Physics. — *"On the mass of a material system according to the gravitation theory of* EINSTEIN." By Dr. G. NORDSTRÖM. (Communicated by Prof. H. A. LORENTZ).

(Communicated in the meeting of December 29, 1917).

§ 1. In this paper some formulae will be deduced for the mass of a material system according to EINSTEIN's gravitation theory. The principal purpose of these formulae is to express the mass firstly by a volume integral over the material system and secondly by a surface integral over a surface surrounding the system.

First I shall indicate in this paragraph the general formulae which will be used further on. The following calculations are principally based upon EINSTEIN's paper: "HAMILTONsches Prinzip und allgemeine Relativitätstheorie" [1]) (further cited as: EINSTEIN, HAMILTONsches Prinzip). His article: "Die Grundlage der allgemeinen Relativitätstheorie" [2]) (further denoted by EINSTEIN, Grundlage) will also be referred to.

In the first paper EINSTEIN points out that the formulae in his gravitation theory can be deduced from a variation principle of this form:

$$\delta \iiiint (\mathfrak{G}^* + \varkappa \mathfrak{M})\, dx_1\, dx_2\, dx_3\, dx_4 = 0, \quad . \quad . \quad . \quad . \quad (1)$$

where the first part $\mathfrak{G}^*$ of the integrand refers to the gravitation field and the second part $\varkappa \mathfrak{M}$ to the matter (inclusively the electromagnetic field), $\varkappa$ is the gravitation constant, which in EINSTEIN's paper has been put equal to 1. $\mathfrak{G}^*$ is a function of $g^{\mu\nu}$ and

$$g_\alpha^{\mu\nu} = \frac{\partial g^{\mu\nu}}{\partial x_\alpha}.$$

$\mathfrak{M}$ is a function of $g^{\mu\nu}$ and of several parameters which determine the state of the matter.

The components $\mathfrak{T}_\sigma^\nu$ of the stress-energy-tensor for the matter are represented by the following expression (formula (19) EINSTEIN, HAMILTONsches Prinzip):

[1]) A. EINSTEIN, Berl. Ber. 1916, p. 1111.
[2]) A. EINSTEIN, Ann. d. Phys. **49**, p. 769, 1916.

$$\mathfrak{T}_\sigma^\nu = - \sum_\mu \frac{\partial \mathfrak{M}}{\partial g^{\mu\sigma}} g^{\mu\nu} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

According to its behaviour with respect to transformations $\mathfrak{T}$ is a mixed volume tensor, $\mathfrak{M}$ a volume scalar, $\mathfrak{G}^*$ is no volume scalar, but this quantity is formed from the volume scalar $-\sqrt{-g}\, G$ (where G is the *total curvature* of the four-dimensional continuum) by elimination of the second derivative $\frac{\partial^2 g^{\mu\nu}}{\partial x_\alpha \partial x_\beta}$ by partial integration. We have[1])

$$- \mathfrak{G}^* = \sqrt{-g}\, G - \sum_\tau \frac{\partial \mathfrak{A}_\tau}{\partial x_\tau}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where $\mathfrak{A}$ is a four-fold vector in the sense given to it in the special theory of relativity of EINSTEIN-MINKOWSKY. $\mathfrak{A}$ is thus covariant for LORENTZ transformations. The sign of $\mathfrak{G}^*$ and for $\mathfrak{M}$ has been chosen in such a way that the expression (2) gives the density of energy of the matter with the right sign. For $-\mathfrak{G}^*$ and $\mathfrak{A}_\tau$ we have the expressions:

$$\left. \begin{aligned} - \mathfrak{G}^* = \sqrt{-g} \sum_{\substack{\alpha\beta\mu \\ \nu\sigma\tau}} g^{\mu\nu} g^{\sigma\alpha} g^{\tau\beta} \left(\begin{bmatrix} \mu\tau \\ \alpha \end{bmatrix} \begin{bmatrix} \nu\sigma \\ \beta \end{bmatrix} - \begin{bmatrix} \mu\nu \\ \alpha \end{bmatrix} \begin{bmatrix} \sigma\tau \\ \beta \end{bmatrix} \right) - \\ - \sum_{\mu\nu\sigma\tau} g^{\sigma\nu} \begin{bmatrix} \mu\nu \\ \sigma \end{bmatrix} \frac{\partial}{\partial x_\tau} (\sqrt{-g}\, g^{\mu\tau}) - \sum_{\mu\nu\sigma\tau} g^{\sigma\tau} \begin{bmatrix} \mu\nu \\ \sigma \end{bmatrix} \frac{\partial}{\partial x_\tau} (\sqrt{-g}\, g^{\mu\nu}), \end{aligned} \right\} . \quad (4)$$

$$\mathfrak{A}_\tau = \sqrt{-g} \sum_{\mu\nu\sigma} (g^{\mu\tau} g^{\nu\sigma} - g^{\sigma\tau} g^{\mu\nu}) \begin{bmatrix} \mu\nu \\ \sigma \end{bmatrix}, \quad . \quad . \quad . \quad . \quad (5)$$

where the CHRISTOFFEL symbols

$$\begin{bmatrix} \alpha\beta \\ \gamma \end{bmatrix} = \frac{1}{2} \left(\frac{\partial g_{\alpha\gamma}}{\partial x_\beta} + \frac{\partial g_{\beta\gamma}}{\partial x_\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x_\gamma} \right) . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

are used. According to the equations[2])

$$\frac{\partial \sqrt{-g}}{\partial x_\alpha} = - \frac{1}{2} \sqrt{-g} \sum_{\mu\nu} g_{\mu\nu} \frac{\partial g^{\mu\nu}}{\partial x_\alpha}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$\frac{\partial g_{\mu\nu}}{\partial x_\alpha} = - \sum_{\sigma\tau} g_{\mu\sigma} g_{\nu\tau} \frac{\partial g^{\sigma\tau}}{\partial x_\alpha}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

[1]) Because of equation (3) and as at the limits of the domain of integration all variations are taken equal to zero, the variation principle (1) is equivalent with the variation principle expressed by the following equation

$$\delta \iiiint (-\sqrt{-g} \cdot G + \varkappa \mathfrak{M})\, dx_1\, dx_2\, dx_3\, dx_4 = 0\, , \quad . \quad . \quad (1a)$$

from which equation EINSTEIN originally started.

