Mathematics. — "*Surfaces that may be represented in a plane by a linear congruence of rays*". By Prof. JAN DE VRIES.

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1. In order to obtain a representation (1,1) of a cubic surface in a plane we may make use of the bilinear congruence which has two straight lines lying on that surface as directrices. To this purpose the linear congruence (1,3) may also be used, which is formed by the bisecants of a twisted cubic lying on the surface [1]). It would also be possible to make use of the congruence (1,3) of the rays that intersect a twisted cubic and one of its bisecants, provided that those two directrices are lying on the surface.

We shall now consider, more in general, the surfaces that can be represented by means of a *linear congruence of rays* $(1, n)$.

The representation of a surface Φ^{n+p+1} with an n-fold and a p-fold straight line by means of a *bilinear congruence* has been amply treated by GUCCIA and MINEO [2]).

2. Let now be given a surface Φ^{2n+1} with an n-fold twisted cubic α^3. Through a point P of Φ passes a bisecant of α, which intersects the plane of representation τ in P'; in general a point P has *one* image P', and, inversely, a point P' of τ is the image of *one* point P.

Let us now consider the ruled surface $\mathfrak{G}$ formed by the bisecants t projecting the points of a plane section γ^{2n+1}. In the plane of that curve lie three straight lines t, $\mathfrak{G}$ is consequently of order $(2n+4)$.

That cone, which projects α^3 out of one of its points, determines on γ^{2n+1} evidently $(n+2)$ points P, consequently α is an $(n+2)$-fold curve on $\mathfrak{G}^{2n+4}$.

The image of γ^{2n+1} *is therefore a curve of order* $(2n+4)$ *with three* $(n+2)$*-fold points* A; this curve will be indicated by the symbol $c^{2n+4}(3A^{n+2})$.

The images of two plane sections have the $(2n+1)$ points P' in common, which are the images of the points P lying on the intersection of the two planes.

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1) See e.g. R STURM, *Geometrische Verwandtschaften*, IV, 288

2) GUCCIA, *Sur une classe de surfaces, représentables point par point sur un plan* (Ass. francaise pour l'avancement des sciences, 1880). MINEO, *Sopra una classe di superficie unicursali* (Le matematiche pure ed applicate, volume I, p. 220).

Besides the $3(n+2)^2$ intersections lying in the principal points A they have consequently moreover $m=(n^2+2n+3)$ points G in common; they are evidently the intersections of as many bisecants g of a^3 lying on Φ^{2n+1}.

On the surface lie therefore at least (n^2+2n+3) *straight lines.*

From this it ensues in particular that any twisted cubic lying on a cubic surface has six straight lines of Φ^3 as bisecants.[1])

The complex of plane intersections of Φ^{2n+1} is consequently represented by a complex (∞^3) of curves c^{2n+4}, which has three $(n+2)$-fold and (n^2+2n+3) simple base-points.

3. The curve a^3 is represented by the ruled surface $\mathfrak{A}$ of the bisecants t, which touch at Φ^{2n+1} in one of its two points of support. The straight lines g are double generatrices of $\mathfrak{A}$; for they may be considered to touch in two points of a^3.

Let now x be the order of $\mathfrak{A}$, y the multiplicity of a^3 on that ruled surface. The intersection on τ is then a curve $a^x(3A^y, mG^2)$. As a^3 has evidently $3n$ points in common with a plane section γ^{2n+1}, the consideration of their images produces the relation

$$(2n+4)x = 3(n+2)y + 2(n^2+2n+3) + 3n.$$

As two bisecants of a^3 can only intersect on that curve $\mathfrak{A}$ has $2y$ points in common with an arbitrary bisecant; so we have $x=2y$.

We now find $y=2n+3$, $x=4n+6$.

The image of the curve a^3 is therefore a curve $a^{4n+6}(3A^{2n+3}, mG^2)$.

