Mathematics. — *"On Elementary Surfaces of the third order".* (Third communication). By Dr. B. P. HAALMEYER. (Communicated by Prof. L. E. J. BROUWER).

(Communicated in the meeting of September 29, 1917).

It has been proved that F^3 cannot exist if that surface does not contain at least one straight line. It will now be shown that if F^3 contains a line [1]), this surface still cannot exist if in no plane through that line the section consists of three lines.

We start from a line a on F^3 and assume that in no plane through a the restcurve consists of two lines. It will be shown that this assumption leads to contradictory results.

Theorem 1: Every point of line a has a tangent plane.

Let A be an arbitrary point of a and β a plane through A not containing a. A cannot be isolated in β because there are points of F^3 on both sides of β inside any vicinity of A. Hence in β a curve passes through A. On this curve we choose a sequence of points $A_1, A_2 \ldots$ converging towards A from only one side. Let $\alpha_1, \alpha_2 \ldots$ be the planes passing through a and through $A_1, A_2 \ldots$ respectively and let α be their limiting plane (obviously α is the plane through a and the tangent at A to the curve in β). In every one of the planes $\alpha_1, \alpha_2 \ldots$ is situated a curve of the second order, passing respectively through $A_1, A_2 \ldots$

Three possibilities are to be considered:

1. The curves of the second order contract towards a or part of a.

2. The curve in the limiting plane α consists of a and an oval which *intersects* a at A.

3. The curve in the limiting plane α consists of a and an oval which has a *for tangent* at A.

1. The curves of the second order contract towards a or part of a. This part of a anyway contains the point A. Each curve of the second order divides the corresponding plane α_n in two regions [2]).

[1]) Again *line* will be used for *straight line*.

[2]) A_n cannot be isolated in α_n because the curve in β *intersects* the plane α_n. Neither can the restcurve in α_n consist of a line counting double, as we assumed that no second line of F^3 intersects line a.

We call *internal* region that one which contracts towards a or part of a only. Now if A continued to belong to the *external* regions, the curve in plane β would show a cusp in A with both branches arriving from the same side of the tangent. and this is excluded. The possibility might be put forward that for every n the oval in α_n has the line of intersection b_n with β for tangent in A_n, in other words that the two points of F^3, which b_n carries besides A, coincide.

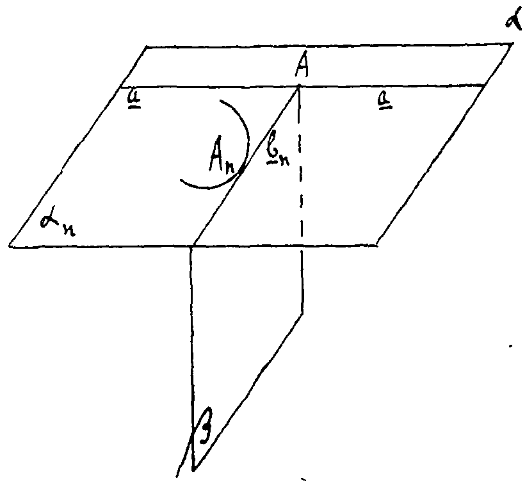

Fig. 11.

As follows can be shown that this possibility is excluded. In the second communication (*Second part, theorem* 1) we proved:

If a line a in a plane α intersects the curve in that plane at an ordinary point A, then lines which converge towards a end up by carrying points of F^3 converging towards A. The demonstration we used there, also holds if A lies on one or more lines of F^3, provided A is not situated on such a line in plane α. We proceed to apply this theorem to the case of fig. 11. In plane β the line b_n intersects the curve (which is no straight line) at the ordinary point A_n. In plane α_n however it would be possible to find a sequence of lines converging towards b_n but carrying no points of F^3 which converge towards A_n: a contradiction.

Hence A will end up by belonging to the internal regions of the ovals [1]) and considering this region together with its boundary contracts towards a or part of a it follows that *every plane through A not containing line a has a point of inflexion at A with tangent in α.*

[1]) We exclude the possibility that A continues to lie on the ovals themselves. The cases in which A belongs to an oval in a plane through a will be dealt with sub 2 and 3.

Sections in planes through a will be dealt with later on.