[2]) EINSTEIN, Grundlage, equations (29) and (32).

we find that $\mathfrak{G}^*$ is a homogeneous quadratic function of the quantities $g_\sigma^{\mu\nu}$, so that we have

$$\sum_{\mu\nu\alpha} \frac{\partial \mathfrak{G}^*}{\partial g^{\mu\nu}_{\alpha}} = 2\, \mathfrak{G}^* \quad \ldots \ldots \ldots \quad (9)$$

The stress-energy-components of the gravitation field t_σ^ν introduced by EINSTEIN are connected with $\mathfrak{G}^*$ by the formula [1])

$$\varkappa\, t_\sigma^\nu = \frac{1}{2}\left(\mathfrak{G}^*\, \delta_\sigma^\nu - \sum_{\mu\tau} \frac{\partial \mathfrak{G}^*}{\partial g_\nu{}^{\mu\tau}}\, g_\sigma{}^{\mu\tau}\right), \quad \ldots \ldots \quad (10)$$

where $\delta_\nu^\nu = 1$, $\delta_\sigma^\nu = 0$ for $\sigma \neq \nu$. For the diagonal summation we find because of (9)

$$\varkappa \sum_\nu t_\nu^\nu = \mathfrak{G}^*. \quad \ldots \ldots \ldots \quad (11)$$

For the diagonal summation of the material stress-energy tensor we have again [2])

$$\varkappa \sum_\nu \mathfrak{T}_\nu^\nu = \sqrt{-g}\, G. \quad \ldots \ldots \ldots \quad (12)$$

By summation we find taking (3) into account

$$\varkappa \sum_\nu (\mathfrak{T}_\nu^\nu + t_\nu^\nu) = \sum_\tau \frac{\partial \mathfrak{A}_\tau}{\partial x_\tau} \quad \ldots \ldots \ldots \quad (13)$$

An equation of quite the same form is obtained from the following formula of EINSTEIN [3])

$$\varkappa\, (\mathfrak{T}_\sigma^\nu + t_\sigma^\nu) = -\sum_{\mu\tau} \frac{\partial}{\partial x_\tau}\left(\frac{\partial \mathfrak{G}^*}{\partial g_\tau{}^{\mu\sigma}}\, g^{\mu\nu}\right) \quad \ldots \ldots \quad (14)$$

We thus find that the four-fold vector $\mathfrak{A}$ and the four-fold vector, the components of which are $-\sum_{\mu\nu} \frac{\partial \mathfrak{G}^*}{\partial g_\tau{}^{\mu\nu}}\, g^{\mu\nu}$ have the same divergency; the notations "four-fold vector" and "divergency" have here the meaning ascribed to them in the special theory of relativity. From a private correspondence with EINSTEIN I learned that he has proved that these two vectors are really identical, at least when the system of coordinates is thus chosen that $\sqrt{-g} = 1$.

Now all general formulae necessary for the following have been cited. We still remark that not yet anything has been said about the units in which the quantities are expressed. In order to obtain the stress- and energy-density in the desired units it may therefore be necessary to introduce in the expressions (2) and (10) a constant

---

[1]) EINSTEIN, Hamiltonsches Prinzip, equation (20).

[2]) See e.g. J. DROSTE. Het zwaartekrachtsveld van een of meer lichamen volgens de theorie van EINSTEIN. (Diss. Leiden 1916) p. 8 and 12.

[3]) EINSTEIN, Hamiltonsches Prinzip, equation (18).

factor depending on the system of units (comp. the next §, equation (15)).

§ 2. *Energy of a stationary system.*

We shall now consider a material system of finite dimensions and especially one for which there exists (at least) *one system of coordinates in which the gravitation field is stationary.* Let us first consider what must be understood by the mass of the system. The material system having finite dimensions it is evident that its gravitation field may be considered as being caused by a material point, the mass of which has a definite meaning, and all that holds with greater accuracy according as the distance to the system is greater. The best way of defining the mass of the system is based on the properties of the created gravitation field at points at a great distance. According to the theory of relativity however the mass of the system is equal to its total energy when at rest divided by the square of the universal constant c which represents the velocity of light in natural units. If according to our assumption we use a system of coordinates in which the gravitation field is stationary we find for the energy at rest the expression

$$\iiint (\mathfrak{T}_4^{\ 4} + \mathfrak{t}_4^{\ 4})\, dx_1\, dx_2\, dx_3$$

where the integration has to be extended over the whole three-dimensional space. Possibly a universal constant factor has to be added in order that we may obtain the energy expressed in the desired units (comp. § 1 end). It is easy to see whether this is necessary. First of all we choose the time-coordinate in such a way that *at an infinite distance* g_{44} *gets the value* c^2. Of course the value of the universal constant c depends again on the system of units, which can be chosen thus that $c = 1$. Further we remark, that together with a change of the unit of the time-coordinate the numerical values of $\sqrt{-g}$ and of all $\mathfrak{t}_\mu^{\ \nu}$'s changes proportionally to the numerical value of c. The energy having the dimensions ML^2T^{-2}, it is now evident that the factor c must be added to our integral expression in order that it may express the energy independently of the choice of the unit of time in the corresponding unit. We thus have for the energy at rest E:

$$E = c \iiint (\mathfrak{T}_4^{\ 4} + \mathfrak{t}_4^{\ 4})\, dx_1\, dx_2\, dx_3, \quad . \quad . \quad . \quad . \quad . \quad (15)$$

integrated over the whole three-dimensional space.

