Mathematics. — *"Some Considerations on Complete Transmutation".* (Sixth Communication). By Dr. H B. A. BOCKWINKEL. (Communicated by Prof. L. E. J. BROUWER).

(Communicated in the meeting of March 31, 1917).

In the preceding communication we treated of the transmutation $T = T_2 T_1$, which is obtained when two complete transmutations T_1 and T_2 are applied to some regular function u. We saw there that the resulting transmutation is likewise *complete* in some pair or other of associated fields, the mutual dependence of the new N. F. O. and the new N. F. F. being to some extent established. We further gave a strong proof of the formula determining the resulting series P, which was furnished by BOURLET without domains of validity being mentioned by this author. As we have seen the formula expresses the so-called operative function of the resulting series P in those of the components P_1 and P_2 and differential coefficients of them. Again in giving some examples to illustrate our theorem of N°. 24, we observed that the method to find the resulting series by means of the just mentioned formula of BOURLET, is often much more difficult in practice than a somewhat more direct method, according to which first the functions $\xi_m(x) = T_2 T_1 (x^m)$ are determined, and then, by the symbolic formula (24)

$$a_m = (\xi - x)^m \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

the coefficients $a_m(x)$ of the resulting series P. BOURLET, however, has been able to apply his formula with success to questions of a more theoretical character.

The examples mentioned give rise to the question whether it is possible by means of the more direct method to find a general formula which expresses the coefficients a_m of the resulting series P in the coefficients λ_m and μ_m of the composing series. We thus arrived at a rather simple symbolic formula, which allowed us to shew again the completeness of P, the statement about corresponding domains being the same as in the foregoing communication. The investigations which led us to this result, gave us an opportunity to establish other more simple formulae, which served us to go on further, and which have moreover a certain interest in themselves. Again it seemed convenient to add some further formulae to those already obtained forming with the latter a more or less complete system. The development and the discussion of all these formulae is the subject of the following pages.

30. In all symbolic formulae to be treated of, the closed expressions occurring in the right-hand members of them must be developed according to ascending powers of one or more letters, these powers having no meaning in themselves, but obtaining one when the exponents (upper indices) are replaced by (lower) indices. Now, often certain reductions are allowed which would also be valid if the letters denoted variable *quantities,* whether or not being restricted to certain domains. Such reductions we shall call *analytical reductions.* The principal condition which should be noticed in order to be able to perform an analytical reduction with symbolic expressions is that *equal* symbols occurring in different parts of them, have the *same* meaning, this being the same fundamental condition if the letters denote numbers.

Generally speaking an analytic reduction is permitted if the *proper meaning of the result is the result of the proper meanings,* when by the latter the result is meant which would be obtained if the proper meanings were introduced *before* the reduction mentioned is performed. Thus we may have an *analytic sum* of symbols or a *product.* In the first case we shall often have to apply the rule that in a polynomial consisting of symbolic powers of the same letter, before substituting indices for exponents, terms involving *equal* powers a^k may be added analytically. For the proper meanings of such terms are quantities involving *equal* factors a_k, the coefficients in the symbolic terms being respectively equal to those in the proper ones. The sum of these coefficients multiplied by a^k is the analytic sum of the symbols and the same sum multiplied by a_k is the sum of the corresponding proper expressions. Thus the latter sum is indeed the proper meaning of the former.

If we have a *product* of symbolic powers of the same letter a, we should carefully state whether *the product of their proper meanings* is meant by it, or *the proper meaning of the analytic product,* that is of the single power which is obtained by multiplying the powers of a according to the ordinary rule giving as new exponent the sum of the partial exponents. For the proper meaning of the analytic product of a certain number of powers of the same letter a is not in general equal to the product of proper meanings of all factors.[1]) We shall always have to deal with such products of

[1]) See, however, the example in N⁰. 35.

powers of a letter a that have to be multiplied analytically before the proper meanings are substituted. In other words, the proper expressions will always be *linear* functions of quantities involving the same letter a and different indices k.