4. Each of the n planes that touch Φ^{2n+1} in a point R of a^3, contains a generator t of the ruled surface $\mathfrak{A}$, which moreover intersects a^3 in a point S. The remaining $(n+3)$ straight lines t meeting in R touch in Φ^{2n+1} in another point of a^3. The pairs of points R, S belong to a correspondence with characteristic numbers $(n+3)$ and n. The points S belonging to the same point R form pairs of an involutory correspondence with characteristic number $(n+3)(n-1)$; the coincidences originate from points R, where two of the tangent planes coincide. On a^3 lie therefore $2(n+3)(n-1)$ *cuspidal points*

To each point R correspond n points of the image a, so the points of a are arranged in an involution I_n.

5 Let in the plane τ a curve f be given of order p, which passes a_k times through the principal point A_k, and g_k times through the principal point G_k. With the image $c^{2n+4}(A_k^{n+2}, G_k)$ it has apart from the principal points a number of points in common, indicated by

$$p^{+} = (n+2)(2p - \Sigma a_k) - \Sigma g_k.$$

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[1]) For $n=2$ we find the surface Φ^5 with nodal curve a^3 amply discussed by R. STURM, (*Geom. Verw.* IV, 311).

This number is evidently the order of the twisted curve Φ, which has f as image.

As f has apart from the principal points, a number of points in common with $a^{4n+6}(A_k^{2n+3}, G_k^2)$, represented by

$$n^* = (2n + 3)(2p - \Sigma a_k) - 2\Sigma g_k,$$

the curve Φ rests in n^* points on the curve α^3. A straight line l is therefore the image of a λ^{2n+4}, which intersects α^3 in $(4n+6)$ points.

6. For the simplification of the representation we submit the figures in τ to a quadratic transformation, which has A_k as principal points. By this the curve c^{2n+4} is transformed into a curve c^{n+2}, which does not pass through A_k, but does pass through the (n^2+2n+3) points $\mathfrak{G}$, in which the principal points G are transformed.

To the curves γ^{2n+1}, in which the surface Φ^{2n+1} is intersected by the planes γ of a pencil, correspond now the curves of a pencil (c^{n+2}). Among them there are $3(n+1)^2$, which possess a nodal point, which, is then at the same time the case with the corresponding curves γ^{2n+1}.

The surface Φ^{2n+1} *is consequently of class* $3(n+1)^2$.

The straight line $\mathfrak{G}_1\mathfrak{G}_2$ is transformed by the quadratic transformation into the conic $f^2(A_k\, G_1 G_2)$ and the latter is the image of a twisted curve Φ^n, which rests on α^3 in $(2n-1)$ points For through f^2 and α^3 passes a hyperboloid, which has the curve f^2 and n times the curve α^3 in common with Φ^{2n+1}; the residual section is the Φ^n in question.

7. *We shall now consider a surface* Φ^{n+p+1} *that passes* n *times through a twisted curve* α^q *of order* q, *and* p *times through a straight line* β, *which is* $(q-1)$ *times intersected by* α^q.

The straight lines t, which intersect α and β, form a linear congruence $(1,q)$, by which Φ is represented in a plane τ, for t intersects Φ, except on α and β, only in *one* point P more.

The ruled surface $\mathfrak{C}$, which represents the plane section γ^{n+p+1} is of order $(n+p+q+1)$; for in the plane γ lie q generators.

Out of a point of β the curve α^q is projected by a cone of order q with $(q+1)$ fold edge β, which intersects γ^{n+p+1} in $(p+q)$ points P. Consequently $\mathfrak{C}$ passes $(p+q)$ times through β.

Out of a point of α the line β is projected by a plane that determines $(n+1)$ points P on γ. Consequently α^q is an $(n+1)$ fold directrix.

The image of γ^{n+p+1} *is consequently a curve* $c^{n+p+q+1}(qA^{n+1}, B^{p+q})$.

