Mathematics. — "*On hyperelliptic integrals of deficiency $p = 2$, reducible by a transformation of order $r = 4$.*" By Prof. J. C. KLUYVER.

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The conditions that an hyperelliptic integral of deficiency $p = 2$ is reducible to an elliptic integral by a transformation of order $r = 4$, have been assigned by BOLZA [1]), who used direct algebraic methods and also by IGEL [2]), who based his deductions on the transformation of the double thetas. I will show that the geometry of a linear system of conics affords the means to solve the problem, and that geometric considerations enable us to add some results to those previously obtained [3]).

Let the integral be of the form

$$\int \frac{X dx}{\sqrt{\psi_1 \psi_2 \psi_3}},$$

where X is a linear quantic and ψ_1, ψ_2, ψ_3 are binary quadrics of the variable $x = x_1 : x_2$, then the integral is reducible under the following three conditions:

1. There are three quadrics ξ_1, ξ_2, ξ_3, each of which is a perfect square, such that the quartics $\xi_1\psi_1, \xi_2\psi_2, \xi_3\psi_3$ are linearly connected. Otherwise stated, these quartics are elements of an involution J of order 4.

2. The involution J contains an element T^2, a quartic being a perfect square.

3. The numerator X of the integral has a determinate form depending on ψ_1, ψ_2, ψ_3.

In fact, supposing the first and the second of these conditions to be fulfilled, we can take in J any two elements whatever R_1 and R_2, and k_1, k_2, k_3, h being certain constants, we have

$$\xi_1\psi_1 = R_1 - k_1 R_2 \ , \ \xi_2\psi_2 = R_1 - k_2 R_2 \ , \ \xi_3\psi_3 = R_1 - k_3 R_2 \ , \ T^2 = R_1 - h R_2 \ ,$$

then putting

$$t = \frac{R_1}{R_2},$$

1) Math. Ann., Bd. 28, p. 447.

2) Monatshefte für Math. u. Phys., II, p. 157.

3) For a summary of the researches on reducible Abelian integrals see: W. C. POST, Dissertation, Leyden, 1917.

we get

$$dt = \frac{1}{R_2^2}(R_2R_1' - R_2'R_1)\,dx\,,$$

$$\sqrt{(t-k_1)(t-k_2)(t-k_3)(t-h)} = \frac{T}{R_2^2}\sqrt{\xi_1\xi_2\xi_3\psi_1\psi_2\psi_3}$$

and hence

$$\frac{dt}{\sqrt{(t-k_1)(t-k_2)(t-k_3)(t-h)}} = \frac{dx}{\sqrt{\psi_1\psi_2\psi_3}} \times \frac{(R_2R_1'-R_2'R_1)}{T\sqrt{\xi_1\xi_2\xi_3}}\,.$$

Now the sextic $(R_2R_1'-R_2'R_1)$ is evidently divisible by the quintic $T\sqrt{\xi_1\xi_2\xi_3}$, therefore the given integral is reducible, if we take

$$X = \frac{(R_2R_1'-R_2'R_1)}{T\sqrt{\xi_1\xi_2\xi_3}}\,.$$

Thus it is seen that the reducibility of the integral in the first instance depends upon the existence of the involution J, and on the possibility of determining the quadrics ξ_1, ξ_2, ξ_3. The investigation of the involution J and of its characteristic properties may be conducted as follows.

With three binary quadrics

$$\psi_1 = a_0x^2 + 2a_1x + a_2\ ,\quad \psi_2 = a_0'x^2 + 2a_1'x + a_2'\ ,\quad \psi_3 = a_0''x^2 + 2a_1''x + a_2''$$

six common invariants are associated. As such we get in the first place the discriminants

$$A_{11} = 2(a_0a_2 - a_1^2)\ ,\quad A_{22} = 2(a_0'a_2' - a_1'^2)\ ,\quad A_{33} = 2(a_0''a_2'' - a_1''^2)$$

and further the harmonic invariants

$$A_{23} = (a_0'a_2'' + a_0''a_2' - 2a_1'a_1'')\ ,\quad A_{31} = (a_0''a_2 + a_0a_2'' - 2a_1''a_1)\,,$$
$$A_{12} = (a_0a_2' + a_0'a_2 - 2a_1a_1')\,.$$

The three quadrics themselves are connected by the identical relation

$$\begin{vmatrix} A_{11} & A_{12} & A_{31} & \psi_1 \\ A_{12} & A_{22} & A_{23} & \psi_2 \\ A_{31} & A_{23} & A_{33} & \psi_3 \\ \psi_1 & \psi_2 & \psi_3 & 0 \end{vmatrix} = 0\,,$$

which I write in the form

$$K \equiv a\psi_1^2 + b\psi_2^2 + c\psi_3^2 + 2f\psi_2\psi_3 + 2g\psi_3\psi_1 + 2h\psi_1\psi_2 = 0\,.$$