To begin with we observe that the functional theorem of MAC-LAURIN, treated in the 3rd communication leads to a generalization of the symbolic formula (23)

$$\xi_m \equiv T(x^m) = (x + a)^m, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

which expresses the transmuted ξ_m of the rational integral functions x^m in terms of the coefficients a_m of the series P answering to the normal transmutation T. Formula (23) is valid in any circular domain to which belong all functions a_m and ξ_m; the existence of such domains is one of the characteristics which make a transmutation normal, according to the definition we gave in N°. 15.

When the series P is complete in the domain (α) then, according to the just mentioned theorem

$$Tu = Pu \equiv \sum_0^\infty{}_m \frac{a_m u^{(m)}(x)}{m!},$$

for functions u which belong to the domain (β) corresponding to (α).

The right-hand member may apparently be denoted by the symbol $u(x + a)$, provided we interpret this in the following way: substitute for the symbol the power-series in the letter a which answers to the function $u(x + a)$ if that letter means a complex number. This power-series is unique, since x is a point in the domain (α) and u a function belonging to (β) and thus certainly to (α). [1]) We therefore obtain the symbolic formula

$$Tu(x) = u(x + a), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (67)$$

valid in (α) and of which (23) forms a particular case. [2])

[1]) Considerations of uniqueness were really of use already when in the 3rd communication we put for the formula

$$\xi_m = \sum_0^m{}_k\, m_k\, x^{m-k}\, a_k$$

the symbolic formula (23); in fact, if the expansion of $(x+a)^m$ in a power series according to a were not unique, special reference should be made to the fact that the series in the right-hand member is meant and no other. But no one thinks of uniqueness in the development of a binomial, nor did we in writing our 3rd communication. Nevertheless, in the light of the present general developments, in which the uniqueness of a power-series forms the principal part, it seemed convenient to us to mention this point.

[2]) We have to take care that in the first term of the expansion the factor a^0

For the m^{th} derivative of Tu we may as well give a symbolic formula. We saw in the 4th communication that this quantity, defined by PINCHERLE by means of (45), may in the domain (α) also be found by formula (39):

$$T^{(m)}(u) = \sum_{0}^{\infty} {}_k \frac{a_{m+k}\, u^{(k)}(x)}{k!} \quad . \quad . \quad . \quad . \quad . \quad (39)$$

of course for functions u belonging to (β). Instead of this formula we may write symbolically

$$T^{(m)}(u) = a^m\, u\,(x+a) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (68)$$

which formula has (67) as a particular case ($m = 0$). This might perhaps give occasion to make the mistake of substituting in the factor a^m index for exponent before developing the form $u\,(x+a)$ in a power-series of the letter a; this should first be done, then multiplication by a^m should be performed, and finally exponents should be replaced by indices.

We now come to the symbolic representation of the more general functional theorem of TAYLOR, dealt with in the 4th communication. Applying (67) to the product of the functions v and u both belonging to (β) we get

$$T(v(x)u(x)) = v(x+a)\; u(x+a) \quad . \quad . \quad . \quad . \quad . \quad . \quad (69)$$

provided no other meaning be as yet assigned to it than that the right-hand member be regarded *as a whole*, according to which it has to be replaced by the power-series in a which corresponds to the function $w\,(x+a) = v\,(x+a)\,u\,(x+a)$, if a denotes a number. This power-series, however, is to be obtained by multiplying the partial series corresponding to $v\,(x+a)$ and to $u\,(x+a)$ according to the well-known rule, and then ordering the resulting aggregate so that terms involving the same power of a are combined. If, now, we collect into one all terms of the aggregate containing the *same* factor

$$\frac{a^m u^{(m)}(x)}{m!}$$

which is due to the expansion of $u\,(x+a)$, the result for all values of m is the functional series of TAYLOR. For the whole of those terms corresponding to a definite value of m is represented by

$$\frac{a^m\, v\,(x+a)}{m!}\, u^{(m)}(x),$$

which, through (68), is equal to

is not omitted, as was the case with (23) and m.m. more general with all symbolic expansions we shall treat of.