Two curves c have apart from the points A and B and the images of the $(n+p+1)$ points in the section of their planes a number of points G in common represented by

$$m=(n+p+q+1)^2-(n+1)^2q-(p+q)^2-(n+p+1)=$$
$$=n(n+1)+(2n+1)p-(n^2-1)q.$$

From this it ensues that the surface Φ^{n+p+1} contains at least m straight lines.

If we take here $q=1$, it appears *that a surface Φ^{n+p+1} with an n-fold and a p-fold straight line contains $2np+n+p+1$ straight lines resting on the multiple straight lines* [1]). For $n=1$, the number m appears to be independent of q; we find that a surface Φ^{p+2} with p-fold straight line contains $m=3p+2$ straight lines g, resting on α and β. A plane passing through β and one of those straight lines contains one more straight line h, wich does not rest on α; on Φ^{p+2} lie therefore at least $6p+4$ straight lines.

8. Let us now determine the image of the straight line β. A plane γ passing through β contains one more curve γ^{n+1}, which has on α but not on β, an n-fold point Q. As it intersects β in $(n+1)$ points, the plane contains $(n+1)$ straight lines t, which touch the surface in points of β, consequently are generators of the ruled surface $\mathfrak{B}$ by which β is represented; α is therefore an $(n+1)$ fold directrix. As Q comes to lie on β for $(q-1)$ different positions of the plane γ, t will as many times coincide with β, consequently β is a $(q-1)$ fold torsal line. But each of the p planes that touch at Φ in a point of β, contains *one* straight line t; consequently the multiplicity of β on $\mathfrak{B}$ is equal to $(p+q-1)$ and the order of $\mathfrak{B}$ equal to $(n+p+q)$.

The image of the straight line β is therefore a curve b^{n+p+q} $(qA^{n+1}$ B^{n+q-1}, $mG)$.

This result may be controlled by determining the number of points that b has in common with a curve c apart from the principal points A, B, G; we promptly find then the number p, being the number of common points of β^p and γ^{n+p+1}.

9. Let us now determine the image of α^q. Each of the n planes that touch at Φ in a point Q of α, contains *one* straight t of the ruled surface $\mathfrak{A}$ that represents α, consequently α is an n-fold directrix.

If β is a y-fold directrix a plane passing through β contains a section of order $(n+y)$, and the image of α has as symbol a^{n+y} $(qA^n$, B^y, $mG)$.

By combining with $c^{n+p+q+1}$ $(qA^{n+1}$, B^{p+q}, $mG)$ we find for the determination of y, the relation $(n+p+q+1)(n+y)=n(n+1)q+$ $+(p+q)y+m+nq$, in which it has been taken into account that a plane section has nq points in common with α^q.

1) See MINEO ibid. page 221 or J. DE VRIES, *Surfaces algébriques renfermant un nombre fini de droites* (Archives Teyler, serie II, tome VIII, p. 262).

From this relation it ensues that $y = p + q$.

Consequently the image of a^q *is a curve* $a^{n+p+q}(qA', B^{p+q}, mG)$.

The combination with the image of β produces a control; from which it appears that the curves a and b have promptly $n(q-1)$ points in common, apart from the principal points.

10. If we write in the results arrived at, $n = 1$, $q = 3$, $p = 1$, we obtain the representation of the cubic surface to which we referred in § 1. The directrices of the linear congruence (1,3) are then a twisted curve α^3 of Φ^3 and one of the bisecants of α^3 lying on Φ^3.

The image of a plane γ^3 is then a $c^6(3A^2, B^4)$. If the six bisecants mentioned above are indicated by b_k, and if b_1 is the directrix of the (1,3), the five straight lines c_{1k} are represented by points.

The image of α^3 is a curve a^3 $(3A, B^4, 5C)$ the image of b_1 a curve $b^5(3A^2, B^3, 5C)$; these curves have, as they ought to have, two more points in common, which are the images of the points of support of the bisecant b_1.

It is easy to determine from these data the images of the remaining 21 straight lines of Φ^3.