Now this relation between binary quadrics can also be conceived as the equation in trilinear coordinates ψ_1, ψ_2, ψ_3 of a conic K, and since each of the coordinates is a quadric in x, this variable procures a parametric representation of the curve. From the same point of view any homogeneous polynomial $F(\psi_1\, \psi_2\, \psi_3)$ on the

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one side is a binary quantic in x, on the other side it represents a curve in the plane of the conic K. An arbitrary quadric, for instance, can always be thrown into the form $h_1\psi_1 + h_2\psi_2 + h_3\psi_3$, and therefore it represents a right line meeting the conic K in two points, at which the parameter x is equal to either of the roots of the quadric $h_1\psi_1 + h_2\psi_2 + h_3\psi_3$. In particular, since the quadrics ξ_1, ξ_2, ξ_3, are perfect squares, the right lines ξ_1, ξ_2, ξ_3, are tangents of K with the points of contact, say, A_1, A_2, A_3. In this way each element of the involution J corresponds with a conic and the involution J itself defines a linear system S of conics, such that the system is determined by three of its elements. Evidently the system S thus constructed must contain: the conic K, the three pairs of right lines $\xi_1\psi_1$, $\xi_2\psi_2$, $\xi_3\psi_3$ and lastly the double line T. Since S contains a double line, it is not a wholly general system. Its Jacobian breaks up into the right line T and into a conic H. The Jacobian passes through every point of contact of two conics belonging to S, hence the conic H passes through the points A_1, A_2, A_3, and meets K in a fourth point A_4. The tangents to K at the points A_1, A_2, A_3, i.e. the right lines ξ_1, ξ_2, ξ_3, and also the tangent ξ_4 at the point A_4 intersect H again respectively in the points B_1, B_2, B_3, B_4. The latter points, lying on the Jacobian, are the centres of degenerate conics $\xi_1\psi_1, \xi_2\psi_2, \xi_3\psi_3$ and of a fourth degenerate conic $\xi_4\psi_4$, and thus we have proved that the involution J besides the three elements $\xi_1\psi_1$, $\xi_2\psi_2, \xi_3\psi_3$ each having a double root, necessarily must contain a fourth element $\xi_4\psi_4$ that has the same peculiarity.

In a certain sense the tangent ξ_4 considered as a binary quadric is directly connected with the reducible integral. Let us seek for the points in which the conic K is touched by any conic of the system S. If R_1 and R_2 are two arbitrary elements of S, the equation of the system is

$$R_1 + \lambda R_2 + \mu K = 0\,.$$

In order to find the values of the parameter x at the points of contact, we must again conceive R_1 and R_2 as binary quartics and the required parametric values are the roots of the sextic

$$(R_2 R_1' - R_2' R_1) = 0\,.$$

Now, as was said before, the points of contact in question are the points in which K meets the Jacobian. Hence the roots of the sextic are the parametric values of x at the points A_1, A_2, A_3, A_4 and at the points of intersection of K with the right line T. Therefore the sextic is the product of the five quantics $\sqrt{\xi_1}, \sqrt{\xi_2}, \sqrt{\xi_3}, \sqrt{\xi_4}$

and T, and the numerator X of the integral, which we have found to be

$$\frac{(R_2 R_1' - R_2' R_1)}{T\sqrt{\xi_1 \xi_2 \xi_3}},$$

is identical with the linear quantic $\sqrt{\xi_4}$.

Obviously, we may now conclude that as soon as the given integral is reducible, there are three other integrals of deficiency $p = 2$ connected with the involution J, i.e. the integrals

$$\int \frac{\sqrt{\xi_1}}{\sqrt{\psi_2 \psi_3 \psi_4}} dx \quad , \quad \int \frac{\sqrt{\xi_2}}{\sqrt{\psi_1 \psi_3 \psi_4}} dx \quad , \quad \int \frac{\sqrt{\xi_3}}{\sqrt{\psi_1 \psi_2 \psi_4}} dx ,$$

which can be reduced to elliptic integrals. Moreover, it will be evident that the transformation of the four integrals will be effected by one and the same transformation formula, and we may notice that likewise the integral

$$\int \frac{T}{\sqrt{\psi_1 \psi_2 \psi_3 \psi_4}} dx$$

of deficiency $p = 3$ becomes elliptic by that transformation.

In order to find how the involution J can be constructed from the given quadrics ψ_1, ψ_2, ψ_3, I will proceed with the analytical investigation of the system S. It is always possible by adjoining suitable constant factors to the quadrics ξ_1, ξ_2, ξ_3 to ensure the identical relation

$$\sqrt{\xi_1} + \sqrt{\xi_2} + \sqrt{\xi_3} = 0,$$

and hence the relation

$$\xi_1^2 + \xi_1^2 + \xi_3^2 - 2\xi_2\xi_3 - 2\xi_3\xi_1 - 2\xi_1\xi_2 = 0,$$

an identity in the variable x that denotes at the same time the conic K in the trilinear coordinates ξ_1, ξ_2, ξ_3.