2. The restcurve in α is an oval which intersects a at A. In α four branches depart from A: AB and AC on a and AE and AD on the oval. Regarding the connection of these branches the JORDAN theorem for space leaves only two possibilities.

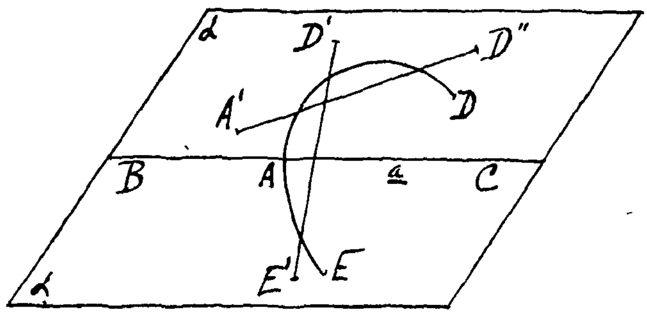

Fig 12.

First possibility: AC and AD are connected by I, AD and AB by II, AB and AE by III and lastly AE and AC by IV. If I and IV were situated on the same side of α then a parallel linesegment converging from that side towards $E'\,D'$ would end up by having two points in common with I and two with IV: a contradiction.

If I and II were situated on the same side of α, then a parallel linesegment, converging from that side towards $A'\,D''$ would end up by having two points in common with I and also two with II: a contradiction.

In the same way it can be shown that III and IV cannot lie on the same side of α. Combining these results it appears that the connecting sets of points are situated alternately above and below α.

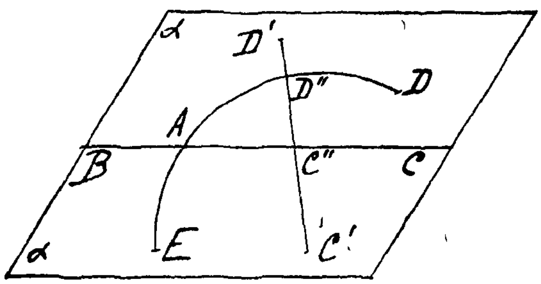

Fig 13.

Second possibility. The following is a representative case: I connects AB and AC above α and III connects AE with AD below α. AC is connected with AD above or below α by II and lastly AB with AE above or below α by IV. Let parallel linesegments converge towards $D'C'$ from that side on which II is situated. This line ends up by carrying two points of II. Besides it has a point in common with either I or III converging towards C'' or D'' and lastly it carries a point of F^3 converging towards the second point of intersection of $D'C'$ and the oval in α. Altogether four points.

It thus appears that the second possibility is excluded and we need only consider the first.

In § 3 of the first communication we proved: *If A is double point in a plane α, and cusp in not more than one plane, then α is tangent plane,* assuming that no line of F^3 passes through A. Here however one of the branches passing through A, is a straight line. This is the only one, as we assumed that no second line of F^3 intersects the line a on which A is situated. Hence in no plane through A except those passing through a, can the curve contain a line through A and the demonstration of § 3 still holds. The results obtained for planes through the tangents at A in α also remain valid for the planes through the tangent at A to the oval in α. Regarding the curves in planes through the *line* in α however (which line corresponds to the second tangent of the former case) the demonstration says nothing. These shall be dealt with later on. Also the first part of § 3 where the connection of the branches is examined, has to be slightly altered, but this has been done already above.

In order to be able to use the former results here, it remains to prove the following theorem:

If a line of F^3 passes through A, which line is not intersected by a second one, then A cannot be cusp in more than one plane (we give a fresh demonstration as the former one must be altered a good deal).

A is situated on the line a of F^3 and is cusp in a plane β which of course does not contain a. Let α be an arbitrary plane through a not containing the cuspidal tangent in β and b the line of intersection of α and β. Line b carries except A only one point B of F^3. In plane α the point B cannot be isolated, as the curve in β *crosses* α. Neither can the restcurve in α (that is: the curve minus a), according to our assumption, consist of two lines and the only remaining possibility is that the restcurve is an oval through B. This oval also passes through A, because b has only the points A and B in common with F^3 (that the oval cannot have b for tangent in B follows from the same reasoning which shows that fig. 11 represents an impossibility.) [1])

[1]) Here we are not entitled to use the theorem given at the end of the first communication, because this was only proved for points not situated on a line of F^3, and it is not excluded that B lies on such a line.