$$\frac{T^{(m)}(v)u^{(m)}(x)}{m!}$$

But this is, if we consider $v(x)$ as "original point" and $u(x)$ as "increment", exactly the general term of the series in question, the validity of which we proved in the 4^{th} communication. This proof[1]), as a matter of course, consists in shewing that the change in the term-grouping is permitted, the convergence of the aggregate being absolute. It may therefore be grouped in an arbitrary manner so that the symbolic formula (69) admits of the following interpretation: replace both functions $v(x+a)$ and $u(x+a)$ by their power-series in the letter a, then form the aggregate arising from the multiplicative combination of the series-terms, and substitute indices for exponents. If the so obtained aggregate be ordered according to indices of a we simply get the functional series of MAC-LAURIN for $Tw = T(vu)$; if it be ordered according to powers of Du, the functional series of TAYLOR for Tw in a "neighbourhood" of $w = v$ is obtained; if, lastly, the aggregate in question should be ordered according to powers of Dv, we should find the functional series of TAYLOR for a "neigbourhood" of $w = u$. The symbolic formula (69) contains all these different cases; we only wish to observe that, if we expand the right-hand member according to powers of Du, the general coefficient in that expansion, which is, except for the factor $1/m!$, equal to

$$a^m v(x+a),$$

or to $T^{(m)}(v)$, has in *this very form* a meaning only in domains (α) *smaller* than (r_1), where r_1 is the α-value to which the radius of convergence r of the function v corresponds as a β-value; whereas in N°. 20 we saw that the other form of the coefficients in question, viz. that defined by (45), possibly has a meaning in domains greater than (r_1).

31. We now come to our principal object; to construct a symbolic formula which expresses the coefficients a_m of the series P answering to the composed transmutation $T = T_2 T_1$ in terms of the coefficients λ_m and μ_m of the partial series P_1 and P_2. As we said already, first the functions

$$\xi_k = T_2 T_1(x^k),$$

into which T transforms the integral powers of x, are determined for the purpose, in order to derive from them, by means of

1) We wish to insert here the remark that the proof we refer to becomes simpler if the majorant-functions $\overline{a_m}$ of a_m are used, as we did in the 5^{th} communication.

formula (24) (mentioned again in the beginning of the present communication), the functions a_m. The difference from the course followed in the previous communication consists therefore in the determination of $T_2 T_1$ for the *particular* function x^k instead of at once for the arbitrary function u. This can but lead to simplification.

We retain all notations and suppositions of N°. 24, and thus especially assume the existence of three numbers α, γ, β, having the properties explained there. To begin with, we observe that x^k belongs to the circle (β), hence $T_1(x^k)$ to (γ), hence $T_2 T_1(x^k)$ to (α). In other words ξ_k is a function that is regular in the closed domain (α), and we at once add the remark that the regularity of a_m follows from this by means of (24). We further develop $T_1(x^k)$, as $T_1 u$ in N°. 24, in the series of MAC-LAURIN, which, however, here simply becomes the finite series (23) (copied in the previous paragraph). Thus the transmutation T_2 may be without any addition applied *term by term* to that series, whereas the same operation in N°. 24 wanted some further explanation the series in question being there infinite. We therefore have, in terms with *proper* meanings,

$$\xi_k = \sum_0^k {}_i\, k_i\, T_2(x^{k-i}\lambda_i),$$

valid in (α). The quantity $T_2(x^{k-i}\lambda_i)$ may in this domain be determined by means of (69), since $\lambda_i(x)$ as well as x^{k-i} belong to (γ); this gives

$$T_2(x^{k-i}\lambda_i) = (x+\mu)^{k-i}\,\lambda_i(x+\mu).$$

Substituting this result in the foregoing formula we find

$$\xi_k = \sum_0^k {}_i\, k_i\,(x+\mu)^{k-i}\,\lambda_i(x+\mu), \quad . \quad . \quad . \quad . \quad . \qquad (70)$$

without anything wanting to be proved, provided we replace each of the $k+1$ terms of this series separately by its own proper meaning, and add them after this being done. The proper meaning in question is: substitute for the expressions $(x+\mu)^{k-i}$ and $\lambda_i(x+\mu)$ their power-series in μ, multiply those series term by term, and finally replace the exponents of μ by indices: then, the so obtained aggregate converges absolutely and uniformly in (α). But the same holds for each new aggregate that arises from the collection of a *finite* number of suchlike aggregates. Thus the $k+1$ aggregates corresponding to the right-hand member of (70) need not be kept apart from one another.