The point A_1 on K, the coordinates of which in the system ξ_1, ξ_2, ξ_3 are $(0, 1, 1)$, has its conjugate with respect to the system S at the point A'_1 where the tangent ξ_1 of K meets the double line T of the system.

Supposing T to have the equation

$$T \equiv L\xi_1 + M\xi_2 + N\xi_3 = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

this point A'_1 has the coordinates $(0, -N, M)$, hence the coefficients of the equation

$$A\xi_1^2 + B\xi_2^2 + C\xi_3^2 + 2F\xi_2\xi_3 + 2G\xi_3\xi_1 + 2H\xi_1\xi_2 = 0,$$

representing any conic of S, underly the condition

$$BN - FM + F(N - M) = 0,$$

or

$$F = B\frac{N}{M-N} - C\frac{M}{M-N}.$$

In like manner the points A_2 and A'_2, A_3 and A'_3 are conjugate points of S, therefore we have also

$$G = C\frac{L}{N-L} - A\frac{N}{N-L},$$

$$H = A\frac{M}{L-M} - B\frac{L}{L-M},$$

and the equation of the system itself may be written in the form

$$A\xi_1\left[-\xi_1 - \frac{2M}{L-M}\xi_2 + \frac{2N}{N-L}\xi_3\right] + B\xi_2\left[\frac{2L}{L-M}\xi_1 - \xi_2 - \frac{2N}{M-N}\xi_3\right] +$$
$$+ C\xi_3\left[-\frac{2L}{N-L}\xi_1 + \frac{2M}{M-N}\xi_2 - \xi_3\right] = 0.$$

Since S contains the three degenerate conics $\xi_1\psi_1$, $\xi_2\psi_2$, $\xi_3\psi_3$, it is seen that the expressions between brackets in the above equation denote the right lines ψ_1, ψ_2, ψ_3, and we may write

$$\left.\begin{aligned} P_1\psi_1 &= \left[-\xi_1 - \frac{2M}{L-M}\xi_2 + \frac{2N}{N-L}\xi_3\right], \\ P_2\psi_2 &= \left[\frac{2L}{L-M}\xi_1 - \xi_2 - \frac{2N}{M-N}\xi_3\right], \\ P_3\psi_3 &= \left[-\frac{2L}{N-L}\xi_1 + \frac{2M}{M-N}\xi_2 - \xi_3\right], \end{aligned}\right\} \quad . \; . \; . \; . \quad (2)$$

where P_1, P_2, P_3 are determinate constants.

From these equations we deduce, always using the coordinates ξ_1, ξ_2, ξ_3, the coordinates of the points B_1, B_2, B_3, the centres of the conics $\xi_1\psi_1$, $\xi_2\psi_2$, $\xi_3\psi_3$.

Putting

$$L(M-N) = q_1 \quad , \quad M(N-L) = q_2 \quad , \quad N(L-M) = q_3, \quad . \quad (3)$$

so that the constants q_1, q_2, q_3 are related by the equation

$$q_1 + q_2 + q_3 = 0,$$

we find for the coordinates of B_1, B_2, B_3 respectively $(0, q_3, q_2)$, $(q_3, 0, q_1)$, $(q_2, q_1, 0)$. Incidentally we may remark that at the points B_1, B_2, B_3 the right lines ξ_1, ξ_2, ξ_3 are touched by the conic

$$K_1 \equiv q_1{}^2\xi_1{}^2 + q_2{}^2\xi_2{}^2 + q_3{}^2\xi_3{}^2 - 2q_2q_3\,\xi_2\xi_3 - 2q_3q_1\,\xi_3\xi_1 - 2\,q_1q_2\,\xi_1\xi_2 = 0,$$

and at the same time we conclude that the equation of the conic H, that forms part of the Jacobian, must be

$$H \equiv q_1\xi_1{}^2 + q_2\xi_2{}^2 + q_3\xi_3{}^2 + q_1\xi_2\xi_3 + q_2\xi_3\xi_1 + q_3\xi_1\xi_2 = 0.$$

For plainly this conic passes through the points A_1, A_2, A_3 and also through B_1, B_2, B_3.