This expression gives the total energy at rest for a definite material system when this is the only one within the domain of integration.

Dividing by c^2 we then obtain the mass of the system and in § 4 we shall find that this mass is identical with the one we obtain by considering the gravitation field at points at a very great distance.

As has been said, the integral in (15) must be extended over the whole infinite space. It is however desirable to express the mass of a material system by an integral taken over the material system itself and we shall now show how this can be done. According to a law of v. LAUE we have for the energy E also this expression

$$E = c \iiint \sum_{\nu} (\mathfrak{T}_\nu{}^\nu + \mathfrak{t}_\nu{}^\nu)\, dx_1\, dx_2\, dx_3, \quad \ldots \ldots \quad (16)$$

integrated over the whole three-dimensional space. We subtract this equation from equation (15) after having multiplied the latter by 2. As in a stationary field $g_4{}^{\mu\tau} = 0$, we have because of (10) and (11)

$$\varkappa\, \mathfrak{t}_4{}^4 = \tfrac{1}{2}\, \mathfrak{G}^* = \tfrac{1}{2}\, \varkappa \sum_{\nu} \mathfrak{t}_\nu{}^\nu, \quad \ldots \ldots \quad (17)$$

and we obtain

$$E = c \iiint (\mathfrak{T}_4{}^4 - \mathfrak{T}_1{}^1 - \mathfrak{T}_2{}^2 - \mathfrak{T}_3{}^3)\, dx_1\, dx_2\, dx_3 \quad \ldots \quad (18)$$

The integrand being zero at every point outside the material system, the integral here has only to be extended over the material system itself.

By means of formula (18) we have expressed the mass of a material system by a space-integral extended over the material system. This space-integral can again be transformed into a surface integral extended over a surface enclosing the material system. This may be made evident in the following way. From formula (14) we see that $\varkappa (\mathfrak{T}_4{}^4 + \mathfrak{t}_4{}^4)$ can be expressed as the divergency of a three-dimensional quasivector $\mathfrak{B}$:

$$\varkappa (\mathfrak{T}_4{}^4 + \mathfrak{t}_4{}^4) = \sum_{\tau} \frac{\partial \mathfrak{B}_\tau}{\partial x_\tau}, \quad \ldots \ldots \quad (19)$$

where

$$\mathfrak{B}_\tau = - \sum_{\mu} \frac{\partial \mathfrak{G}^*}{\partial g_\tau{}^{\mu 4}}\, g^{\mu 4}. \quad \ldots \ldots \quad (19a)$$

Multiplying (19) by 2 and subtracting (13) from this product while also (17) is taken into account, we obtain

$$\varkappa (\mathfrak{T}_4{}^4 - \mathfrak{T}_1{}^1 - \mathfrak{T}_2{}^2 - \mathfrak{T}_3{}^3) = \sum_{\tau} \frac{\partial}{\partial x_\tau} (2\, \mathfrak{B}_\tau - \mathfrak{A}_\tau) = \sum_{\tau} \frac{\partial \mathfrak{C}_\tau}{\partial x_\tau}. \quad (20)$$

According to this equation the application of GAUSS's law to (18) gives

$$E = \frac{c}{\varkappa} \int \mathfrak{C}_n\, df. \quad \ldots \ldots \quad (21)$$

integrated over a surface f, enclosing the material system. Therefore the mass of the system is also expressed by a surface integral over a surface enclosing the system. Unfortunately the quasivector $\mathfrak{C}$, the normal component $\mathfrak{C}_n$ of which occurs in the integral expression is not covariant, even not with respect to LORENTZ-transformations.

§ 3. *Application to a field with spherical symmetry.*

In our discussion on a system with spherical symmetry we shall principally introduce the same notation as J. DROSTE in his article: Het zwaartekrachtsveld van een of meer lichamen volgens de theorie van EINSTEIN (further cited as: DROSTE, Het zwaartekrachtsveld) chapter II § 1. In contradiction with DROSTE we shall however consider also the field within a material body. Introducing as space-coordinates the polar coordinates r, ϑ, φ we can at any rate represent the line-element ds by the same expression as DROSTE viz.:

$$ds^2 = w^2\, dt^2 - u^2\, dr^2 - v^2\, (d\vartheta^2 + \sin^2 \vartheta\, d\varphi^2), \quad . \quad . \quad . \quad (22)$$

where u, v, w are functions of r only. Here the time-coordinate $x_4 = t$ has thus been chosen that everywhere

$$g_{14} = g_{24} = g_{34} = 0,$$

which is always possible in a stationary field with spherical symmetry [1]).

