Physics. — "*Adiabatic Invariants of Mechanical Systems.* III". By J. M. BURGERS. Supplement N°. 41*e* to the Communications from the Physical Laboratory at Leiden. (Communicated by Prof. H. KAMERLINGH ONNES).

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In the two preceding papers [1]) on this subject the question was investigated as to what quantities possess the property of being adiabatic invariants for those mechanical systems in which the variables can be separated, i.e. where the momenta can be expressed by formulae of the form:

$$p_k = \sqrt{F_k(q_k, \alpha^1 \ldots \alpha^n, a)}$$

The result obtained was that the "phase-integrals": $I_k = \int dq_k p_k$ *do not change during an adiabatic disturbance of the system*; this conclusion is closely connected with the quantum formulae as introduced by EPSTEIN, DEBYE and SOMMERFELD, who put these integrals equal to integral multiples of PLANCK's constant. SCHWARZSCHILD [2]), however, has put the quantum formulae into another form, which is far more general. He supposes that by means of certain transformations it is possible to express the original coördinates and momenta (q,p) as functions of a new system (Q,P), possessing the following properties:

1. The Q are *linear* functions of the time;
2. the P are constants;
3. the q and p are periodic functions of the Q with a period 2π; hence for instance:

$$q(Q_1 + 2l_1\pi, \ldots \; . \; Q_n + 2l_n\pi) = q(Q_1, \ldots Q_n).$$

These variables Q are the so-called "angular variables" ("Winkelkoordinaten"). He then introduces the quantum formulae:

$$\int_0^{2\pi} dQ_k \cdot P_k = 2\pi P_k = n_k \cdot h + \text{constant}. \quad . \quad . \quad . \quad . \quad (A).$$

If the character of the system is such that the variables can be

[1]) These Proceedings p. 149 and 158.

[2]) K. SCHWARZSCHILD, Sitz. Ber. Berl. Akad. p. 548, 1916.

separated, it is always possible to introduce angular variables; in that case the formulae of SCHWARZSCHILD and those of EPSTEIN coincide [1]). In this paper, however, it will be shown without making use of the separation of the variables, that — provided certain conditions mentioned below are fulfilled — it is always possible to choose the quantities P_k in such a way that they are adiabatic invariants. This is of importance as the possiblity of introducing angular variables is not limited to those systems.

§ 1. We consider a mechanical system possessing solutions of the following form: the coordinates and momenta q and p can be expanded into trigonometric series (multiple Fourier series) proceeding according to sines and cosines of multiples of n variables $Q_1 \dots Q_n$:

$$\left.\begin{aligned} q_k &= \sum_{-\infty}^{\infty} A^k_{m_1 \dots m_n} \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} (m_1 Q_1 + \dots m_n Q_n) \\ p_k &= \sum_{-\infty}^{\infty} B^k_{m_1 \dots m_n} \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} (m_1 Q_1 + \dots m_n Q_n) \end{aligned}\right\} \quad . \quad . \quad . \quad (1)$$

These variables are *linear* functions of the time:

$$Q_i = \omega_i t + \varepsilon_i; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

we limit ourselves to the case that the *mean motions* ω_i *are all incommensurable*. $\varepsilon_1 \dots \varepsilon_n$ are n constants of integration; the ω_i and the coefficients of the trigonometric series are functions of the parameters a occurring in the equations of the system (masses, intensity of a field of force, &c.) and of n other integration constants $P_1 \dots P_n$, chosen in such a way that together with the Q they form a system of canonical variables; the transformation of the q and p into the new variables Q and P is a *contact-transformation* [2]).

We suppose that for a given domain of values of the P the series considered are uniformly convergent, independent of the value of t.