The equations of the tangent ξ_4 and of the right line ψ_4 remain to be found, and to this end I consider the pencil of quartic curves

$$\lambda H^2 + \xi_1 \xi_2 \xi_3 \xi_4 = 0.$$

These curves have the right lines $\xi_1, \xi_2, \xi_3, \xi_4$ as bitangents and the eight points of contact are obviously the points $A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4$. Now the product KK_1 is a quartic curve that passes through 14 of the 16 fixed points that are common to the curves of the pencil. For K touches the bitangents at A_1, A_2, A_3, A_4 and K_1 has contact with ξ_1, ξ_2, ξ_3 at the points B_1, B_2, B_3. Hence the curve KK_1 belongs to the pencil, K_1 touches ξ_4 at the point B_4, and there is a certain value λ_1 of λ such that

$$\lambda_1 H^2 + \xi_1 \xi_2 \xi_3 \xi_4 \equiv \mu KK_1$$

It is readily seen that $\lambda_1 = 1$, that we must have $\mu = 1$, and from

$$\xi_1 \xi_2 \xi_3 \xi_4 \equiv KK_1 - H^2$$

we find

$$\xi_4 \equiv (q_3-q_1)(q_1-q_2)\,\xi_1 + (q_1-q_2)(q_2-q_3)\,\xi_2 + (q_2-q_3)(q_3-q_1)\,\xi_3 = 0. \quad (4)$$

Putting

$$\psi_4 = \mu_1 \xi_1 + \mu_2 \xi_2 + \mu_3 \xi_3,$$

we write down that $A_1(0,1,1)$ and $A_1'(0,-N_1,M)$ are conjugate points with respect to the conic $\xi_4\psi_4$. Thus we find the relation

$$\frac{\mu_2}{-3M + \frac{1}{M}(MN + NL + LM)} = \frac{\mu_3}{-3N + \frac{1}{N}(MN + NL + LM)}$$

and a similar relation is obtained by means of the conjugate points A_2 and A_2'. In this way it appears that the right line ψ_4 will be denoted by

$$\psi_4 = -3T + \left(\frac{\xi_1}{L} + \frac{\xi_2}{M} + \frac{\xi_3}{N}\right)(MN + NL + LM). \quad . \quad . \quad (5)$$

In the preceding we took for granted that it was possible to represent one and the same conic K in two different systems of coordinates by the two equations

$$K \equiv a\psi_1^2 + b\psi_2^2 + c\psi_3^2 + 2f\psi_2\psi_3 + 2g\psi_3\psi_1 + 2h\psi_1\psi_2 = 0, \quad . \quad (6)$$

$$K \equiv \xi_1^2 + \xi_2^2 + \xi_3^2 - 2\xi_2\xi_3 - 2\xi_3\xi_1 - 2\xi_1\xi_2 = 0, \quad . \quad . \quad (7)$$

the ψ-coordinates depending on the ξ-coordinates as is indicated by the equations (2), and we have now to examine if these two equations are really consistent.

Here it is noteworthy that, after introducing certain constants f_1, g_1, h_1, the lefthand side of equation (6) becomes

$$\psi_1[a\psi_1 + (h-h_1)\psi_2 + (g+g_1)\varphi_3] + \psi_2[(h+h_1)\psi_1 + b\psi_2 + (f-f_1)\psi_3] +$$
$$+ \psi_3[(g-g_1)\psi_1 + (f + f_1)\psi_2 + c\psi_3],$$

and that the lefthand side of equation (7) can be written

$$-(P_1\xi_1\psi_1 + P_2\xi_2\psi_2 + P_3\xi_3\psi_3).$$

The equations (6) and (7) therefore have the same meaning, as soon as we have

$$\frac{P_1\xi_1}{a\psi_1+(h-h_1)\psi_2+(g+g_1)\psi_3} = \frac{P_2\xi_2}{(h+h_1)\psi_1+b\psi_2+(f-f_1)\psi_3} =$$
$$= \frac{P_3\xi_3}{(g-g_1)\psi_1+(f+f_1)\psi_2+c\psi_3} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and it is only when these relations hold that either of the equations (6) and (7) is a direct consequence, of the other.

To simplify somewhat the notation I put

$$\alpha = \frac{M+N}{M-N}\ ,\quad \beta = \frac{N+L}{N-L}\ ,\quad \gamma = \frac{L+M}{L-M}, \quad . \quad . \quad (9)$$

the new constants α, β, γ being related by the equation

$$(1+\alpha)(1+\beta)(1+\gamma) + (1-\alpha)(1-\beta)(1-\gamma) = 0\ ,$$

or by

$$1 + \beta\gamma + \gamma\alpha + \alpha\beta = 0\ , \quad . \quad . \quad . \quad . \quad . \quad (10)$$

and consequently instead of (2) we may write

$$\left.\begin{aligned} P_1\psi_1 &= -\xi_1 + (1-\gamma)\xi_2 + (1+\beta)\xi_3\ , \\ P_2\psi_2 &= (1+\gamma)\xi_1 - \xi_2 + (1-\alpha)\xi_3\ , \\ P_3\psi_3 &= (1-\beta)\xi_1 + (1+\alpha)\xi_2 - \xi_3\ . \end{aligned}\right\} \quad . \quad . \quad (11)$$

Comparing now the two sets of equations (8) and (11), we observe that (11) defines a homographic transformation expressing the quantities $P\psi$ in the quantities ξ, and that the inverse transformation is given by (8).

