Mathematics. – *"Some Considerations on Complete Transmutation"*. (Seventh communication). By Dr. H. B. A. BOCKWINKEL. (Communicated by Prof. L. E. J. BROUWER).

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34. As a counterpart of the formula that expresses the coefficients of the resultant of a certain number of complete transmuting series in terms of the coefficients of the components we shall treat of the formula which expresses the quantities ξ for the resultant in terms of the same quantities for the components. It is convenient always to consider a neighbourhood of the origin; or, if not, to appropriate the symbol ξ_m to the transmuted of $(x-x_0)^m$, instead of to that of x^m, x_0 being the common centre of the circular domains considered; if, in the latter case, we assume $x-x_0$ as a new variable, all is reduced to the former.

We denote the functions mentioned for the series $P_1, P_2, \ldots$ respectively by $\xi_{1,m}\, \xi_{2,m}, \ldots$, the resultant for two, three, ... series by $\xi_{II,m}, \xi_{III,m}, \ldots$; further we retain the notations and suppositions of the preceding paragraph. Taking first two components, we have

$$\xi_{II,m} = P_2\, P_1\, (x^m) = P_2\, (\xi_{1,m})$$

The function $\xi_{1,m}(x)$ belongs to the circle (α_1), thus we have by (67) in the domain (α_2)

$$\xi_{II,m} = \xi_{1,m}(x + a_2) \equiv \sum_0^\infty{}_i \frac{a_{2,i}\, \xi_{1,m}^{(i)}}{i!}, \qquad (84)$$

where the last member expresses the proper meaning. Now a_{2i} can be expressed by means of the functions $\xi_{2,m}$ according to the symbolic formula (23), giving

$$\xi_{II,m} = \sum_0^\infty{}_i \frac{(\xi_2 - x)^i\, \xi_{1,m}^{(i)}(x)}{i!} \qquad (84')$$

provided the substitution of $(\xi_2 - x)^i$ for $a_{2,i}$ provisionally does not mean any other thing than a *definite mode of calculation* of $a_2{}^{,i}$ considered as a whole. We now notice again that a corresponding formula, *with the very same domain of validity*, applies in the case that arises from the present one by replacing the functions

$a_{1,m}, a_{2,m}, \ldots$ by their natural majorants; and we infer from (23) that the functions $\xi_{1,m}, \xi_{2,m}, \ldots$ are in this case also replaced by majorant-functions[1]), though, these need not be the *natural* majorants of the former functions. In order to avoid reproductions of formulae taking up much room we propose to imagine for a moment that formula (84) relates to the last-mentioned case. But even so we are not justified yet in considering the aggregate arising from the expansion of each binomial of the series in (84′) as an absolutely converging one, if x lies in the domain (α_2), because it is not known whether we may replace $-x$ by x in that expansion. We therefore make an estimate of the magnitude of the sum

$$\sum_{0}^{i}{}_{h}\, i_h\, x^h\, \xi_{2,\,i-h},$$

which we may denote by the symbol $(\xi + x)^i$, and to this purpose remark in the first place that the quantities ξ_m for a *complete* transmutation satisfy the same characteristic property as the coefficients a_m of the corresponding series: to be smaller, as to their moduli, than the m^{th} power of a number which is independent of m. This follows from formula (23). For from and after some value of k $(k = E)$ we have in a domain of completeness (α), $|a_k| < (a + \varepsilon)^k$, where a depends on α, and ε may be chosen arbitrarily small. Then by (23) and supposing $m \gtrless E$

$$|\xi_m| \leq \sum_{0}^{E-1}{}_{k}\, m_k\, \alpha^{m-k}\, |a_k| + \sum_{E}^{m}{}_{k}\, m_k\, \alpha^{m-k}\, (a + \varepsilon)^k$$

$$\leq \longleftarrow\!\!\longrightarrow + \sum_{0}^{m}{}_{k} \longleftarrow\!\!\longrightarrow$$