One method of grouping the elements of the aggregate consists in taking all those elements with the same index of μ, or, if indices have not yet been substituted for exponents, with the same exponent, together, and thus we may interpret the right-hand member of (70) as follows: replace it by its expansion in a power-series of the letter μ and substitute indices for exponents. That a uniquely determined power-series corresponds to the right-hand member of (70) hardly needs any further mentioning, this having already been stated for each of the $k+1$ terms separately. The manner of grouping considered here makes it clear, however, that the expression in question may be transformed analytically before proceeding to its interpretation, owing to the fact that a function in the neighbourhood of a regular point can but be expanded in *one* power-series. This remark will be of use when λ's of *different* indices are in some relation to each other so that further reductions of (70) are possible. But a *general* reduction of (70) is not possible since in none of the $k+1$ terms of the series occur terms with the same index at λ.

But further symbolization of the formula for ξ_k is possible if we replace the index at the letter λ by an exponent; if, at the same time, we omit for a moment the form $(x+\mu)$ from λ, we way write

$$\xi_k = \sum_0^k {}_i\, k_i\,(x+\mu)^{k-i}\lambda^i, \quad . \quad . \quad . \quad . \quad . \quad . \quad (70')$$

If this be interpreted such that, before performing other reductions, the exponent of λ be replaced by an index and the form $(x+\mu)$ be added, the foregoing formula is produced again and there is nothing to be established. But a new result is obtained if we do not consider each of the $k+1$ nembers of the sum *as a whole*, but every product $(x+\mu)^{k-i+1}\lambda$, where λ^i stands for $\lambda_i\,(x+\mu)$, as the sum of $k-i+1$ magnitudes the symbolic representation of which is obtained by the development of the binomial $(x+\mu)^{k-i+1}$ and the multiplication of each of its terms by λ^i as if λ and μ were numbers [1]). The total symbolic aggregate obtained in that way from (70') is an *ending* power-series in λ and μ, so that any other development of (70') than the special one mentioned leads to the same power-series. Now the expression in question can be analytically reduced to $(\lambda+\mu+x)^k$, so that finally we have the symbolic formula

$$\xi_k = (\lambda+\mu+x)^k, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (71)$$

[1]) The correctness of this interpretation of the product mentioned has been pointed out at the end of the previous paragraph, and, as is evident from the exposition there, the interpretation consists in considering the product as the symbolic representation of the expansion in the TAYLOR series of $T_0[x^{k-i}\lambda_i(x)]$, $\lambda_i(x)$ being the "origin" and x^{k-i} the "increment". This is contrary to what in N⁰. 24 led to the formula of BOURLET, where we took $\lambda_i(x)$ as the "increment" and $u^{(i)}(x)$ as the "origin".

the interpretation of which is implied in what precedes. We only wish to call the attention to the characteristic fact that the letters λ and μ must not at all be treated in the same manner: first comes the change of exponents into indices of λ, then the same change with regard to μ.

Finally the last step: the determination of the coefficients a_m from the quantities ξ_k by means of (24). If we put in this formula the right-hand member of (71) we find

$$a_m = \sum_0^m {}_k\, m_k\,(-x)^{m-k}\,(\lambda+\mu+x)^k\,. \quad . \quad . \quad . \quad . \qquad (72)$$

and there is nothing to be proved, if we substitute in each of the $m+1$ members of this sum separately for $(\lambda+\mu+x)^k$ its proper meaning. In order to get this latter we must expand the trinomial in its power-series in λ and μ: each of the terms then has its own real value as is explained above, and the same therefore holds for the product of such a term by the factor $m_k(-x)^{m-k}$. We thus obtain for each of the $m+1$ members of (72) an aggregate consisting of a finite number of elements each of which is characterized by a definite symbolic power of λ and μ. The total number of elements arising from the $m+1$ members is therefore also finite, so that it forms a new aggregate that may be arranged arbitrarily. If this be done in such a way that terms involving the same powers of λ and μ are collected — these may be added analytically, the meaning of a product $\lambda^p\mu^q$ depending only on the exponents p and q and not on its source — then we obtain a power-series in λ and μ. But the *same* power-series evidently corresponds to all expressions which can be derived analytically from the right-hand member of (72). Since, now, this latter is equal to $(\lambda+\mu)^m$, we may finally write