Writing down the determinant

$$\begin{vmatrix} -1 & 1-\gamma & 1+\beta \\ 1+\gamma & -1 & 1-\alpha \\ 1-\beta & 1+\alpha & -1 \end{vmatrix}\cdot$$

of the first transformation and also the determinant

$$\begin{vmatrix} \dfrac{a}{P_1^2} & \dfrac{h+h_1}{P_1P_2} & \dfrac{g-g_1}{P_3P_1} \\[2ex] \dfrac{h-h_1}{P_1P_2} & \dfrac{b}{P_2^2} & \dfrac{f+f_1}{P_2P_3} \\[2ex] \dfrac{g+g_1}{P_3P_1} & \dfrac{f-f_1}{P_2P_3} & \dfrac{c}{P_3^2} \end{vmatrix}$$

of the inverse transformation, in which I have interchanged lines and columns, a known proposition says that the elements of the latter determinant are proportional to the corresponding minors of the former. Nine ratios therefore are equal to one and the same quantity λ, and so we have the equations

$$\lambda = \frac{b}{P_2^2\beta^2} = \frac{c}{P_3^2\gamma^2} = \frac{f+f_1}{P_2P_3(2+\alpha-\beta-\gamma+\beta\gamma)} = \frac{f-f_1}{P_2P_3(2-\alpha+\beta+\gamma+\beta\gamma)},$$

from which we infer

$$\frac{f-\sqrt{bc}}{P_2P_3} = \frac{f+\sqrt{bc}}{P_2P_3(1+\beta\gamma)},$$

or

$$1+\beta\gamma = \frac{f+\sqrt{bc}}{f-\sqrt{bc}}.$$

Similarly we obtain

$$1+\gamma\alpha = \frac{g+\sqrt{ca}}{g-\sqrt{ca}},$$

$$1+\alpha\beta = \frac{h+\sqrt{ab}}{h-\sqrt{ab}},$$

and then by equation (10)

$$2 = \frac{f+\sqrt{bc}}{f-\sqrt{bc}} + \frac{g+\sqrt{ca}}{g-\sqrt{ca}} + \frac{h+\sqrt{ab}}{h-\sqrt{ab}}. \quad . \quad . \quad . \quad . \qquad (12)$$

Thus it appears that the reducibility of the given hyperelliptic integral implies the relation (12) between the invariants common to ψ_1, ψ_2, ψ_3, and conversely as soon as these invariants, with a suitable determination of the surds, satisfy the relation (12) the involution J can be realised, and the given integral will degenerate.

Supposing that the condition (12) is fulfilled, we have

$$\lambda = \frac{a}{P_1^2\alpha^2} = \frac{b}{P_2^2\beta^2} = \frac{c}{P_3^2\gamma^2} = \frac{f}{P_2P_3(2+\beta\gamma)} = \frac{g}{P_3P_1(2+\gamma\alpha)} =$$

$$= \frac{h}{P_1P_2(2+\alpha\beta)} = \frac{f_1}{P_2P_3(\alpha-\beta-\gamma)} = \frac{g_1}{P_3P_1(-\alpha+\beta-\gamma)} = \frac{h_1}{P_1P_2(-\alpha-\beta+\gamma)}, \qquad (13)$$

and

$$\alpha P_1\sqrt{bc} = \beta P_2\sqrt{ca} = \gamma P_3\sqrt{ab} \quad . \quad . \quad . \quad . \quad . \qquad (14)$$

From these equations the constants $\alpha, \beta, \gamma, P_1, P_2, P_3, L, M, N, q_1, q_2, q_3$ can be successively evaluated, we can find the quantics $\xi_1, \xi_2, \xi_3, \xi_4, T, \psi_4$ and finally the transformation that reduces the integral.

To illustrate the method described, I will consider a numerical example. Let the given integral be

$$\int \frac{X\,dx}{\sqrt{(5x^2-12x+4)(5x^2-2x+2)(7x^2-6x+2)}},$$

that is, let us assume

$$\psi_1 = 5x^2-12x+4, \qquad \psi_2 = 5x^2-2x+2, \qquad \psi_3 = 7x^2-6x+2.$$

Calculating the invariants, we have

$$\frac{A_{11}}{-16}=\frac{A_{22}}{9}=\frac{A_{33}}{5}=\frac{A_{23}}{9}=\frac{A_{31}}{1}=\frac{A_{12}}{9},$$

$$\lambda=\frac{a}{4}=\frac{b}{9}=\frac{c}{25}=\frac{f}{-17}=\frac{g}{-8}=\frac{h}{4}.$$