Hence in every plane through a, not containing the cuspidal tangent in β, the restcurve is an oval through A. Passing on the limiting case, it appears that in the plane through a and the cuspidal tangent in β the curve consists of a only, or a together with an oval through A.

Furthermore it appears that an arbitrary line through A ($\neq a$) carries at the utmost one point of F^3 different from A. But in no plane can A be isolated (because a furnishes points of F^3 inside any vicinity of A, on both sides of every plane not containing a), hence in any plane which does not contain a the point A is either cusp or double point. Concerning the planes through a it appeared that A is double point in every one of these with the possible exception of the one through a and the cuspidal tangent in β, in which case the curve in that plane consists of a only.

So far we only assumed A to be cusp in a single plane β. Now let A be cusp in two planes β and γ. We shall consider separately the cases that only one or more than one line can be cuspidal tangent at A.

First case. A is situated on the line a of F^3. Let b denote the only line through A which can be cuspidal tangent and let α be the plane through a and b. The foregoing results show that there are only two possibilities:

I. The curve in α consists of a and an oval through A.

II. The curve in α consists of a only.

I. Let c be a line through A in α, not being tangent to the oval and not coinciding with a or b. Only line b can be cuspidal tangent at A hence in every plane through c ($\neq \alpha$) A is double point, but A is double point in α also, hence A would be double point in *every* plane through c, and c cannot be tangent in any of these planes because c carries besides A a second point of F^3. This however cannot be, as may be shown in the same way as in § 3 of the first communication. The fact that here A lies on a line of F^3 makes no difference as the demonstration merely depended on the connection of the branches dictated by the assumption that F^3 is a twodimensional continuum.

II. Again let c be a line through A in α not coinciding with a or b. In every plane through c ($\neq \alpha$) A is ordinary double point and in α the curve consists of a only.

Let δ be an arbitrary plane through c ($\neq \alpha$). In this plane δ the line c is tangent at the double point A, hence in δ on both sides of c at least one branch joins A with the line at infinity (on one side there even can be three, when the loop reaches the line at infinity, but in any case there is at least one on either side). Now let δ revolve round c. The curve in a limiting plane is the limiting set of the curves in the converging planes (no isolated points are possible here). Besides a sequence of infinite branches has an infinite limiting branch. Hence it follows that in every plane through c ($\neq \alpha$) we can choose on both sides of c an infinite branch such that they merge in each other in continuous fashion when δ revolves round c. If we add the line a in α these branches are just sufficient to give F^3 the character of a twodimensional continuum in the neighbourhood of A and the branches departing from A, which we have left out, cannot be fitted in anymore. This contradicts our assumption that F^3 is a twodimensional continuum (of course the neighbourhood of a point on a twodimensial continuum can in an infinite number of ways be represented on the neighbourhood of a point in a plane, but the neighbourhood of a point in a plane can by (1,1) continuous transformation in the plane never be transformed in anything but the neighbourhood of a point).

Second case. A is situated on the line a of F^3 and is cusp in β and γ. The cuspidal tangents do not coincide, hence the line of intersection b of β and γ cannot be cuspidal tangent in either of these planes. It follows that b carries besides A a second point B of F^3 and the curve in the plane α through a and b consists of a and an oval through A and B. The line b divides β in two semiplanes: in the one the cuspidal branches depart from A, hence in the other A is isolated. In the same way A is isolated in one of the semiplanes in which b divides γ.

Now a foregoing demonstration (§ 5, second communication) shows that in this case A is isolated inside the entire angle ($< 180°$) between these semiplanes. Hence the line a belonging to F^3 cannot pass through this angle and it follows that the semiplanes of β and γ in which the cuspidal branches depart from A, are situated on the same side of the plane α through a and b, let us say below α. In α four branches arrive at A, consecutively AP, AQ, AR and AS (two on a and two on the oval). Suppose above α, AP is connected with AQ and AR with AS. Then line b must lie inside the angles QAR and PAS, because planes pass through b in which A is isolated *above* α. Let c be a line in α through A inside the angles PAQ and SAR (this is impossible when the oval in α has a for tangent at A, which case we shall consider separately). The foregoing results show that A is double point or cusp in every plane through c. In any such plane however two branches arrive at A from above α, one on the set which joins AP to AQ and the other on the set by which AR is connected with AS. Now if A were cusp in a plane through c then, considering the branches arrive from above α, at the utmost one plane could pass through b in which A is isolated above α and this contradicts the above results. Hence A must be double point in *every* plane through c: a contradiction.