$$a_m = (\lambda+\mu)^m = (\{\lambda\}_{x+\mu}+\mu)^m\,. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (73)$$

where the last member shews more explicitly the signification which is to be assigned to the formula. This is as follows: expand the binomial $(\lambda+\mu)^m$ analytically in its power-series in λ and μ; substitute indices for the exponents of λ and in $\lambda_i(x)$ replace x by $x+\mu$; again develop the so obtained functional expression in a series according to ascending powers of μ and finally substitute in these powers indices for exponents.

This is the symbolic formula we had in view, expressing the coefficients a_m of the resulting series P in terms of the coefficients λ_m and μ_m of the components P_1 and P_2. In deriving this formula we have met with another, viz (71), which expresses the resulting quantities ξ_m also in λ_m and μ_m. But if we want this formula as a *final* result we had better write it in the following simpler form

$$\xi_m = (\lambda + x)^m = \{(\lambda + x)^m\}_{x+\mu}. \quad . \quad . \quad . \quad . \quad . \quad (74)$$

32. Formulae (73) and (74) are valid in the domain (α), as it has been shewn in the foregoing paragraph. It still remains to be proved by means of (73) that the resulting series P is complete in (α) with a corresponding domain that is at most equal to (β), a statement we gave in the previous communication. To do this we shall make use of the following proposition, the proof of which we do not give, first because it is very easy, and secondly because the proposition may perhaps be established elsewhere:

The upper limit for $m = \infty$

$$\overline{\lim_{m=\infty}} \mid P_m + Q_m \mid^{\frac{1}{m}},$$

of the m^{th} root of the modulus of the sum of two complex quantities P_m and Q_m, both defined in the aggregate of positive integral m-values, is equal to the greatest of the two upper limits

$$\overline{\lim_{m=\infty}} \mid P_m \mid^{\frac{1}{m}} \quad , \quad \overline{\lim_{m=\infty}} \mid Q_m \mid^{\frac{1}{m}}$$

of the m^{th} roots of the moduli of those two quantities separately. If the two latter limits be equal then the former is never greater than each of them.

An analogous proposition is, as a corollary of the one just mentioned, valid for a sum consisting of an arbitrary finite number of terms, this number not depending on m.

The proposition will serve us to investigate the m^{th} root of the modulus of the coefficient $a_m(x)$ of the resulting series P. If we work out the righthand member of (73) in the prescribed manner, we obtain

$$a_m = \sum_0^m {}_k \, m_k \, \mu^{m-k} \left[\lambda_k + \frac{\mu}{1} \lambda_k' + \frac{\mu^2}{2!} \lambda_k'' + \ldots \right] =$$

$$= \sum_0^m {}_k \left[m_k \sum_0^\infty {}_i \frac{\mu_{m-k+i} \, \lambda_k^{(i)}}{i!} \right], \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (75)$$

this equality containing only *proper* expressions. We assume again, as in the previous communication, that (γ) is not the *maximum* domain of completeness for the series P_1, so that there is a domain (γ') > (γ), in which P_1 is likewise complete; let the domain corresponding to this latter be denoted by (β'). We may suppose β' to be *arbitrarily little* greater than β — provided γ' be chosen *sufficiently little* greater than γ — if we assume at the same time, as we did in the previous communication, that α and β *increase and decrease continuously* with each other. Let further $L_k(\gamma')$ be the maximum modulus of λ_k on the circumference of the circle (γ'). There is, on account of the completeness of P_1 and P_2 mentioned above, corresponding to any arbitrarily small chosen number ε a whole number E such that for $n > E$

$$L_n(\gamma') \leq (\beta' - \gamma' + \varepsilon)^n , \quad \dots \quad (76)$$

together with

$$|\mu_n| \lesseqgtr (\gamma - \alpha + \varepsilon)^n . \quad \text{if} \quad |x| \leq \alpha \quad \dots \quad (77)$$