Instead of the polar coordinates r, ϑ, φ we shall now introduce as space-coordinates the corresponding orthogonal coordinates

$$\left.\begin{aligned} x_1 &= r \cos \vartheta \cos \varphi, \\ x_2 &= r \cos \vartheta \sin \varphi, \\ x_3 &= r \sin \vartheta, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

while we keep the same time-coordinate as DROSTE. We put

$$v = rp. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

1) Because of the spherical symmetry $g_{\vartheta 4}$ and $g_{\varphi 4}$ must be zero. The system of coordinates may however be chosen in such a way that g_{r4} is not zero. We then have

$$ds^2 = w^2\, dt^2 + 2g_{r4}\, dt\, dr - u^2\, dr^2 - v^2\, (d\vartheta^2 + \sin^2 \vartheta\, d\varphi^2). \qquad (22a)$$

If however the time coordinate is transformed in the following way, while r is left unchanged:

$$dt = d\bar{t} + \psi(r)\, dr,$$

we obtain

$$ds^2 = w^2 d\bar{t}^2 + 2(g_{r4} + \psi w^2) d\bar{t}\, dr - (u^2 - \psi^2 w^2 - 2\psi g_{r4}) dr^2 - v^2 (d\vartheta^2 + \sin^2 \vartheta\, d\varphi^2).$$

If now the function $\psi(r)$ is defined thus that

$$g_{r4} + \psi w^2 = 0,$$

g_{r4} will be zero in the new system of coordinates.

for the components of the fundamental tensor we then have the following expressions

$$\left.\begin{aligned} g_{\mu\nu} &= -\frac{x_\mu x_\nu}{r^2}(u^2-p^2) \quad \text{for} \quad \mu \neq \nu, \\ g_{\mu\mu} &= -p^2 - \frac{x_\mu^2}{r^2}(u^2-p^2),\; g_{\mu 4} = 0,\; g_{44} = w^2, \end{aligned}\right\} \quad . \quad . \quad (25)$$

where $\mu, \nu = 1, 2, 3$.

For the components of the contravariant fundamental tensor we have:

$$\left.\begin{aligned} g^{\mu\nu} &= -\frac{x_\mu x_\nu}{r^2}\left(\frac{1}{u^2}-\frac{1}{p^2}\right) \quad \text{for} \quad \mu \neq \nu, \\ g^{\mu\mu} &= -\frac{1}{p^2} - \frac{x_\mu^2}{r^2}\left(\frac{1}{u^2}-\frac{1}{p^2}\right),\; g^{\mu 4} = 0 \; g^{44} = \frac{1}{w^2}. \end{aligned}\right\} \quad . \quad . \quad (26)$$

As we consider also the field inside the matter, the material stress-energy-tensor $\mathfrak{T}_\mu^\nu$ occurs now too in our formulae. Because of the spherical symmetry we can write for its components:

$$\left.\begin{aligned} \mathfrak{T}_\mu^\nu &= \frac{x_\mu x_\nu}{r^2}(\mathfrak{T}_r^r - \mathfrak{T}_p^p) \quad \text{for} \quad \mu \neq \nu, \\ \mathfrak{T}_\mu^\mu &= \mathfrak{T}_p^p + \frac{x_\mu^2}{r^2}(\mathfrak{T}_r^r - \mathfrak{T}_p^p),\; \mathfrak{T}_\mu^4 = \mathfrak{T}_4^\mu = 0, \\ \mu,\; \nu &= 1, 2, 3. \end{aligned}\right\} \quad . \quad . \quad (27)$$

That here $\mathfrak{T}_\mu^4 = \mathfrak{T}_4^\mu = 0$ rests on our assumption that the energy of the system remains constant. No radial energy-current can exist then.

Now we shall deduce formulae for the gravitation field from the variation principle of the form ($1a$). We chose this form of the variation principle with a view to a better correspondence with the article of J. DROSTE.

By a right choice of the limits of integration the equation ($1a$) becomes:

$$4\pi\delta\int_{t_1}^{t_2} dt \int_{r_1}^{r_2} (-\sqrt{-g}\,G + \varkappa\,\mathfrak{M})\, r^2\, dr = 0$$

or by division by $4\pi(t_1 - t_2)$

$$\delta\int_{r_1}^{r_2} \sqrt{-g}\,G\, r^2\, dr = \varkappa\,\delta\int_{r_1}^{r_2} \mathfrak{M}\, r^2\, dr \quad . \quad . \quad . \quad . \quad . \quad (28)$$

The integral on the lefthand side, which multiplied by 4π evidently gives the space-integral $\int \sqrt{-g}\, G\, dV$ over an empty spherical space V, has been calculated by DROSTE. He used polar coordinates, but, the integral multiplied by $4\pi\,(t_2-t_1)$ giving a scalar, the result is independent of the choice of the space-coordinates. First DROSTE finds for G, which evidently is also a scalar, (see DROSTE, "Het zwaartekrachtsveld" p. 16)

$$G=\frac{2}{v^2}-\frac{2\,v'^2}{u^2v^2}-\frac{4\,v'w}{u^2vw}-\frac{4\,v''}{u^2\,v}+\frac{4\,u'v'}{u^3v}-\frac{2\,w''}{u^2w}+\frac{2\,u'w'}{u^3w}, \quad . \quad (29)$$

where $u'\,v'\,w'$ are derivatives with respect to r. Further DROSTE finds:

$$\int \sqrt{-g}\, G\, r^2\, dr = 2\int \left\{ -\frac{d}{dr}\left(\frac{v^2w'+2vwv'}{u}\right)+\frac{wv'^2+2\,vv'w'}{u}+uw \right\} dr. \quad . \quad (30)$$

All variations being taken zero at the limits $r=r_1$ and $r=r_2$, we have

$$\delta\int_{r_1}^{r_2} \sqrt{-g}\, G\, r^2\, dr = 2\,\delta\int_{r_1}^{r_2}\left\{\frac{wv'^2+2\,v\,v'w'}{u}+uw\right\} dr. \quad (31)$$