$$\leq \longleftarrow\!\!\longrightarrow + (a + \alpha + \varepsilon)^m$$

The latter right-hand member consists of a finite number of terms, viz. $E + 1$, which is independent of m; thus it is sufficient, in order to calculate a majorant value for $\overline{lim}\, |\, \xi_m\, |^{\frac{1}{m}}$, to determine the corresponding limit for each of the terms individually: the greatest among these limits will, according to the lemma of N°. 32, be a majorant value as required. Now for each of the first E terms the limit in question is clearly not greater than α; and for the last term it is $a + \alpha = \beta$; thus β is a majorant-value as was required. We have now proved the following proposition:

[1]) This, if an arbitrary point x_0 is the centre of the domains, is valid only if by ξ_m be denoted the transmuted of $(x - x_0)^m$.

If in all points x of a domain (α), centre the origin

$$\overline{\lim_{m=\infty}} |a_m|^{\frac{1}{m}} \leq \alpha, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (85)$$

where a_m for every integral value of m is a regular function of x in that domain, then in the same points we have for the symbolic binomial expression $(x + a)^m$.

$$\overline{\lim_{m=\infty}} |(x + a)^m|^{\frac{1}{m}} \leq \alpha + a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (86)$$

And as a consequence of this:

If a transmutation be complete in a circular domain (α), centre the origin, and the domain corresponding to (α) be (β), then the upper limit,

$$\overline{\lim_{m=\infty}} |\xi_m|^{\frac{1}{m}}$$

of the m^{th} root of the modulus of the transmuted ξ_m of x^m is not greater than β. [1])

The same proposition holds if for ξ_m the above mentioned majorant-function be substituted, the value of β being unaltered as it follows from the lemma in the last section of N°. 23 (5th communication).

If, now, we apply the preceding result to the above case, we find that in all points x of a domain (α) not greater than (α_2)

$$\overline{\lim_{m=\infty}} |\xi_{2,m}|^{\frac{1}{m}} \leq \alpha + (\alpha_1 - \alpha_2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (87)$$

since α_1 is the number that for the series P_2 corresponds to α_2 and the difference between corresponding radii α and β does not increase if α diminishes. Further it may be inferred from (87) in the same way as (86) followed from (85)

$$\overline{\lim_{m=\infty}} |(\xi_{2,m} + x)^m|^{\frac{1}{m}} \leq 2\alpha + \alpha_1 - \alpha_2.$$

This is the inequality we wished to obtain. It follows that the series arising from (84') by substituting the sign $+$ for $-$ in $(\xi_2 - x)^i$, and by replacing the functions $\xi_{1,m}$ and $\xi_{2,m}$ by the above mentioned majorant-functions, converges in any domain (α) when

$$3\alpha + \alpha_1 - \alpha_2 < \alpha_1, \quad \text{or} \quad \alpha < \tfrac{1}{3}\alpha_2,$$

because $\xi_{1,m}$ is regular in (α_1); a further explanation may be superfluous since it would be a repetition of what has been stated more

[1]) For a domain of an arbitrary point x_0 the same holds as to the transmuted of $(x - x_0)^m$.

than once in the course of our developments of the theory of complete series. Thus the aggregate arising after the expansion of the symbolic binomial in every term of the series in (84') will converge *absolutely* in a domain (a) satisfying the foregoing condition; and in this case it may therefore be ordered arbitrarily. We do this in such a way that elements involving symbols $\xi_{2,i}$ with the same index i — or symbolic powers $\xi_2{}^i$ with the same exponent i, if exponents have not yet been replaced by indices — are collected into one, and then may find the new arising coefficient in a simple manner. First we remark that, if ξ_2 denoted a certain *number*, the right-hand member of (84') would represent the formal expansion of the magnitude $\xi_{1,m}(\xi_2)$ at the point $\xi_2 = x$. Now it is a well-known truth in the theory of functions that the *formal* development of the function $f(y)$ at the point $y=0$,