Further we have in the domain (α) for all integral not negative values of i and k

$$\left| \frac{\lambda_k^{(i)}}{i!} \right| < \frac{\alpha L_k(\gamma')}{(\gamma' - \alpha)^{i+1}} \quad \dots \quad (78)$$

We now suppose m to be chosen greater than $2E$ and on that supposition divide the double sum (75) into the following four parts, which we denote for brevity by their limits only,

$$s_1 = \sum_{E}^{m-E}{}_k \sum_{0}^{\infty}{}_i \quad , \quad s_2 = \sum_{0}^{E-1}{}_k \sum_{0}^{\infty}{}_i \quad , \quad s_3 = \sum_{m-E+1}^{m}{}_k \sum_{E}^{\infty}{}_i \quad , \quad s_4 = \sum_{m-E+1}^{m}{}_k \sum_{0}^{E-1}{}_i \, .$$

Further we assume ε, *after* γ', to be so chosen that $\gamma + \varepsilon < \gamma'$, say $\gamma' = \gamma + \varepsilon + \delta$. Then we find for the first three sums by means of the inequalities (77) and (78)

$$|s_1| < \frac{\alpha}{\delta} \sum_{E}^{m-E}{}_k \, m_k \, L_k(\gamma') \, (\gamma - \alpha + \varepsilon)^{m-k}$$

$$|s_2| < \frac{\alpha}{\delta} \sum_{0}^{E-1}{}_k \, m_k \, L_k(\gamma') \, (\gamma - \alpha + \varepsilon)^{m-k}$$

$$|s_3| < \frac{\alpha}{\delta} \left(\frac{\gamma - \alpha + \varepsilon}{\gamma' - \alpha} \right)^E \sum_{m-E+1}^{m}{}_k \, m_k \, L_k(\gamma') \, (\gamma - \alpha + \varepsilon)^{m-k}.$$

In the first and third sums moreover the inequality (76) can be applied, and thus we find for s_1

$$|s_1| < \frac{\alpha}{\delta} \sum_{E}^{m-E}{}_k \, m_k \, (\beta' - \gamma' + \varepsilon)^k (\gamma - \alpha + \varepsilon)^{m-k} < \frac{\alpha}{\delta} \sum_{0}^{m}{}_k <$$

$$< \frac{\alpha}{\delta} (\beta - \gamma' + \gamma - \alpha + 2\varepsilon)^m < \frac{\alpha}{\delta} (\beta' - \alpha + 2\varepsilon)^m$$

For s_3 we find by analogous reductions

$$|s_3| < \frac{a}{\delta}\left(\frac{\gamma-\alpha+\varepsilon}{\gamma'-\alpha}\right)^E (\beta'-\alpha+2\varepsilon)^m$$

$$< \frac{a}{\delta}\left(\frac{\gamma-\alpha+\varepsilon}{\gamma-\alpha+\varepsilon+\delta}\right)^E (\beta'-\alpha+2\varepsilon)^m < \frac{a}{\delta}(\beta'-\alpha+2\varepsilon)^m$$

From this it may be inferred

$$\overline{\lim_{m=\infty}} |s_1|^{\frac{1}{m}} \leq \beta-\alpha \quad , \quad \overline{\lim_{m=\infty}} |s_3|^{\frac{1}{m}} \leq \beta-\alpha,$$

since ε and $\beta'-\beta$ may be supposed arbitrarily small.

As for s_2, in this sum we cannot assign a majorant-value for the quantity $L_k(\gamma')$. But the number of terms of s_2 is a *fixed* one *not depending on m*. Thus we need only calculate, according to the lemma at the beginning of this paragraph, the required limit for each term separately and then for the whole that limit is in any case not greater. In no one of the terms the factor $L_k(\gamma')$ depends on m, so that this factor gives the amount 1 for the required limit and, therefore, does not influence it.