It now remains to consider the case that the oval in α has a for tangent at A. We shall consider separately the following possibilities:

I. There exists a semiplane through b above α in which A is not isolated.

II. No such semiplane exists.

I. A is not isolated above α in a plane δ through b. Then in the semiplane of δ above α two branches depart from A, because A is cusp or double point in every plane not containing a and line b has a second point B in common with F^3. From the way in which the branches meeting at A in α are connected, follows that in every plane through b two branches arrive from below α, hence A is ordinary double point in δ. Here, there is no danger of a line as we assumed that no second line of F^3 intersects a. If δ revolves round b then in one of the two directions A will remain ordinary double point till δ coincides with α.

From this it follows that the semiplanes of β and γ in which the cuspidal branches depart from A, are situated on the same side of δ, let us say below δ. In δ we now choose a line d through A separated from b by the tangents at A. The same reasoning used before shows that A would have to be double point in every plane through d. Only for the plane through a and d a slight alteration is required, which however is selfevident. The impossibility of assumption I has thus been proved.

II. A is isolated in every semiplane through b above α. In every plane through A not containing a the point A is double point or cusp, hence in every plane through b ($\neq \alpha$) A is cusp and all the branches arrive at A from below α. It follows that A must be cusp in *every* plane except α, all branches arriving at A from below α. This however is only possible, if the cuspidal tangents form one plane ε through a, which plane has nothing but line a in common with F^3. Let a sequence of planes $\varepsilon_1, \varepsilon_2, \ldots\ldots$ all passing through a converge towards ε. In each of these an oval passes through A. Now suppose the oval in ε_n *crosses* the line a at A. Then four branches arrive at A in ε_n, forming finite angles. These branches are connected alternately on different sides of ε_n. Through A in ε_n we can at once find a line through which pass two planes having a cusp in A and such that in both the cuspidal branches arrive from the same side of ε_n. Then in the same way as before we can once more obtain a contradiction.

It now remains to consider the possibility that for every n the oval in ε_n has a for *tangent* at A. For increasing n these ovals contract either towards A only or towards a connected part of a containing A.

If A is the only limiting point, then the contracting ovals would give to A the character of a point of a twodimensional continuum and a sequence of points on a having A for limiting point, could not be fitted in anymore.

If on the other hand the limiting set is an interval on line a then the internal points of this interval would, in planes not containing a, be cusps with both branches arriving from the same side of the tangent and this is also excluded.

We thus have proved that *every plane through A, containing neither line a, nor the tangent at A to the oval in α, has an ordinary point in A with tangent in α. The planes through the tangent at A to the oval in α have point of inflexion in A with tangent in α.*

There remain to be considered the curves in planes through a. These shall be dealt with presently.

3. We now come to the third possibility mentioned on page 736. The restcurve in α consists of an oval having a for tangent in A. In α there depart from A two branches AB and AC on a and AE and AD on the oval. In almost the same way as before it appears that here AC is connected with AD, AD with AE, AE with AB and lastly AB with AC. The connecting sets of points are again situated alternately above and below α. This being established the further reasoning used for case 2 holds here without any alteration (again we remind the reader of the assumption that no second line of F^3 intersects a). Results: *In every plane which does not contain line a, the point A is ordinary point with tangent in α* (in all these planes the branches depart from A to the same side of α).

The curves in planes through a must be considered still.

In each of the three above cases, a was found to possess the character of tangent plane, only we had no certainty with regard to the curves in planes through a. Now all possibilities have been considered it appears that for no point A two different planes can pass through a both possessing one of the examined characters (we obtain an immediate contradiction by considering a plane through A not containing a). It follows that in the three above cases no plane through a ($=\!|\!=\alpha$) can contain branches departing from A (except a itself). This completes the demonstration that α is tangent plane.