$$f(y) = \sum_0^\infty {}_i \frac{y^k f^{(k)}(0)}{k!}$$

may be obtained from that at the point $y = x$,

$$f(y) = \sum_0^\infty {}_i \frac{(y-x)^i f^{(i)}(x)}{i!},$$

by expanding every binomial expression in the latter series and then collecting in the so obtained aggregate terms involving the same power y^k of y: the resulting coefficient of y^k is the *formal* expansion of

$$\frac{f^{(k)}(0)}{k!}$$

at the point x. We intentionally speak of *formal* expansion, because it may happen that there is no value of y for which the two series converge, the circles of convergence of $f(y)$ lying wholly outside each other. Or, if there does exist a value as indicated, it may happen that the just-mentioned power-series for the expression $f^{(k)}(0): k!$ diverges because the circle of convergence of the function $f(z)$ for the value $z = x$ does not contain the point $z = 0$. The statement applies in any case since the *general form* of the expansions is independent of the particular character of the function in question and functions do occur, viz. the integral transcendental functions, for which the two series converge in the whole plane.

If now we apply the foregoing considerations in the present case, we infer that the required coefficient of $\xi_2{}^k$ is the formal expansion in a power-series of $\xi_{1,m}^{(k)}(0): k!$ at the point x. But this *formal*

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expansion is here also *essential* if x lies in the domain $(\frac{1}{3}\alpha_2)$, for the convergence of the series is in this case included as a special result in the one just obtained viz. that the aggregate in question is absolutely converging in $(\frac{1}{3}\alpha_2)$. The coefficient we treat of is therefore equal in value to $\xi_{1,m}^{(k)}(0):k!$, so that we may write

$$\xi_{II,m} = \{\xi_{1,m}\}_{\xi_2} \equiv \sum_0^\infty {}_k \frac{\xi_{2,k}(x)\xi_{1,m}{}^{(k)}(0)}{k!}, \quad . \quad . \quad . \quad . \quad (88)$$

where the last member shews the meaning of the symbolic second member: in $\xi_{1,m}(x)$ we have to replace x by ξ_2, then to expand the expression $\xi_{1,m}(\xi_2)$, as if ξ_2 were a number, in its power-series of MAC-LAURIN and finally to substitute indices for exponents in the symbolic powers of ξ_2. As already remarked the formula is valid in any domain (α) the radius of which is not greater than $\frac{1}{3}\alpha_2$. The series in the last member, however, converges in the whole domain (α_2), since the limit in the left-hand member of (87) for $\alpha = \alpha_2$ is, according to the very same inequality, less than α_1 and $\xi_{1,m}(x)$ is regular in the closed domain (α_1); further explanation may, as above in an analogous case, be omitted. The convergence of the series is also *uniform* in (α_2), its terms are regular functions of x in that domain, hence its sum represents also a regular function there. This latter must be identical with $\xi_{II,m}(x)$ since this function is also regular in (α_2) and the two functions agree already in a finite part of (α_2) viz. $(\frac{1}{3}\alpha_2)$. Thus, finally, the symbolic formula (88) is valid in the same domain as its counterpart (73).

The generalization of (88) is at once obvious. Evidently we have

$$\xi_{III,m} = \{\xi_{II,m}\}_{\xi_3}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (89)$$

valid in the domain (α_3), because $\xi_{II,m}$ belongs to the domain (α_2). In connection with (88) this gives

$$\xi_{III,m} = \{\{\xi_{1,m}\}_{\xi_2}\}_{\xi_3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (90)$$

meaning: $\xi_{1,m}(x)$ has to be developed according to powers of x and x^i must be replaced by $\xi_{2,i}(x)$, then the resulting series, which is absolutely and uniformly convergent in (α_2), represents the function $\xi_{II,m}(x)$, which is regular in the same domain; again expand this function according to powers of x and replace x^i by $\xi_{3,i}(x)$; the resulting series, which is absolutely and uniformly converging in the domain (α_3), represents the function $\xi_{III,m}(x)$, regular in that domain.