Now we may take

$$\lambda=\frac{\sqrt{bc}}{-15}=\frac{\sqrt{ca}}{-10}=\frac{\sqrt{ab}}{6}$$

and with this determination of the surds we get

$$\frac{f+\sqrt{bc}}{f-\sqrt{bc}}=1+\beta\gamma=16,\ \frac{g+\sqrt{ca}}{g-\sqrt{ca}}=1+\gamma\alpha=-9,\ \frac{h+\sqrt{ab}}{h-\sqrt{ab}}=1+\alpha\beta=-5.$$

The sum of the three fractions is equal to 2, therefore the integral is reducible. At the same time we have found

$$\beta\gamma=15, \qquad \gamma\alpha=-10, \qquad \alpha\beta=-6,$$

or

$$\alpha^2=4, \qquad \beta^2=9, \qquad \gamma^2=25,$$

so that we have either

$$\alpha=2, \qquad \beta=-3, \qquad \gamma=-5,$$

or

$$\alpha=-2, \qquad \beta=3, \qquad \gamma=5.$$

Two sets of values for the constants α, β, γ being admissible, we infer that the given quadrics ψ_1, ψ_2, ψ_3 allow us to us build up two entirely distinct involutions J, and instead of a single reducible integral of the given form, two such integrals are possible. This is obviously in accordance with the known proposition that, as soon as an integral belonging to an algebraic function of deficiency $p=2$ is reducible, that function possesses a second degenerate integral.

I will take up the case

$$\alpha=2, \qquad \beta=-3, \qquad \gamma=-5.$$

Then we have from (14) and from (13)

$$\frac{P_1}{1}=\frac{P_2}{-1}=\frac{P_3}{1},$$

$$\frac{a}{4}=\frac{f_1}{-10}=\frac{g_1}{0}=\frac{h_1}{1}$$

hence from (8)

$$\frac{\xi_1}{4\psi_1-8\psi_3}=\frac{-\xi_2}{8\psi_1+9\psi_2-7\psi_3}=\frac{\xi_3}{-8\psi_1-27\psi_2+25\psi_3},$$

or

$$\frac{\xi_1}{x^2}=\frac{\xi_2}{(x-1)^2}=\frac{\xi_3}{1},$$

$$\frac{\sqrt{\xi_1}}{x}=\frac{\sqrt{\xi_2}}{1-x}=\frac{\sqrt{\xi}}{-1}.$$

Now equations (9) and (3) give

$$L=2,\quad M=3,\quad N=1,\quad q_1=4,\quad q_2=-3,\quad q_3=-1$$

and then by equations (1), (4) and (5) we find

$$T=const.(5x^2-6x+4),\quad \xi_4=const.(7x-2)^2,\quad \psi_4=const.(35x^2-64x-16))$$

The elements $\xi_1\psi_1, \xi_2\psi_2, \xi_3\psi_3, \xi_4\psi_4, T^2$ of the involution J being known, we may put for instance

$$t=\frac{\xi_1\psi_1}{\xi_2\psi_2}=\frac{x^2(5x^2-12x+4)}{(x-1)^2(5x^2-2x+2)}$$

and obtain consequently

$$\frac{t}{x^2(5x^2-12x+4)}=\frac{1}{(x-1)^2(5x^2-2x+2)}=\frac{1-t}{(7x^2-6x+2)}=$$

$$=\frac{375\,t-32}{(7x-2)^2(35x^2-64x-16)}=\frac{8-3t}{(5x^2-6x+4)^2}.$$

The above transformation now will reduce four integrals of deficiency $p=2$, connected with the involution J, and we may write down at once

$$\int\frac{(7x-2)\,dx}{\sqrt{(5x^2-12x+4)(5x^2-2x+2)(7x^2-6x+2)}}=-\frac{1}{\sqrt{2}}\int\frac{dt}{\sqrt{t(1-t)(8-3t)}},$$

$$\int\frac{dx}{\sqrt{(5x^2-12x+4)(5x^2-2x+2)(-35x^2+64x+16)}}=\sqrt{2}\int\frac{dt}{\sqrt{t(32-375t)(8-3t)}},$$

$$\int\frac{(x-1)\,dx}{\sqrt{(5x^2-12x+4)(7x^2-6x+2)(-35x^2+64x+16)}}=$$

$$=-\sqrt{2}\int\frac{dt}{\sqrt{t(1-t)(32-375t)(8-3t)}},$$

$$\int\frac{x\,dx}{\sqrt{(5x^2-2x+2)(7x^2-6x+2)(-35x^2+64x+16)}}=$$

$$=\int\frac{dt}{\sqrt{(1-t)(32-375t)(8-3t)}}.$$

the constants $-\frac{1}{\sqrt{2}}$, etc. at the right hand sides being easily found by observing, that for small values of x we must have $t=x^2$.

As I have remarked before, the same transformation will also reduce an integral of deficiency $p=3$, connected with the involution J.