This is now our expression for the lefthand side of equation (28). Now we must consider the righthand side of this equation, and we shall begin by proving the following relations:

$$\left.\begin{aligned}
\sum_{\mu\nu}\frac{\partial \mathfrak{M}}{\partial g^{\mu\nu}}\frac{\partial g^{\mu\nu}}{\partial u}&=\frac{2}{u}\,\mathfrak{T}_r^{\,r},\\
\sum_{\mu\nu}\frac{\partial \mathfrak{M}}{\partial g^{\mu\nu}}\frac{\partial g^{\mu\nu}}{\partial v}&=\frac{4}{v}\,\mathfrak{T}_p^{\,p},\\
\sum_{\mu\nu}\frac{\partial \mathfrak{M}}{\partial g^{\mu\nu}}\frac{\partial g^{\mu\nu}}{\partial w}&=\frac{2}{w}\,\mathfrak{T}_4^{\,4},
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (32)$$

where $\mathfrak{T}_r^r$ and $\mathfrak{T}_p^p$ are connected with the tensor $\mathfrak{T}$ in the way indicated by the equations (27). In order to prove the validity of the equations (32), we first remark that because of the spherical symmetry both the lefthand and the righthand side depend on r only. If the equations hold for an arbitrary point on the X_1-axis ($x_1=r,\ x_2=x_3=0$), they are always valid.

According to (26) and (27) we have for points on the X_1-axis:

$$g^{11}=-\frac{1}{u^2},\quad g^{22}=g^{33}=-\frac{1}{p^2}=-\frac{r^2}{v^2},\quad g^{44}=\frac{1}{w^2}, \qquad (33)$$

$$\mathfrak{T}_1^{\,1}=\mathfrak{T}_r^{\,r},\quad \mathfrak{T}_2^{\,2}=\mathfrak{T}_3^{\,3}=\mathfrak{T}_p^{\,p} \quad . \quad . \quad . \quad . \quad (34)$$

All quantities $g^{\mu\nu}$ and $\mathfrak{T}_\mu^\nu$ for which $\mu \neq \nu$ are equal to zero. Consequently we have for points on the X_1-axis

$$\frac{\partial g^{11}}{\partial u} = \frac{2}{u^3}, \quad \frac{\partial g^{22}}{\partial v} = \frac{\partial g^{33}}{\partial v} = \frac{2r^2}{v^3}, \quad \frac{\partial g^{44}}{\partial w} = -\frac{2}{w^3}; \quad . \quad . \quad (35)$$

the other derivatives of $g^{\mu\nu}$ with respect to u, v, w are zero. According to the formulae (33) equation (2) gives

$$\mathfrak{T}_1{}^1 = -\frac{\partial \mathfrak{M}}{\partial g^{11}} g^{11} = \frac{\partial \mathfrak{M}}{\partial g^{11}} \frac{1}{u^2}, \quad \mathfrak{T}_2{}^2 = -\frac{\partial \mathfrak{M}}{\partial g^{22}} g^{22} = \frac{\partial \mathfrak{M}}{\partial g^{22}} \frac{r^2}{v^2} (= \mathfrak{T}_3{}^3),$$

further we have because of (35)

$$\sum_{\mu\nu} \frac{\partial \mathfrak{M}}{\partial g^{\mu\nu}} \frac{\partial g^{\mu\nu}}{\partial u} = \frac{\partial \mathfrak{M}}{\partial g^{11}} \frac{2}{u^3}, \quad \sum_{\mu\nu} \frac{\partial \mathfrak{M}}{\partial g^{\mu\nu}} \frac{\partial g^{\mu\nu}}{\partial v} = \left(\frac{\partial \mathfrak{M}}{\partial g^{22}} + \frac{\partial \mathfrak{M}}{\partial g^{33}}\right) \frac{2r^2}{v^3}.$$

All these equations hold for points on the X_1-axis and consequently the two first equations (32) are valid for these points. The general validity of the equations follows from the above. The proof for the third formula (32) is given in the same way; this latter proof directly holds for points not on the X_1-axis, as everywhere $g^{14} = g^{24} = g^{34} = 0$.

Because of the equations (32) the righthand side of (28) can be written in the form

$$\varkappa \delta \int_{r_1}^{r_2} \mathfrak{M} \, r^2 \, dr = 2\varkappa \int_{r_1}^{r_2} \left(\mathfrak{T}_1{}^1 \frac{\delta u}{u} + 2\,\mathfrak{T}_p{}^p \frac{\delta v}{v} + \mathfrak{T}_4{}^4 \frac{\delta w}{w}\right) r^2 \, dr. \quad . \quad (36)$$

Introducing the expressions (31) and (36) for both sides of equation (28) and dividing by 2 the variation principle for a field with spherical symmetry finally becomes

$$\delta \int_{r_1}^{r_2} \left\{ \frac{w v'^2 + v v' w'}{u} + u w \right\} dr = \varkappa \int_{r_1}^{r_2} \left(\mathfrak{T}_1{}^1 \frac{\delta u}{u} + 2\,\mathfrak{T}_p{}^p \frac{\delta v}{v} + \mathfrak{T}_4{}^4 \frac{\delta w}{w}\right) r^2 dr. \; (37)$$