It will be convenient to observe that the interpretation given just now does not at all differ from the one corresponding to the pair of formulae (88) and (89) before they are replaced by (90), so that there is nothing to be proved in doing this. Matters would be different if we wanted to effect the final operation of the given rule on the infinite series which we have obtained for $\xi_{II,m}(x)$, and not on this function considered as a whole. But that the second interpretation will do as well, follows again from the consideration of the natural majorants of $a_{1,m}$, $a_{2,m}$, ...

35. In applying the preceding symbolic formulae to particular cases special reductions are often necessary which have to be justified individually. Only in order to call the attention to this point we shall discuss one or two examples, but for the rest further explanations by means of examples of the symbolic formulae may be omitted after the detailed consideration of examples with the transmutations D^{-1} and S_ω in the previous communication, the more so as the formulae in question provide the same *general* results relative to resultant magnitudes as the formula of BOURLET.

We first take the case $T_1 = T_2 = S_\omega$. Here

$$a_{1,i} = a_{2,i} = (\omega - x)^i.$$

Applying (73) we successively obtain

$$a_{II,m} = \sum_0^m{}_k\, m_k\, a_2^{m-k} a_{1,k}(x + a_2) = \sum_0^m{}_k\, m_k\, a_2^{m-k}\left[\omega(x + a_2) - (x + a_2)\right]^k$$
$$= \left[\omega(x + a_2) - x\right]^m.$$

Here the reduction of the last member but one to the last is allowed in consequence of the particular form of the given transmutation. Nevertheless the exactness of the reduction is included in the *general* considerations according to which analytic reductions are permitted. The resulting expression must now be expanded in a power-series of a_2, which must really exist, according to the general theory, if x lies in the domain (α_2); this is in fact the case since the circle of convergence of ω is greater than a domain of completeness (a_1) of S_ω and thus a fortiori greater than (a_2). We may also obtain the power-series in question by raising that for $m = 1$ to the m^{th} power, according to the common rule for the involution of an infinite series; this is meant, when we write

$$a_{II,m} = \left[\sum_0^\infty{}_i\, \frac{a_2{}^i\, \omega^{(i)}(x)}{i\,!} - x\right]^m, \quad \ldots \ldots \quad (91)$$

where the term $-x$ has to be combined with the term corresponding to $i = 0$ under the sign of summation.

When the involution has been performed we have to replace a^i_2 by $a_{2,i}$. The peculiar thing to be noticed is now that we may invert the order of the last two operations; this is caused by the fact that here a_2^i is not only a symbolic but a real i^{th} power, viz. of the number $\omega - x$, which is independent of i. In consequence of this the proper meaning of the product of two symbolic powers is in the present case equal to the product of their proper meanings and this has the effect that a_2^i may be replaced by $a_{2,i}$ before performing the involution [1]). The result is

$$a_{II,m} = \left[\sum_0^{\infty} \frac{(\omega - x)^i \, \omega^{(i)}(x)}{i\,!} - x \right]^m$$

The infinite series within the brackets (without the term $-x$) is clearly a formal expansion of the expression

$$\omega[\omega(x)]$$

and if N°. 26 of the preceding communication is consulted, especially formula (60), it will be evident that the present expansion does represent the last mentioned quantity in the domain (α_2). Finally we have therefore

$$a_{II,m} = [\omega\,[\omega\,(x)] - x]^m\,,$$

which of course could be derived in a much simpler manner (observing that $S_\omega S_\omega$ is itself an operation of substitution in which the function $\omega[\omega(x)]$ has to be substituted for x).