If we further notice that for a given value of k not depending on m the limit for $m=\infty$ of $m_k^{\frac{1}{m}}$ is also 1, and that, finally, ε may be chosen arbitrarily small, we infer that

$$\overline{\lim_{m=\infty}} |s_2|^{\frac{1}{m}} \leq \gamma-\alpha.$$

Lastly we consider s_4; substituting $k=m-k'$, and then omitting again the accent at the letter k, we find

$$s_4 = \sum_0^{E-1}{}_k \, m_k \sum_0^{E-1}{}_i \frac{\mu_{k+i} \, \lambda_{m-k}^{(i)}}{i\,!}$$

The double summation extends over a finite number of terms, which number is independent of m; each of the terms may be identified by fixed values of i and k, likewise independent of m, so that it is sufficient for our purpose to consider the terms separately. To such a term we may apply the inequalities (76) and (78), giving

$$\left|\frac{m_k \mu_{k+i} \lambda_{m-k}^{(i)}}{i\,!}\right| < \frac{a\,|\mu_{k+i}|\, m_k (\beta'-\gamma'+\varepsilon)^{m-k}}{(\gamma'-\alpha)^{i+1}}$$

By remarks analogous to those made with regard to the preceding sum we infer from this

$$\overline{\lim_{m=\infty}} |s_4|^{\frac{1}{m}} \leq \beta-\gamma$$

None of the four limits is, therefore, greater than $\beta - \alpha$, since γ is at least equal to α and at most equal to β. Hence for the whole sum (75) the limit in question is not greater either than $\beta - \alpha$. Thus the radius corresponding to α is for the resulting series P at most equal to $\alpha + (\beta - \alpha) = \beta$; the required result has therefore been established.

33. We may say that with the foregoing developments our original object has been performed: to find a symbolic formula expressing the coefficients of the *resultant* of two complete transmuting series in the coefficients of these two; to fix the domain of validity of this formula; finally to derive from it the statement that the resultant transmutation is likewise complete; as to the last point, we found the same result with regard to the dependence between two corresponding domains as was the case in the proof we gave of the formula of BOURLET.

Before, however, finishing our considerations on the subject we wish to establish a few other formulae constituting with those already found a sort of closed system. In the first place we have in view the generalization of the formulae found in N°. 31 for more than two transmutations. It will appear to be sufficient if we take only *three* transmutations, represented by the series P_1, P_2 and P_3. We thereby assume that it is possible to assign *four* numbers: α_3, α_2, α_1, α_0 such that P_1 is complete in a circular domain (α_1) with corresponding domain (α_0), P_2 in a domain (α_2) with corresponding domain $\alpha_1)$, P_3 in a domain (α_3) with corresponding domain (α_2). Let the coefficients of the series be denoted respectively by $a_{1,m}(x)$, $a_{2,m}(x)$, $a_{3,m}(x)$, those of the resultant P_{II} of P_1 and P_2 by $a_{II,m}(x)$ and those of the total resultant P_{III} by $a_{III,m}(x)$.
Then we have

$$a_{II,m} = (\{a_1\}_{x+a_2} + a_2)^m \quad . \; . \; . \; . \; . \; . \; . \quad (79)$$

valid in (α_2); further, since the series P_{II} is complete in (α_2), with a corresponding domain not greater than (α_0), we also have

$$a_{III,m} = (\{a_{II}\}_{x+a_3} + a_3)^m \quad . \; . \; . \; . \; . \; . \; . \quad (80)$$

valid in (α_3), and P_{III} is complete in the domain (α_3), with a corresponding domain which is at most equal to (α_0); all this is to be inferred from (73) and what has been stated about this formula. The statement that the resultant P_{III} is *complete* so that as a domain of completeness comes into account that of the *last* component, the corresponding domain being at most equal to that which for the *first* component corresponds to its domain of completeness, is thus easily proved, the generalisation for n components being at once evident. It remains only to combine the two preceding formulae, so that the resulting coefficient $a_{III,m}$ may be expressed in terms of the coefficients $a_{1,m}$, $a_{2,m}$, $a_{3,m}$.