As the variations $\delta u, \delta v, \delta w$ are independent of each other, and as u, v, w, v', w' are not varied at the limits r_1 and r_2, we find (comp. DROSTE, Het zwaartekrachtsveld, equations (24) which hold for the field outside the matter),

$$\left.\begin{aligned} -\frac{w v'^2 + 2v v' w'}{u^2} + w &= \frac{r^2}{u} \varkappa \mathfrak{T}_1{}^1, \\ -\frac{w v'' + v' w' + v w''}{u} + (v w' + w v') \frac{u'}{u^2} &= \frac{r^2}{v} \varkappa \mathfrak{T}_p^p \\ -\frac{2 v v'' + v'^2}{u} + u + 2 v v' \frac{u'}{u^2} &= \frac{r^2}{w} \varkappa \mathfrak{T}_4{}^4 \end{aligned}\right\} \quad . \quad . \quad (38)$$

These equations are the fundamental formulae for a gravitation field with spherical symmetry. We can easily deduce from it the following equation

$$r^2\left(\frac{u'}{u}\mathfrak{T}_r{}^r + 2\frac{v'}{v}\mathfrak{T}_p{}^p + \frac{w'}{w}\mathfrak{T}_4{}^4\right) = \frac{d}{dr}(r^2\mathfrak{T}_r{}^r), \quad . \quad . \quad . \quad (39)$$

which can also be found immediately by applying formula (22) of EINSTEIN (Hamiltonsches Prinzip) to our case. Formula (39) expresses that the spherically symmetrical material system is in equilibrium when the gravitation is taken into consideration [1]).

Starting from equation (18) we shall now deduce a formula for the energy and the mass of the system. We put

$$\Psi = \mathfrak{T}_4{}^4 - \mathfrak{T}_1{}^1 - \mathfrak{T}_2{}^2 - \mathfrak{T}_3{}^3 = \mathfrak{T}_4{}^4 - \mathfrak{T}_r{}^r - 2\mathfrak{T}_p{}^p \quad . \quad . \quad (40)$$

and calculate $r^2\varkappa\Psi$. Putting for $r^2\varkappa\mathfrak{T}_4{}^4$, $r^2\varkappa\mathfrak{T}_r{}^r$, $r^2\varkappa\mathfrak{T}_p^p$ the expressions following from (38), we find that most terms neutralize each other and we obtain

$$r^2\varkappa\Psi = 4\frac{vv'w'}{u} + 2\frac{v^2w''}{u} - \frac{2v^2w'u'}{u^2},$$

$$r^2\varkappa\Psi = 2\frac{d}{dr}\left(\frac{v^2w'}{u}\right) = 2\frac{d}{dr}\left(\frac{r^2p^2w'}{u}\right) \quad . \quad . \quad . \quad . \quad (41)$$

Outside the material system is $\Psi = 0$ and we thus have for $r > R$ (R being the radius of the body)

$$2r^2\frac{p^2w'}{u} = constant \qquad (r > R) \quad . \quad . \quad . \quad . \quad . \quad (42)$$

The meaning of the constant will be examined later on.

Equation (41) suggests a connexion with our former equation (20) and we shall directly see that this really exists.

Excluding the theoretically possible case that $\frac{p^2w'}{u}$ is ∞ at the centre of the system we find by integration of (41) from $r = 0$ to an arbitrary upper limit r

$$\int_0^r r^2\varkappa\Psi\,dr = 2r^2\frac{p^2w'}{u}.$$

1) If we put

$$u = w = 1 \quad , \quad v = r,$$

viz. if we neglect the gravitation (39) becomes

$$2r\mathfrak{T}_p{}^p = \frac{d}{dr}(r^2\mathfrak{T}_r{}^r),$$

which equation expresses the equilibrium between the ponderomotoric forces given by the stress-tensor $\mathfrak{T}$ for the case that there is no gravitation.

For the volume integral $\int \varkappa \Psi d V$ over a spherical space with radius r we obtain

$$4 \pi \varkappa \int_0^r r^2 \, \Psi \, dr = 4 \pi r^2 \frac{2 p^2 w'}{u}.$$

If we integrate over the same sphere and apply the law of Gauss, equation (20) gives again

$$4 \pi \varkappa \int_0^r r^2 \, \Psi \, dr = 4 \pi r^2 \mathfrak{E}_r,$$

where $\mathfrak{E}_r$ is the component of the quasi-vector $\mathfrak{E}$ directed radially outward. In consequence of the spherical symmetry there does not exist a component of $\mathfrak{E}$ perpendicular to the radius. Thus we have

$$\mathfrak{E}_r = \frac{2 p^2 w'}{u} \quad \text{. . . . .} \quad (43)$$

In our orthogonal system of coordinates we have as component in the direction of the X_τ-axis

$$\mathfrak{E}_\tau = \frac{x_\tau}{r} \frac{2 p^2 w'}{u} \qquad \tau = 1, 2, 3 \quad \text{. . . .} \quad (44)$$

Combined with our former formula (18) or with (21) our last formulae give also an expression for the total energy at rest and for the mass of the system. Taking r greater than the radius R of the material body we obtain

$$\varkappa E = 4 \pi \varkappa c \int_0^r r^2 \, \Psi \, dr = 4 \pi c r^2 \, \mathfrak{E}_r = 4 \pi c r^2 \frac{2 p^2 w'}{u},$$

$$E = \frac{4 \pi c r^2}{\varkappa} \frac{2 p^2 w'}{u} \qquad (r > R) \quad \text{. . . . .} \quad (45)$$

This formula expresses the mass of the body by means of the gravitation field outside the body. This shows at the same time the meaning of the constant on the right-hand side of equation (42).