In the second place we take $T_1 = S_\omega$, $T_2 = D^{-1}$. Here $a_{1,i}$ is the same as in the preceding case and

$$a_{2,i} = \frac{(-1)^i x^{i+1}}{i+1},$$

as we have already utilized several times (see for instance N°. 16, 3[rd] communication). Since the first component is the same as in the first case, formula (91) will apply here as well. But the order of the involution and the replacement of a_2^i by $a_{2,i}$ must not be inverted now, the symbol a_2^i not being the i^{th} power of a number independent of i. However, an other reduction is permitted. For we also have

$$a_{2,i} = \int_0^x (t - x)^i \, dt$$

so that $a_{2,i}$ is at least the *integral* of an i^{th} power. We may therefore give as a further rule for the reduction of the power-series in the right-hand member of (91) the following one: replace a_2^i by

[1]) It is for the same reason that an exact result is obtained if in applying formula (24), that is $a_m = (\xi - x)^m$, to the operation of substitution, ξ is replaced by ω before the expansion of the binomial.

$(t-x)_i$ and integrate every term from $t=0$ to $t=x$. *This* replacement may now be performed *before* the involution, on the same ground as above, so that we may say: the power-series in $(t-x)$

$$\sum_0^\infty \frac{(t-x)^i\,\omega^{(i)}(x)}{i\,!} - x\,, \quad . \; . \; . \; . \; . \; . \; . \quad (92)$$

where the term $-x$ must be taken together with the term under the sign of summation that corresponds to $i=0$, must be involved to the m^{th} power according to the common rule for the involution of infinite series, and the result is to be integrated term by term between the limits $t=0$ and $t=x$. Now the infinite series in (91) (without the term $-x$) is a formal expansion of $\omega(t)$ and if N°. 28 of the preceding communication is consulted, from which it appears that the radius of convergence of $\omega(x)$ is greater than $2\alpha_2$, we infer that the expansion is *essential* for values of x in the domain (α_2). The involution to the m^{th} power therefore leads to a power-series in $(t-x)$ which, for the x-values mentioned, represents the function $[\omega(t)-x]^m$ and since the convergence of the latter power-series is *uniform* in the integration-interval, its integration term by term produces the integral of the function represented by it. Thus we finally have

$$a_{II,m} = \int_0^x [\omega\,(t) - x]^m\, dt$$

which has also been found in N° 28.

In, applying formula (88), in order to determine $\xi_{II,m}$, analogous peculiarities occur in either of the cases just mentioned.

36. In the two previous communications we have, in considering the resultant of two complete transmutations, been able to simplify our statements by supposing that we had to deal with the following case. If the functions $\bar{a}_m(x)$ or shortly $\bar{a}_m$ be the natural majorants of the coefficients a_m of a series P representing a transmutation which is complete in certain circular domains (α), with common centre x_0 and a radius α varying between 0 and a certain positive value A, then the maximum value $\bar{a}(\alpha)$, for the domain (α), of the upper limit

$$\bar{a}_x \equiv \overline{\lim_{m=\infty}}\, |\bar{a}_m|^{\frac{1}{m}} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (6')$$

is *within* the interval $(0, A)$ a *continuous* function of (α). (Cf. N°. 23). From this it might further be derived that the corresponding quantity $a(\alpha)$, belonging to the given functions a_m themselves, was equal to the first mentioned, and therefore also continuous within $(0, A)$, if we further assumed, as we have done continually, that a certain supposition were realized which we proposed to quote as *the uniformity supposition of* N^0. 4. The above mentioned simplification of statements was a consequence of this identity of $a(\alpha)$ and $\overline{a}(\alpha)$. Again we intimated in No. 23 that we should perhaps recur to the question as to whether the continuity of $a(\alpha)$ represents the only possible case. We now proceed to do so.