A method of obtaining solutions of this kind is treated in the last chapter of WHITTAKER's *Analytical Dynamics* (Cambridge 1904 [3])): *Integration by Trigonometric Series.* — If the Hamiltonian function is a quadratic function of the original variables q and p, the angular variables Q are immediately related to the normal coordinates or principal vibrations of the system [4]); the series then reduce to:

[1]) Cf. for instance P. S. EPSTEIN, Ann. d. Phys. (4) 51 (1916) pg. 176.
[2]) Cf. f. i. E. T. WHITTAKER, Anal. Dynamics, p. 282. (Cambr. 1904).
[3]) A 2nd edition has appeared in 1917 (note added in the English translation).
[4]) WHITTAKER, l. c. p. 399.

$$q_k = q_{k_0} + \Sigma\, \alpha_i^k \cos Q_i + \Sigma\, \beta_i^k \sin Q_i \quad . \quad . \quad . \quad . \quad (3)$$

with analogous expressions for the p_k.

§ 2. Adiabatic disturbances of the system.

As before we shall assume that during the infinitely slow change of the parameters the HAMILTONIAN equations for the original coordinates and momenta q and p remain valid (see, however, below, remark 4, *a*). In order to investigate how the variables Q and P behave during such a process, it is simplest to consider into what expression the differential form:

$$\Sigma p\, dq - H(q, p, a)\, dt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

changes by the transformation from the q, p to the Q, P[1]. As remarked above this transformation is a *contact-transformation*; hence as long as the a are not varied we have:

$$\Sigma p\, dq = \Sigma P\, dQ + dW. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

dW being the complete differential of a function of the Q and P, which may also contain the a. During the variation the a are explicitly given functions of the time; the formula (5) has then to be replaced by

$$\Sigma p\, dq = \Sigma P\, dQ + F \cdot \frac{da}{dt} dt + DW. \quad . \quad . \quad . \quad . \quad (6)$$

where:

$$DW = \Sigma \frac{\partial W}{\partial Q} dQ + \Sigma \frac{\partial W}{\partial P} dP + \frac{\partial W}{\partial a} \frac{da}{dt} dt \text{ [2]} \quad . \quad . \quad . \quad (7)$$

F is a function of Q, P and a, which — if the P have been properly chosen — contains the Q *only in the form of trigonometric functions*:

$$F = \Sigma\, C_{m_1 \ldots m_n} \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} (m_1 Q_1 + \ldots m_n Q_n). \quad . \quad . \quad . \quad (8)$$

The proof of this proposition is given in § 3.

Hence the differential expression (4) changes into:

$$\Sigma\, P\, dQ - \{H^*(Q, P, a) - F \cdot \dot{a}\}\, dt + DW. \quad . \quad . \quad . \quad (9)$$

$H^*(Q, P, a)$ is obtained from $H(q, p, a)$ by replacing the q and p by their expansions in trigonometric series. Now the characteristic property of the angular variables is that H^* *does not contain the* Q:

$$H^* = H^*(P, a) \text{ [3]} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The equations of motion for the Q and P are the canonical equations derived from a HAMILTONIAN function, which is equal to the coefficient of dt in the differential expression (9).[1]

[1]) WHITTAKER, l. c. p. 297.

[2]) In order to simplify the formulae it is assumed that only one parameter a is varied.

[3]) WHITTAKER, l. c. p. 407.

Hence we have for P_k:

$$\frac{dP_k}{dt} = \frac{\partial F}{\partial Q_k}\dot{a} = \dot{a}\left[\sum{}' \pm m_k . C_{m_1} \ldots _{m_n} \begin{Bmatrix} sin \\ cos \end{Bmatrix} (m_1 Q_1 + \ldots m_n Q_n)\right] \quad (11)^{1})$$

If no relations of commensurability exist between the mean motions ω_ι of the Q_ι — as is assumed in § 1 — the mean of this expression with respect to the time is *zero*: hence during the variational process P_k *remains unchanged* [2]). We have thus proved that the expressions which are "quantizised" by SCHWARZSCHILD are invariants for an adiabatic disturbance of the system.