In fact, we have

$$\int\frac{(5x^2-6x+4)\,dx}{\sqrt{(5x^2-12x+4)(5x^2-2x+2)(7x^2-6x+2)(-35x^2+64x+16)}}=$$

$$=\sqrt{2}\int\frac{dt}{\sqrt{t(1-t)(32-375\,t)}}.$$

Again, if we had used the second set of admissible values for α, β, γ

$$\alpha = -2, \qquad \beta = 3, \qquad \gamma = 5,$$

we should have found successively

$$\frac{P_1}{1} = \frac{P_2}{-1} = \frac{P_3}{1}, \qquad \frac{a}{4} = \frac{f_1}{10} = \frac{g_1}{0} = \frac{h_1}{-4},$$

$$\frac{\xi_1}{4\psi_1 + 8\psi_2 - 8\psi_3} = \frac{\xi_2}{9\psi_2 - 27\psi_3} = \frac{\xi_3}{-8\psi_1 - 7\psi_2 + 25\psi_3},$$

$$\frac{\xi_1}{(x-2)^2} = \frac{\xi_2}{(6x-3)^2} = \frac{\xi_3}{(5x-1)^2},$$

$$\frac{\sqrt{\xi_1}}{x-2} = \frac{\sqrt{\xi_2}}{-6x+3} = \frac{\sqrt{\xi_3}}{5x-1},$$

$$L = 3, \quad M = 2, \quad N = 6, \quad q_1 = -12, \quad q_2 = 6, \quad q_3 = 6.$$

$$T = const. \, (25x^2 - 16x + 4), \quad \xi_4 = const \; \xi_1, \quad \psi_4 = const. \; \psi_1.$$

Now we may apply the transformation

$$t = \frac{\xi_1 \psi_1}{\xi_2 \psi_2} = \frac{(x-2)^2 \, (5x^2 - 12x + 4)}{(6x-3)^2 \, (5x^2 - 2x + 2)},$$

whence we have

$$\frac{9t}{(x-2)^2 \, (5x^2 - 12x + 4)} = \frac{9}{(6x-3)^2 \, (5x^2 - 2x + 2)} =$$

$$= \frac{9\,(1-t)}{(5x-1)^2 \, (7x^2 - 6x + 2)} = \frac{32 - 27t}{(25x^2 - 16x + 4)^2}$$

and we shall obtain

$$\int \frac{(x-2)dx}{\sqrt{(5x^2 - 12x + 4)(5x^2 - 2x + 2)(7x^2 - 6x + 2)}} = \tfrac{1}{6} \int \frac{dt}{\sqrt{t(1-t)(32 - 27t)}},$$

where the constant $\frac{1}{6}$ is found by observing that $x = 2 + \delta$ implies $t = \frac{4}{729} \delta^3$.

The involution J, in this case, is somewhat special, because we have now

$$\psi_4 = \psi_1 \quad , \quad \xi_4 = \xi_1 \, .$$

In the corresponding system S the points A_1 and A_4 coincide, and the right line ψ_1 passes through A_1. Hence of the four reducible integrals of deficiency $p = 2$ in the general case connected with the involution J, three degenerate here at once into ordinary logarithmic integrals. The integral

$$\int \frac{T dx}{\sqrt{\psi_1 \psi_2 \psi_3 \psi_4}},$$

in the general case of deficiency $p = 3$, reduces here to an elliptic integral of the third kind, but the transformation indicated above effects still a further reduction, and we obtain another logarithmic integral.

In fact, we shall have

$$\int \frac{(25x^2-16x+4)dx}{(5x^2-12x+4)\sqrt{(7x^2-6x+2)(5x^2-2x+3)}} = \tfrac{1}{6}\int \frac{dt}{t\sqrt{1-t}}.$$

As I remarked in the beginning, the principal condition for the reducibility has been given by Bolza and by Igel. I will now show that the invariant relations they deduced, may be derived without difficulty from the results obtained here.

Bolza and Igel both introduce the anharmonic ratios $\lambda_{23}, \lambda_{31}, \lambda_{12}$, formed by the roots of each pair of the quadrics ψ_1, ψ_2, ψ_3.

The anharmonic ratio λ_{23}, formed by the roots of ψ_2 and ψ_3, is given by the equation

$$\frac{(\lambda_{23}-1)^2}{A_{22}A_{33}} = \frac{(\lambda_{23}+1)^2}{A_{23}^2},$$

and putting

$$\mu_1 = \frac{\sqrt{\lambda_{23}}+1}{\sqrt{\lambda_{23}}-1}, \quad \sqrt{\lambda_{23}} = \frac{\mu_1+1}{\mu_1-1},$$

the constant μ_1 is related to the invariants A_{22}, A_{33}, A_{23} by the equation

$$\frac{2\mu_1}{\sqrt{A_{22}A_{33}}} = \frac{\mu_1^2+1}{A_{23}}.$$

Now we have

$$\frac{A_{22}}{ca-g^2} = \frac{A_{33}}{ab-h^2} = \frac{A_{23}}{gh-af}$$

and hence by equations (13)

$$\frac{A_{22}}{2P_3^2\beta(s-\beta)} = \frac{A_{33}}{2P_2^2\gamma(s-\gamma)} = \frac{A_{23}}{-P_2P_3(2\beta\gamma+\alpha s)},$$

where s stands for $\alpha+\beta+\gamma$.