We may for shortness of notation and without loss of generality consider a neighbourhood of the origin; further we may, as long as only the natural majorants of $a_m(x)$ are considered, denote these quantities without the lines above the letters which were used hitherto; the same thing may be done in denoting quantities connected with the first mentioned, as for example the left-hand member of (6′). This latter attains in the domain (α) its maximum value for the real positive value $x = \alpha$, so that we have

$$a(\alpha) = \overline{lim}_{m=\infty} [a_m(\alpha)]^{\frac{1}{m}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (93)$$

The supposition that the series P is complete in (α) implies that all functions a_m belong to (α), that is, that they are regular within that circle and on its circumference. Let R be the upper limit of the radii α for which this holds. Then we may put the question as follows: If the function $a_m(\alpha)$ of the *real* variable α for all integral positive values of m can within the interval $(0, R)$ be developed in a power-series of α with *real positive* coefficients, to investigate the question as to whether the function $a(\alpha)$ defined by (93) is *continuous* in the same interval.

Since $a(\alpha)$ evidently increases together with α, we may at once infer the following: 1. If $a(\alpha)$ be *finite* for a certain value of α in the interval $(0, R)$, then $a(\alpha)$ is also finite for all smaller values of α, belonging to the same interval. 2. If $a(\alpha)$ be infinite for a certain value of α in the interval $(0, R)$, then $a(\alpha)$ is also infinite for all greater values of α in $(0, R)$. Thus there is in $(0, R)$ a point A forming the section between those values of the interval for which $a(\alpha)$ is finite and those for which it is infinite. The point A may coincide with $\alpha = 0$ or with $\alpha = R$; in the first case the corresponding transmutation is not complete in any domain of the origin, however small, so that this case need not be regarded. (An example is furnished by $a_m = m!\, x^m$, where $R = \infty$, but $A = 0$).

We therefore take the case that there is a certain sub-interval $(0, A)$ of $(0, R)$, which may eventually coincide with the latter, such that $a(\alpha)$ has a finite value within it; and we shall prove that *discontinuity of* $a(\alpha)$ *within that interval is not possible.*

We divide all power-series of $a_m(x)$, for the different values of m, into two parts, the first of which contains a number of terms proportional to m, say km; thus we write

$$a_m(\alpha) = \sum_{0}^{\infty} {}_n\, c_{m,n}\, \alpha^n = P_m(\alpha) + Q_m(\alpha)\,, \quad . \quad . \quad . \quad . \quad (94)$$

where

$$P_m(\alpha) = \sum_{0}^{km-1} {}_n\, c_{m,n}\, \alpha^n \quad , \quad Q_m(\alpha) = \sum_{km}^{\infty} {}_n\, c_{m,n}\, \alpha^n\,,$$

and k is a number, independent of m, which is at our disposal. To either of the parts P_m and Q_m there corresponds, as to their sum, an upper limit as exhibited in (93); these we shall respectively denote by the names *first limit* and *second limit*, whereas we may call *total limit* that corresponding to the whole series. We may now again use the proposition stated in N°. 32 of the preceding communication, which has already served us a few times in estimating limits such as we have to deal with here. In virtue of this the greater of the limits calculated for P_m and Q_m separately is equal to the total limit, and if the first two limits are equal, the total limit is either equal to them or less: the latter, however, cannot be realized here, since all the terms of the series are positive.

If now the total limit $a(\alpha)$ is *zero* in all internal points of $(0, A)$, then $a(\alpha)$ is also *continuous* in those points and there remains nothing to be proved then. Thus we take the case that there is a point α_1 in $(0, A)$ for which the limit in question is a certain *positive* number λ_1. We may state then: There is for the point $\alpha = \alpha_1$ a value k_1 of the above number k such that the first limit is not less than the second and thus equal to the total limit λ_1. For if we suppose for a moment that the second limit were greater than the first and thus equal to λ_1 for an *arbitrary* value of k, that limit would for a value of α in $(0, A)$ greater than α_1 be *at least* $\left(\frac{\alpha}{\alpha_1}\right)^k$ times as great as λ_1 and thus, as k could be taken *arbitrarily great*, the limit in question would be necessarily *infinite*, contrary to the hypothesis. Thus there is a value k_1 of the property mentioned.