In our considerations of this § we assumed the field to be thus, that there exists at least one system of coordinates in which the field is stationary and to have spherical symmetry; and our formulae hold for such a system of coordinates that has its origin in the centre of symmetry of the material system and that has such a time-coordinate that $g_{14} = g_{24} = g_{34} = 0$. If however there exists one system of coordinates of the above mentioned properties there exists an infinite quantity of such systems of coordinates, and our formulae hold for all these systems. Not alone the directions of the X_1-, X_2-, X_3-axes can be chosen in an infinite number of ways, but we are still free to chose the method of measurement for the length of the radius vector in space. Without destroying the validity of our formulae we may thus pass from a system of coordinates x_1, x_2, x_3, x_4 to an other one $x'_1\, x'_2\, x_3'\, x_4$ with the same time coordinate, but for which

$$\frac{x_1'}{x_1} = \frac{x_2'}{x_2} = \frac{x_3'}{x_3} = \frac{r'}{r},$$

where $r' = \sqrt{x'^2_1 + x'^2_2 + x'^2_3}$ is a function of r (comp. Droste, Het zwaartekrachtsveld p. 16). For such a transformation of coordinates u, p, w change of course. If therefore we have to calculate u, p, w (which according to (25) determine all $g_{\mu\nu}$'s) we must first fix the system of coordinates. This may e. g. be chosen in this way that everywhere $p = 1$ (corresponding to $v = r$ of Droste). If then still the unit of time is chosen so that the universal constant c has the value 1 the system of coordinates is determined except as to the directions of the three axes in space, which for spherical symmetry are of no importance. For the thus specially fixed system of coordinates we have outside the body (see Droste, Het zwaartekrachtsveld, p. 18)

$$w^2 = \frac{1}{u^2} = 1 - \frac{a}{r} \quad , \quad p = 1 \qquad (r > R). \quad . \quad . \quad (46)$$

where a is a constant.

That these formulae are right can easily be verified from the formulae (38); they are also found more directly from more general formulae which will be deduced in a following paper. The constant a must of course be connected with the mass of the body. Formula (45) gives for this relation, c being equal to 1,

$$m = E = \frac{4\pi\, a}{\varkappa} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46a)$$

In this special system of coordinates we have according to the formulae (25) outside the body [1])

$$\sqrt{-g} = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46b)$$

Inside the body however this value of $\sqrt{-g}$ need not hold. If

[1]) This is seen most clearly by considering a point on one of the axes of coordinates We then find first $\sqrt{-g} = u\, w\, p^2$.

the system of coordinates is fixed by the condition that everywhere $\sqrt{-g} = 1$, then we have $p \neq 1$.

§ 4. *Generalization of the obtained result.*

In the preceding § we have chosen the time-coordinate so that everywhere $g_{14} = g_{24} = g_{34} = 0$. Now we shall show how the formulae (41)—(45) can be generalized, so that they also hold when this condition is not fulfilled. Because of the spherical symmetry we can write

$$g_{\mu 4} = \frac{x_\mu}{r} g_{r4}, \qquad \mu = 1, 2, 3, \quad . \quad . \quad . \quad . \quad . \quad (47)$$

where g_{r4} has the same meaning as in formula (22*a*) of the note on p. 1081. To generalize one formulae to the case $g_{r4} \neq 0$ we must evidently transform the time-coordinate in the opposite way as in the note on p. 1081. The quantities referring to the original four-dimensional system of coordinates, in which $g_{r4} = 0$, will now be denoted by letters with a dash over them. The expression of the line-element in polar coordinates from which we start becomes then:

$$\overline{ds}^2 = \overline{w}^2 \, \overline{dt}^2 - \overline{u}^2 \, \overline{dr}^2 - \overline{p}^2 \, \overline{r}^2 \, (\overline{d\vartheta}^2 + sin^2 \, \overline{\vartheta} \, \overline{d\varphi}^2).$$

We transform the time-coordinate by putting

$$\overline{dt} = dt - \psi(r) \, dr,$$

while $\overline{r} = r$, $\overline{\vartheta} = \vartheta$, $\overline{\varphi} = \varphi$ are left unchanged.

ds^2 being an invariant, we obtain by substitution

$$ds^2 = \overline{w} \, dt^2 - 2\psi \, \overline{w}^2 \, dt \, dr - (\overline{u}^2 - \psi^2 \, \overline{w}^2) \, dr^2 - \overline{p}^2 \, r^2 \, (d\vartheta^2 + sin^2 \, \vartheta \, d\varphi^2).$$

The components of the fundamental tensor are then transformed according to the formulae

$$w^2 = \overline{w}^2, \quad g_{r4} = -\psi \, \overline{w}^2, \quad u^2 = \overline{u}^2 - \psi^2 \, \overline{w}^2, \quad p^2 = \overline{p}^2.$$

These formulae firstly give

$$\overline{u}^2 \, \overline{w}^2 \, \overline{p}^4 = (u^2 \, w^2 + g_{r4}^2) \, p^4.$$

This equation shows that the determinant g of the components $g_{\mu\nu}$ is not changed by our transformation of the time-coordinate. We have namely

$$\overline{u}^2 \, \overline{w}^2 \, \overline{p}^4 = -\overline{g} \quad , \quad (u^2 \, w^2 + g_{r4}^2) \, p^4 = -g, \quad . \quad . \quad . \quad (48)$$

where $\overline{g}$ and g denote the above mentioned determinant for the *orthogonal* system of coordinates (which through the formulae (23) is connected with the polar system of coordinates) before and after the transformation of the time-coordinate. That both members have the meaning we ascribed to them, is evident from the consideration of a point on one of the axes of coordinates.