Theorem 2: *If A moves continuously along a, then the tangent plane also changes in continuous fashion.*

Let the points $A_1, A_2 \ldots .$ on a converge towards A. Tangent planes $\alpha_1, \alpha_2 \ldots . \alpha$ all passing through a. We assume that $\alpha_1, \alpha_2 \ldots .$ have a limiting plane α' different from α and shall prove that this leads to a contradiction. Let $\beta_1, \beta_2 \ldots . \beta$ be planes respectively passing through $A_1, A_2 \ldots . A$ and all $\perp a$. The line of intersection of α_1, and β_1 we denote by b_1, the one of α_2 and β_2 by b_2 etc. Lastly let b be the line of intersection of α and β and b' the one of α' and β. According to our assumptions b' and b do not coincide and b' is the limiting line of $b_1, b_2 \ldots .$

Now b is tangent at A to the curve in β and in the converging planes $\beta_1, \beta_2 \ldots .$ the curves have for tangents at $A_1, A_2 \ldots .$ respectively the lines $b_1, b_2 \ldots .$ converging towards b' in β.

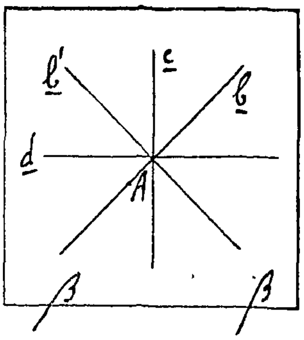

Fig. 14

According to *theorem* 2 of the second communication the curve in β is the limiting set of the curves in $\beta_1, \beta_2 \ldots .$ (with the possible exception of an isolated point).

Let c and d be lines through A in β separating b from b'. The corresponding planes through a shall be denoted by γ and δ.

For n large enough a branch departs in β_n from A_n in both directions inside those opposite angles between γ and δ in which b' is situated. Loops contracting towards A are evidently excluded, hence in order that in β no branch departs from A inside those angles of c and d which contain b', it is unavoidable that in the converging planes the above mentioned branches leave these angles via points of the planes γ and δ (or one of these) converging towards A. Hence in at least one of the planes γ and δ the point A would be limiting point of points of F^3 not situated on a. This means that in one of those planes a branch departs from A different from a, but this is a contradiction considering that only in the tangent plane α a second branch can pass through A. This completes the demonstration of *theorem* 2.

Let an oval in α *cross* the line a at A. This oval and the line a have a second point of intersection B and the points A and B have the tangent plane α in common. If A moves continuously along a then, according to *theorem* 2, the tangent plane α also changes in continuous fashion and the point B also moves continuously.[1]) From this follows that a point A at which an oval crosses a, can only be limiting point of points of a possessing the same character. Besides it is easy to prove that the tangent to the oval at A also changes continuously. This result however will not be needed, but we do want the following:

Let $A_1, A_2 \ldots$ on a converge towards A. Tangent planes $\alpha_1, \alpha_2 \ldots \alpha$. If the oval in α crosses a at A it follows from the above that for n larger than some finite value the plane α_n also shows an oval which crosses a at A_n.

Now suppose all these ovals in α_n turn at A_n their concave sides to the left. The oval in α is the limiting set of the ovals in α_n and considering a sequence of finite concave branches cannot converge towards a finite convex branch, it follows that the branch in α through A also turns its concave side to the left.

Taking these results together we obtain:

Theorem 3: *A point of line a in the tangent plane of which an oval crosses a, can only be limiting point on a of points having the same kind of tangent plane also with regard to the side to which the ovals through those points are concave or convex.*

Theorem 4: F^3 *cannot exist if the restcurve does not degenerate in any plane through a.*

We consider the case in which the curves of the second order in the planes $\alpha_1, \alpha_2 \ldots$ (passing through a and converging towards α), contract towards part of a. We call *internal* region of these ovals *that* region which contracts to nothing but a. We found that the points of a belonging to this limiting part must be situated in the internal region of the oval in α_n for n larger than some finite number. From this follows that the part of a belonging to the

[1]) This theorem and some others which shall be formulated presently concerning the directions in which A and B move, have already been given by JUEL, Math. Ann. 76, p. 552. The existence and continuous changing of tangent planes is simply postulated by that author.

internal region of the oval in α_n must diminish for increasin
because if the oval in α_n crosses a at A_n and B_n then α_n is tan
plane at A_n and in case A_n ended up by being situated inside
ovals, α also would be tangent plane at A_n: a contradiction.