We consequently work out the right-hand member of (80) according to the rule at the end of N°. 31, and then find the following new symbolic formula which answers to the earlier form (70) of formula (73),

$$a_{III,m} = \sum_0^m {}_k\, m_k\, a_3^{\,m-k}\, a_{II,k}\,(x+a_3) \quad . \quad . \quad . \quad . \quad (80')$$

The right-hand member must be developed according to ascending powers of a_3; that this power-series is entirely determined follows from our detailed investigation in N°. 31, according to which the function $a_{II,m}(x)$ is regular in the domain (α_2) and therefore also in (α_3). Further the exponents of a_3 have to be replaced by indices; the so obtained aggregate converges, according to the investigation mentioned, absolutely and uniformly in the domain (α_3) and the same holds for each of the $m+1$ aggregates that can separately be derived from the members of the sum (80'): this latter assertion, moreover, corresponds to an *earlier* stage of the interpretation of (70). The function $a_{II,m}(x)$ has been considered in this as a whole, but now it must be determined by means of (79), making formula (80') pass into

$$a_{III,m} = \sum_0^m {}_k\, m_k a_3^{\,m-k} \{(\{a_1\}_{x+a_2} + a_2)^k\}_{x+a_3} \quad . \quad . \quad . \quad . \quad (81)$$

with the same signification, provided the factor between braces be interpreted as a whole, according to the rule prescribed for the working out of (79), substituting *at the end* of the process $x + a_3$ for x [1]). The latter formula we may write in a more simple manner thus

$$a_{III,m} = \sum_0^m {}_k\, m_k a_3^{\,m-k} (a_1 + a_2)^k \quad . \quad . \quad . \quad . \quad . \quad (81')$$

[1]) By the words *at the end* we mean: *after the function* $a_{II}(x)$ *having been constructed as a whole.* Formula (81) would not express the same thing as (80') if we interpreted it such that *in each element* of the infinite aggregate that is, in general, obtained from (79) for $a_{II,k}$, x were to be replaced by $x + a_3$. Meanwhile we may observe that the latter mode of calculating $a_{II,k}$ might really be applied, as it will be seen on noticing that the preceding investigations remain intact if the coefficients a_1, a_2, a_3 be replaced by their natural majorants. This remark may be of use in theoretical questions.

this form having the same meaning as the foregoing. But the latter formula suggests the idea that we shall finally have

$$a_{III,m} = (a_1 + a_2 + a_3)^m \equiv [\{(a_1)_{x+a_2} + a_2\}_{x+a_3} + a_3]^m, \quad . \quad . \quad (82)$$

the last member of which points out the signification in a more detailed manner. This is: expand the trinomial $(a_1 + a_2 + a_3)^m$, as if a_1, a_2, a_3 were numbers; replace in each individual term of that expansion, viz.

$$C a_1^g a_2^h a_3^i, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (83)$$

where C is a whole number only depending on the exponents g, h, i, the exponent g by an index and in $a_{1,g}(x)$ the letter x by $x + a_2$; then expand every expression

$$a_2^h a_{1,g}(x + a_2)$$

in a series of powers of a_2 and replace the exponents of a_2 by indices; in the functional expression represented by the now obtained aggregate, or in the aggregate itself, [1]) replace the letter x by $x + a_3$; expand the product of the latter expression by a_3^i in a power-series of a_3 and replace the exponents of a_3 by indices; then there results an aggregate that, together with those obtained from the other terms arising, like (83), from the trinomial (82), represents the required function $a_{III,m}$ in the domain (α_3).

We can easily see that the transition from (81) to (82) is allowed. For looking into the matter thoroughly we see that the interpretation of (82) may be obtained from that of (81) by changing the latter only so far as to consider, in the development of the trinomial in question, all terms involving the same power of a_3 as an irresoluble whole. *It is not at all a matter of course* that the two points of view agree, nor even that the aggregates corresponding separately to each of the terms mentioned *converge*. But we may again state that the whole reasoning remains valid if we substitute for the functions a_1, a_2, a_3 their *natural majorants* $\overline{a_1}, \overline{a_2}, \overline{a_3}$, and thus we infer that there is no difference in the results afforded by the two points of view.

[1]) See the foregoing footnote.