We shall now transform formula (41). In the original four-dimensional system of coordinates this is after a slight variation

$$\overline{r}^2 \varkappa \overline{\Psi} = 2 \frac{d}{d\overline{r}} \frac{\overline{r}^2 \overline{p}^4 \overline{w} \overline{w}'}{\sqrt{-\overline{g}}}.$$

Now, we shall prove that the lefthand side remains covariant at the transformation of the time-coordinate. As also the righthand side remains invariant, the formula holds in this form also in the four-dimensional system of coordinates. According to (40) we have for every system of coordinates

$$\Psi = 2 \ {}_4{}^4 - \Sigma \mathfrak{T}_\mu{}^\mu.$$

$\mathfrak{T}$ being a mixed volume-tensor, $\mathfrak{T}_4{}^4$ is transformed according to this formula:

$$\mathfrak{T}_4{}^4 = \frac{\sqrt{-g}}{\sqrt{-\overline{g}}} \sum_{\alpha\beta} \frac{\partial x_4}{\partial \overline{x}_\beta} \frac{\partial \overline{x}_\alpha}{\partial x_4} \overline{\mathfrak{T}}_\alpha{}^\beta. \quad . \quad . \quad . \quad . \quad (49)$$

If we consider a point on the X_1-axis, then $dx_1 = dr$. At our transformation of the time-coordinate $\frac{\partial \overline{x}_4}{\partial x_4} = 1$ is the only one of all $\frac{\partial \overline{x}_\alpha}{\partial x_4}$ which is not zero. Of all $\frac{\partial x_4}{\partial \overline{x}_\beta}$ only $\frac{\partial x_4}{\partial \overline{x}_4} = 1$ and $\frac{\partial x_4}{\partial \overline{x}_1} = \psi$ are different from zero. As further $g = \overline{g}$ we find

$$\mathfrak{T}_4{}^4 = \overline{\mathfrak{T}}_4{}^4 + \psi \, \overline{\mathfrak{T}}_4{}^1.$$

It $\overline{\mathfrak{T}}_4{}^1$ was not zero this would mean that there existed a radial energy-current and the energy of the system would change continually. As we assumed the field to be stationary, we have $\overline{\mathfrak{T}}_4{}^1 = 0$ and therefore $\mathfrak{T}_4{}^4 = \overline{\mathfrak{T}}_4{}^4$. As $\sum_\mu \mathfrak{T}_\mu{}^\mu$ is a volume-scalar and as the determinant g does not change by the transformation, $\sum_\mu \mathfrak{T}_\mu{}^\mu$ does not undergo a change by the transformation either. Thus at the transformation Ψ remains invariant.

$$\overline{\Psi} = \Psi.$$

As r, p, w too remain constant, we thus obtain for the new four-dimensional system of coordinates also

$$r^2 \varkappa \Psi = \frac{d}{dr} \left(\frac{r^2 p^4}{\sqrt{-g}} \frac{dw^2}{dr} \right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (41a)$$

We here have found a generalization for formula (41) which also holds when $g_{r4} \neq 0$. It must still be remarked that g_{r4} occurs in the expression for $\sqrt{-g}$ (see the last formula (48)).

The more general formulae for (43), (45) can easily be obtained in the same way as above.

$$\mathfrak{E}_r = \frac{p^4}{\sqrt{-g}} \frac{dw^2}{dr}, \quad \dots \dots \dots \quad (43a)$$

$$E = \frac{4\pi c r^2}{\varkappa} \frac{p^4}{\sqrt{-g}} \frac{dw^2}{dr} \quad (r > R). \quad \dots \quad (45a)$$

In this § and in the preceding one we have confined our discussion to bodies with spherical symmetry. If we have a body of finite dimensions, which does not possess spherical symmetry, the corresponding gravitation field is different from that belonging to a body of the same mass but with spherical symmetry. We see however, that the greater the distance from the body in question becomes, the more the two fields must become equal. Therefore we can *define* the mass m of a finite material system of arbitrary form by the formula

$$m = \frac{4\pi}{c\varkappa} \lim_{r=\infty} \left(\frac{r^2 p^4}{\sqrt{-g}} \frac{dw^2}{dr} \right) = \frac{4\pi}{c\varkappa} \lim_{r=\infty} \left(\frac{r^2 g_{pp}{}^2}{\sqrt{-g}} \frac{dg_{44}}{dr} \right) \quad . \quad (50)$$

In the last expression we have introduced $-p^2 = g_{pp}$ analogous to the notation in formula (27). In order that formulae (50) may have a definite meaning, the limit on the right-hand side must of course have the same value for any direction in which we move towards the infinite. Formula (50) supposes therefore the system of coordinates to be chosen in such a way that at an infinite distance the field possesses spherical symmetry.

For the case we are considering formula (43a) gives

$$\lim_{r=\infty} r^2 \mathfrak{E}_r = \lim_{r=\infty} \left(\frac{r^2 p^4}{\sqrt{-g}} \frac{dw^2}{dr} \right), \quad \dots \dots \quad (51)$$

and as formula (21) in § 2 is also valid for a stationary field, which has no spherical symmetry, this equation gives together with (50)

$$E = c^2 m, \quad \dots \dots \dots \quad (52)$$

as is demanded by the theory of relativity. Thus we have shown that the calculation of the mass of a stationary system by means of formula (50) from the field at points at a great distance and the calculation of the mass by means of formula (52) from the total energy at rest give the same result also for bodies without spherical symmetry. From our considerations it also follows, that E/c^2 has the same value in every arbitrary system of coordinates in which the field is stationary and possesses spherical symmetry in the infinite. The mass m is thus a scalar.

In a following article the gravitation field for an electrically charged centre will be calculated by application of the result found in this paper for a field with spherical symmetry.

Further it will be proved, that the density of energy of the gravitation field t_4^4 outside the body is everywhere zero, when the system of coordinates is thus chosen that $\sqrt{-g} = 1$ (or constant).
