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Physics. — "*A paradox in the theory of the Brownian movement*". By Prof. P. Ehrenfest. (Communicated by Prof. H. A. Lorentz).

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§ 1. Suppose a small sphere to be suspended in a fluid. Suppose further that its Brownian movement is observed and consider a moment when the sphere has a rather great velocity, e.g. upward. *May we expect the surrounding fluid to move with the sphere?*

Prof. J. D. van der Waals Jr. and Miss A. Snethlage have recently shown [1]: 1st how the answer to this question is connected with Einstein's theory of the Brownian movement,

2nd that the statistical theory of the molecular movement demands *that such a common motion does not exist.* This theory implies namely that for a given place and velocity of the suspended sphere and for a given configuration of the surrounding particles of the fluid, *equal and opposite velocities of these particles* are always equally probable.

The authors cited remarked already that this result is somewhat paradoxical; and therefore subjected it to a detailed discussion.

In the following we shall make the paradox still more acute by considering an analogous question for an extremely simplified model. This will show us that two closely connected material points m_1 and m_2 of which our model will consist, may on one hand possess mutually independent velocities, while on the other hand they still accompany each other (because of the close connexion).

§ 2. We consider two material particles with the masses m_1 and m_2 and the following properties:

1st Both are constrained kinematically to move along the axis of X,

2nd By a field of force their distance can never become greater than D[2]), where D may be small compared with the displacements of the two points in the course of time along the X-axis:

$$|x_2 - x_1| \leq D \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

3rd Let this pair of points be placed in an infinite extension

[1]) These Proc. p. 1322.

[2]) Let e. g. m_1 be a shell in which m_2 remains enclosed.

filled with a gas which is in molecular-statistical equilibrium. Its molecules collide both with m_1 and m_2.

Considering a corresponding canonical ensemble, the expression

$$const.\ e^{-\frac{2\Phi(x_1,x_2)+m_1u_1^2+m_2u_2^2}{2kT}}\ dx_1 dx_2 du_1 du_2 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

(where x_1, x_2, u_1 and u_2 are the coordinates and velocities of the two particles, and Φ (x_1, x_2) is the potential energy of the force holding them together) gives the number of individuals of the ensemble for which x_1, x_2, u_1 and u_2 have their values between specified infinitely close limits. For *given* values of x_1, x_2 and especially also of u_1 (2) gives the same number of individuals in the ensemble for equal and opposite values of u_2 i. e.: *equal and opposite values of u_2 are still equally probable,* u_2 is *"independent"* of u_1. On the other hand the Brownian movement will in the course of time cause great displacements along the X-axis. At the same time the points will remain close together in virtue of the inequality (1). This is the paradox mentioned at the end of § 1.

§ 3. Let us first leave aside the molecular-statistic side of the problem and put the following *purely kinematical* question. During a long time Θ the two points m_1 and m_2, may be conducted along the x-axis in an arbitrary way, only restricted to the conditions that

a. the inequality (1) shall remain valid;

b. the distance between the final and original positions of the pair of points may be great compared with D. This implies that m_2 „accompanies" m_1. Now we ask: does this imply that always the mean with respect to the time

$$\overline{u_1 u_2} = \frac{1}{\Theta}\int_0^{\Theta} dt\, u_1\, u_2 > 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

will be positive, or is it possible that the integral can be zero or eventually even negative?

The sign of the integral (3) indicates in a natural way in how far the two points move more in the same or in opposite directions. But we can point out that for the above described motion of the pair of points m_1, m_2 the inequality (3) need not be fulfilled. This will become evident by an example of a case for which the integral becomes negative. In fig. 1 the two zigzag lines represent a possible x, t-diagram of the two points m_1, m_2. We see that the conditions

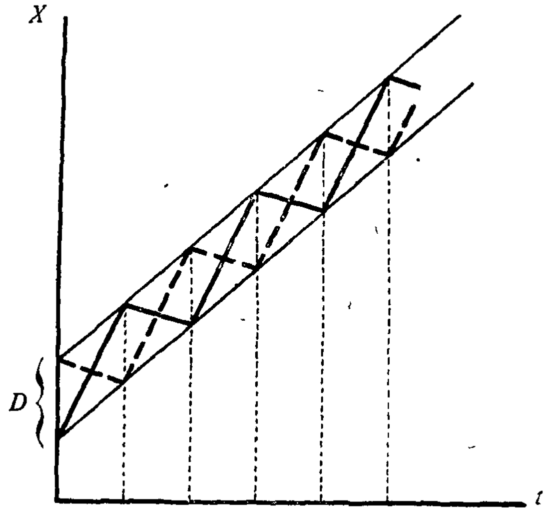

Fig. 1.