As according to formula (10) the *total energy* $E = H^*(P, a)$ *only depends on the* P and on the parameters, it is always possible *to fix the value of the energy by quantizising the* P. [3]) [4])

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1) The meaning of Σ' is: summation over all + and — values of the m, with the exception of *simultaneous zero values of all the m.*

2) This may be formulated more exactly as follows:

For the sake of simplicity suppose $\frac{da}{dt}$ to be constant: then by integrating eq. (11) term by term (which is allowed on account of the uniform convergence):

$$\delta P_k = a\left[\sum{}' \Gamma^k_{m_1 \ldots m_n} \begin{Bmatrix} cos \\ sin \end{Bmatrix} (m_1 Q_1 + \ldots m_n Q_n)\right]_{t_0}^{t_0 + T}$$

Independently of the value of t the value of the term between [ ] always remains below a finite limit g. Hence:

$$|\delta P_k| < 2\dot{a} . g$$

On the other hand:

$$\delta a = \dot{a} . T$$

We thus have:

$$\lim_{T=\infty} \frac{\delta P_k}{\delta a} = 0$$

This reasoning also applies to the demonstration given in the 1st part of this paper (These Proceedings, p. 149).

3) If the original HAMILTONIAN function $H(q,p,a)$ is a *quadratic* function of the q and p, $H^*(P,a)$ will be found to be of the form:

$$\Sigma \omega_k . P_k + constant.$$

Hence if P_k is put equal to $n_k \frac{h}{2\pi}$ the total energy of the system is:

$$E = \frac{h}{2\pi} \Sigma \omega_k . n_k + constant.$$

4) It can be shown that $-\frac{\partial H^*(P,a)}{\partial a}$ is equal to the mean with respect to the time of the force exerted by the system "in the direction of the parameter a".

§ 3. Proof of formula (8).

In the expression $\Sigma p_k dq_k$ q and p are replaced by their expansions (1); in differentiating the Q, P, and t are regarded as independent variables, the parameter a being an explicitly given function of t. This gives:

$$\Sigma p_k . dq_k = \Sigma f_1^k . dQ_k + \Sigma f_2^k . dP_k + f_3 . \frac{da}{dt} . dt$$

f_1^k f_2^k, f_3 are FOURIER series with respect to the Q.

As for $a =$ constant this substitution is a contact transformation, we must have:

$$\Sigma f_1^k . dQ_k + \Sigma f_2^k . dP_k = \Sigma P_k dQ_k + \Sigma \frac{\partial W}{\partial Q_k} dQ_k + \Sigma \frac{\partial W}{\partial P_k} dP_k \quad (12)$$

Hence:

$$\frac{\partial W}{\partial Q_k} = -P_k + f_1^k = -P_k + \gamma_0^k(P, a) + \Sigma' \gamma^k_{m_1 \ldots m_n} \begin{Bmatrix} cos \\ sin \end{Bmatrix} (m_1 Q_1 + \ldots m_n Q_n)$$

and:

$$W = \Sigma (-P_k + \gamma_0^k) Q_k + \Sigma \sigma_{m_1 \ldots m_n} \begin{Bmatrix} cos \\ sin \end{Bmatrix} (m_1 Q_1 + \ldots m_n Q_n)$$

Furthermore we have:

$$\frac{\partial W}{\partial P_k} = f_2^k = -Q_k + \Sigma \frac{\partial \gamma_0^l}{\partial P_k} Q_l + \Sigma \frac{\partial \sigma_{m_1 \ldots m_n}}{\partial P_k} \begin{Bmatrix} cos \\ sin \end{Bmatrix} (m_1 Q_1 + \ldots m_n Q_n)$$

In f_2^k the Q 'occur only under sines and cosines; from this it follows that the coefficient of Q_k on the second side of the equation must be *zero*, and hence:

$$\gamma_0^k = P_k + \pi_k(a).$$

As the condition (12) determines the P and Q all but the additive constants, it is always possible to include the $\pi_k(a)$ in the P. If we suppose this to be the case, we get:

$$\gamma_0^k = P_k,$$

hence:

$$W = \Sigma \sigma_{m_1 \, m_n} \begin{Bmatrix} cos \\ sin \end{Bmatrix} (m_1 Q_1 + \ldots m_n Q_n) \quad . \quad . \quad . \quad . \quad (13)$$

It follows that:

$$F = f_3 - \frac{\partial W}{\partial a}$$

is a FOURIER *series with respect to the* Q, and thus the proposition has been proved.