In this way we get

$$\frac{\mu_1}{\sqrt{\beta\gamma(s-\beta)(s-\gamma)}} = \frac{\mu_1^2+1}{-(2\beta\gamma+\alpha s)} = \frac{\mu_1^2-1}{\alpha s} = \frac{\mu_1^2}{-\beta\gamma}$$

and we may take

$$\mu_1 = -\sqrt{\frac{\beta\gamma}{(\alpha+\beta)(\gamma+\alpha)}},$$

Similarly we obtain

$$\mu_2 = \frac{\sqrt{\lambda_{31}}+1}{\sqrt{\lambda_{31}}-1} = -\sqrt{\frac{\gamma\alpha}{(\beta+\gamma)(\alpha+\beta)}},$$

$$\mu_3 = \frac{\sqrt{\lambda_{12}}+1}{\sqrt{\lambda_{12}}-1} = -\sqrt{\frac{\alpha\beta}{(\gamma+\alpha)(\beta+\gamma)}},$$

and we infer

$$\frac{\mu_2\mu_3}{\mu_1} = -\frac{\alpha}{\beta+\gamma}.$$

Hence we have between $\mu_1\, \mu_2\, \mu_3$ the set of relations

$$\alpha\mu_1 + \beta\mu_2\mu_3 + \gamma\mu_2\mu_3 = 0\,,$$
$$\alpha\mu_3\mu_1 + \beta\mu_2 + \gamma\mu_3\mu_1 = 0\,,$$
$$\alpha\mu_1\mu_2 + \beta\mu_1\mu_2 + \gamma\mu_3 = 0\,,$$

and by eliminating $\alpha, \beta, \gamma,$ we obtain as the invariant relation between the quadrics ψ_1, ψ_2, ψ_3 BOLZA's equation

$$\begin{vmatrix} 1 & \mu_3 & \mu_2 \\ \mu_3 & 1 & \mu_1 \\ \mu_2 & \mu_1 & 1 \end{vmatrix} = 1 - \mu_1^2 - \mu_2^2 - \mu_3^2 + 2\,\mu_1\mu_2\mu_3 = 0\,.$$

When this relation is satisfied for any one of the possible determinations of the constants μ_1, μ_2, μ_3, the quadris $\psi_1, \psi_2\, \psi_3$ are apt to build up a degenerate integral.

As we have

$$\sqrt{\lambda_{23}} = \frac{\mu_1+1}{\mu_1-1} = \frac{(\mu_1+1)^2}{(\mu_1^2-1)},$$

we have also

$$\sqrt[4]{\lambda_{23}\lambda_{31}} = \frac{(\mu_1+1)(\mu_2+1)}{\sqrt{(\mu_1^2-1)(\mu_2^2-1)}}.$$

Now it follows from BOLZA's equation that

$$(\mu_1^2-1)\,(\mu_2^2-1) = (\mu_1\mu_2-\mu_3)^2$$

and since

$$\mu_3 = -\frac{\alpha+\beta}{\gamma}\mu_1\mu_2 \quad , \quad \mu_1\mu_2 - \mu_3 = \frac{s}{\gamma}\mu_1\mu_2\,,$$

we get

$$\sqrt[4]{\lambda_{23}\lambda_{31}} = \frac{\gamma(1+\mu_1+\mu_2+\mu_1\mu_2)}{s\mu_1\mu_2} = \frac{\gamma}{s} + \frac{1}{s\mu_1\mu_2\mu_3}\{\gamma\mu_2\mu_3+\gamma\mu_3\mu_1-(\alpha+\beta)\mu_1\mu_2\}.$$

Writing out similar expressions for $\sqrt[4]{\lambda_{12}\lambda_{23}}$ and $\sqrt[4]{\lambda_{31}\lambda_{23}}$, we find by adding them IGEL's equation

$$\sqrt[4]{\lambda_{12}\lambda_{31}} + \sqrt[4]{\lambda_{23}\lambda_{12}} + \sqrt[4]{\lambda_{31}\lambda_{23}} = 1\,.$$

Again, if for any one of the possible determinations of the surds this relation is satisfied, a degenerate integral can be constructed by means of the quadrics $\psi_1, \psi_2\, \psi_3$. Both the equations of condition given by BOLZA and by IGEL involve rather intricate surds, and I should say that they are less adapted for examining the reducibility of a given integral than the equation (12) deduced in this paper.