Hence if the ovals have the *entire line* a for limiting set, r
can have points in common with a. An idea of this case may
got by imagining a sequence of hyperbolas of which the angl
the asymptotes (inside which the hyperbola is situated) tends tow
180° and such that the centre is situated on a and both asympt
converge towards a.

In this case everything is in favour of counting a as a *triple*
in α. In no plane through a a branch would depart from any p
of a, and except a, F^3 would contain no straight tine.

A second possibility we wish to consider separately is that in
tangent plane of *every* point of a the oval has a for tangent. A
let A be a point of a with tangent plane α. The line a divides α
two semiplanes, in the one A is isolated and in the other an oval
a for tangent at A.

Now let A move along a. The plane α turns round a. If A m
on in the same direction the plane α goes on turning in the s
direction, for otherwise two points of a might be found with
same tangent plane and this cannot be as in either point an
in the tangent plane must have a for tangent.

It A goes round the entire line a the tangent plane meanw
turns 180° round a. The ovals in the tangent planes merge co
nuously into each other, hence after turning 180° the branch hav
a for tangent is situated in the wrong semiplane. This means
on the way the branch must change from the one semiplane
the other and this is only possible either via a tangent plane
which the restcurve consists of two lines through A, or via a tang
plane in which the oval has contracted to nothing but point A.
first possibility is excluded according to our assumption and the
would mean that a sequence of ovals in the converging pla
contract towards a point of a not belonging to the internal regi
of the converging ovals. This was found to be impossible hence
assumption that every point of a has a tangent plane with
having a for tangent, leads to contradictory results.

Leaving apart both cases treated above, there certainly exis
plane through a in which an oval has two different points A
B in common with a. Let this plane α revolve continuously i

certain direction round α The points A and B then also move along a continuously [1]). Two assumptions are possible: A and B can move in the same or in opposite directions. Let the direction be the same. In the time that B has described the original segment BA, the point A has gone further, hence all this time we keep tangent planes with ovals having two different points in common with a.

When B arrives at the original place of A the plane α must have turned an angle of 180°, but if the branch through B has originally been concave to the left, it must now be concave to the righthand side, and this is not possible as on the way the concave side in B cannot jump round and no change from concave to convex can have taken place via a degeneration of the oval in two straight lines (according to our assumptions).

The second possibility was that A and B move in opposite directions. Let the tangent plane successively turn round a in opposite directions, then we obtain two different points in which A and B meet. Such a meeting takes place either when the two points of intersection of a and the oval converge to one point or when the entire oval contracts to nothing but a single point on a. In both cases the concave sides of the branches through A and B face each other. A priori it seems possible that before the meeting the convex side of the branches through A and B face each other, but then these branches would be connected on both sides via the line at infinity and in the limiting plane the oval would degenerate in two straight lines [2]) through the point where A and B meet and this contradicts our assumptions.

Now we start from the original position of A and B and we observe A only. Let the branch through A turn its concave side to the left. If we turn the tangent plane in such a way that A moves to the right, then the concave side goes on being turned to the left. But before the meeting with B takes place the concave side must be turned to the right (that is in the direction in which A moves) and this means a contradiction because the curvature cannot change its sign discontinuously, neither can it change via a degeneration of the oval in two straight lines (according to our assumption). This completes our demonstration.

Remark. Above we spoke about the possibility that the oval

[1]) If α goes on turning in the same direction A and B obviously cannot change the direction in which they move for then points of a would exist with two different tangent planes.

[2]) The oval does not converge towards a.

through A and B contracts to the point where A and B meet. The most rational thing to do is to consider this meeting point as a special kind of oval in the tangent plane. We can also imagine that the oval through A and B contracts to a *segment* of a. All points of this segment would have the same tangent plane (tangent plane of the first kind, examined at the beginning). Now the admission of this possibility has the disadvantage that we should be more or less forced to consider the linesegment in the tangent plane as a special sort of oval and going back to the definition of elementary curves we should not only have to admit isolated points, but linesegments also. This would cause the development of the theory to become a good deal more complicated but the enlargement of the results would probably remain very trivial. To mention an example; to the elementary surfaces of the second order would be added the plane convex regions including the boundary and the linesegment.

Far greater would the changes become if we also dropped the condition that the convex arc is not to contain linesegments. This however would mean an entirely different problem.