Remarks.

1. If $\pi_k(a)$ is not made equal to zero by a proper choice of the additive constant of P_k, it will be found that:

$$P_k + \pi_k(a) = \text{adiabatic invariant.}$$

2. In many cases the P_k can immediately be introduced in such a way that the quantities $\pi_k(a)$ are zero. As examples we may mention:

a. systems the HAMILTONIAN function of which can be expanded according to ascending powers of the q and p, and which are to be treated by a method given by WHITTAKER [1]);

b. systems in which the variables can be separated; the P are then determined by the formulae:

$$2\pi P_k = I_k = \text{phase-integral corresponding to the coordinate } q_k =$$

$$= 2\int_{\xi_k}^{\eta_k} p_k \,.\, dq_k.$$ [2])

3. Suppose the P to be determined as assumed above, so that W is a periodic function of the Q (form. 13). If the parameters are not varied:

$$\sum_l p_l \,.\, dq_l = \sum_k P_k \,.\, dQ_k + dW.$$

Integrating this expression from $Q_k = 0$ to $Q_k = 2\pi$ ($Q_1 \,..\, Q_{k-1}$ $Q_{k+1} \,..\, Q_n$, $P_1 \,..\, P_n$ being kept constant), we find:

$$\int_{Qk=0}^{Qk=2\pi} \Sigma\, pdq = 2\pi P_k = \text{adiabatic invariant.}$$

[If the $\pi_k(a)$ have not been included in the P, it is found that:

$$\int_{Qk=0}^{Qk=2\pi} \Sigma\, pdq = 2\pi\,(P_k + \pi_k) = \text{adiab. inv. according to remark 1].}$$

EPSTEIN has given the quantum formulae in a form which is equivalent to:

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1) WHITTAKER, l.c. p. 398 - 408.

2) The constants ε_k which occur in SCHWARZSCHILD's formulae (l. c. p. 549, 551; see also higher up, form. *A*), and which are determined by the limits of the phase-space, are probably connected with the quantities π_k introduced here; but I have not been able so far to find a general proof.

$$\int_{Qk=0}^{Qk=2\pi} \Sigma\, pdq = n_k \, . \, h$$ [1]

and is therefore in agreement with the above.

4. The following points have still to be mentioned:

a. Probably it will be found sufficient that in passing from $a =$ constant to $a =$ a given function of the time, the Hamiltonian equations remain unchanged only if we neglect terms of the 2nd and higher orders in $\dot{a}$. This has yet to be investigated.

b. In the present paper it has been supposed that the mean motions ω_i are all incommensurable. The ω_i are, however, functions of the parameters. Hence if the a are varied, the ω_i change too, and their ratios pass through rational values. It has still to be investigated, whether this may give rise to difficulties. (This applies also to the demonstrations given in the preceding papers).

SUMMARY.

If a mechanical system of n degrees of freedom possesses solutions which can be expressed by means of multiple trigonometric series, proceeding by the sines and cosines of n angular variables, between the mean motions of which no relations of commensurability exist, it is possible to determine the canonical momenta corresponding to these angular variables in such a way that they are *adiabatic invariants* for an infinitely slow change of the parameters of the system. — (The fact that during an adiabatic disturbance the mean motions change and that their ratios pass through rational values has to be further inquired into.)

[1] P. S. EPSTEIN, Verh. d. D. Physik. Ges. 18 (1916) p. 411.