a and *b* are fulfilled and that still u_1 and u_2 have continually opposite signs, so that the integral (3) becomes *negative.*

§ 4. Now we have:

$$4\,\overline{u_1 u_2} = \overline{(u_1 + u_2)^2} - \overline{(u_1 - u_2)^2}. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The sign of $\overline{u_1\, u_2}$ is determined by which term is the greater of the two.

When the motions of the pair of points obey the equipartition theorem, $\overline{u_1\, u_2}$ is just equal to zero. (See the appendix).

From the above it is evident that a motion of the pair of points is possible, in which they remain close together and at the same time travel through great distances, while still at every moment the velocity u_2 is "independent" of u_1. The paradox mentioned in §§ 1 and 2 proves thus to be apparent only. *Therefore there is no objection against* EINSTEIN's *assumption that a suspended sphere during its* BROWNIAN *movement imparts its motion to the surrounding fluid in the same way as in the case of a systematic motion under the influence of a constant force.*

§ 5. In the *positive* proof however that EINSTEIN's assumption follows from the fundaments of statistical mechanics we meet with the following difficulty: Let us demand, (to stick to our example), that the inequality $u_{10} < u_1 < u_{10} + \varepsilon$ exists

1st at the instant t_0;

2nd also during the interval from $t_0 - r$ till t_0; and let us ask what can be said of the occurrence of different values of u_2. In the first case we have to take from the canonical ensemble an easily defined sub-ensemble M_1, in which u_2 has equally often equal and opposite values (and therefore is "independent" with respect to u_1). For the second demand a more closely limited sub-ensemble M_{12} has to be selected from the mentioned ensemble M_1. It is however hardly possible to determine M_{12}. Still this should be done in order to decide whether the distribution of the values of u_2 in it does or does not agree with EINSTEIN's assumption.

APPENDIX.

Let

$$\frac{m_1 x_1 + m_2 x_2}{M} = q_1 \quad , \quad \frac{m_1 x_1 - m_2 x_2}{M} = q_2 \quad . \quad m_1 + m_2 = M,$$

$$\frac{dx_1}{dt} = u_1 \quad , \quad \frac{dx_2}{dt} = u_2,$$

$$u_1 = \frac{M}{2m_1}(\dot{q}_1 + \dot{q}_2) \quad , \quad u_2 = \frac{M}{2m_2}(\dot{q}_1 - \dot{q}_2) \quad . \quad . \quad . \quad . \quad (\alpha)$$

Then:

$$\tfrac{1}{2}(m_1 u_1^2 + m_2 u_2^2) = \frac{M^2}{8m_1 m_2}\{\dot{q}_1^2 M + \dot{q}_2^2 M + 2\,\dot{q}_1 \dot{q}_2 (m_2 - m_1)\}.$$

Let p_1, p_2 be the momentum corresponding to the coordinates q_1, q_2, then we have

$$\dot{q}_1 p_1 = \frac{M^2}{4m_1 m_2}\{M\dot{q}_1^2 + (m_2 - m_1)\,\dot{q}_1 \dot{q}_2\}, \quad . \quad . \quad . \quad . \quad (\beta)$$

$$\dot{q}_2 p_2 = \frac{M^2}{4m_1 m_2}\{M\dot{q}_2^2 + (m_2 - m_1\,\dot{q}_1 \dot{q}_2\}, \quad . \quad . \quad . \quad . \quad (\gamma)$$

and because of the equipartition theorem the mean values with respect to time of (β) and (γ) are both equal to kT, so that their difference

$$\frac{M^3}{4m_1 m_2}\overline{\{\dot{q}_1^2 \quad \dot{q}_2^2\}} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\delta)$$

On the other hand (α) gives:

$$\overline{u_1 u_2} = \frac{M^2}{4m_1 m_2}\overline{(\dot{q}_1^2 - \dot{q}_2^2)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\varepsilon)$$

From (δ) and (ε) combined we find: $\overline{u_1 u_2} = 0$ (q.e.d.).