in δ, hence the two sectors of the surface, meeting at that branch, would be situated on the same side of δ and this has been shown to be impossible at the beginning of § 3.