For a point α on the left of α_1 the second limit is *for the same value* k_1 of k no more greater than the first, so that the first limit for $k = k_1$ is again equal to the total limit in such a point. In fact, the second limit is there *at least* and the first *at most* $\left(\frac{a}{\alpha_1}\right)^{k_1}$ times as small as in the point α_1. From this it at once follows that the first limit cannot be equal to zero in any such point α nor can the total limit. Further we have for two arbitrary points α and α' of the interval $(0, \alpha_1)$ that the ratio of the values which the first limit assumes there, lies between $\left(\frac{\alpha'}{\alpha}\right)^{k_1}$ and 1, and thus approaches to 1 as α' approaches to α. In other words the first limit and thus the total limit too, which is equal to it, is a continuous function of α in the interval $(0, \alpha_1)$.

From the hypothesis that $a(\alpha_1)$ is *finite* and *different from zero* we have thus inferred that $a(\alpha)$ is *continuous* in the interval $(0, \alpha_1)$ and also different from zero. But if $a(\alpha_1)$ differs from zero the same holds for any value of $a(\alpha)$, corresponding to a point α of the interval (α_1, A). Thus we infer that *the function $a(\alpha)$ is continuous and different from zero in any sub-interval of* $(0, A)$ *which has the left-hand end-point in common with* $(0, A)$, *and thus shortly speaking in* $(0, A)$. The limiting values (93) are thus in the internal points of the interval $(0, A)$ either *all* equal to zero or *all* different from zero, but also in the latter case they form a function of α which is continuous within that interval. This is the result required.

As regards the endpoints of the interval $(0, A)$ the foregoing reasoning does not inform us of anything. We may with a view to these points distinguish the following cases, which all are possible as it appears from the examples added.

1[st]. The function $a(\alpha)$ is continuous in both endpoints, that is to say continuous *on the right* at $\alpha = 0$, and continuous *on the left* at $\alpha = A$.

Examples: $a_m(x) = 1 + x^{m^2}$. The interval $(0, R)$ where the functions $a_m(\alpha)$ can be expanded into a power-series of α, is here the interval $(0, \infty)$; the maximum value A of the α-values for which $a(\alpha)$ is finite is equal to unity. The function $a(\alpha)$ is in the closed interval (0,1) equal to the constant value 1. As another example we may quote $a_m(x) = x^m + x^{m^2}$, for which also $R = \infty$, $A = 1$; here $a(\alpha)$ is in the closed interval (0,1) equal to x. More generally we may take

$$a_m(x) = y^m (1 + x^{m^2}),$$

where $y = f(x)$ is a function of x with a radius of convergence greater than unity, and is identical with its natural majorant. Here we again have $A = 1$ and $R =$ the just-mentioned radius of convergence of y; the function $a(\alpha)$ is in the closed interval (0,1) identical with $f(\alpha)$. An example in which $a(\alpha)$ is everywhere in the closed interval $(0, A)$ equal to zero, is provided by $a_m(x) = m^{-m} x^{m^2}$, where $A = 1$.

2nd. The function $a(\alpha)$ is discontinuous at the end-point on the left of $(0, A)$, and continuous at the end-point on the right of $(0, A)$.

Examples: $a_m(x) = x + x^{m^2}$. We have $R = \infty$, $A = 1$. For $\alpha = 0$, $a(\alpha) = 0$ and in the other points of the closed interval $(0,1)$, $a(\alpha)$ is equal to 1. More generally we may take

$$a_m(x) = c^m + y^m (x + x^{m^2})$$

where c is a positive constant and $y = f(x)$ a function of x of the same kind as above, with the restriction that it is for $x = 0$ greater than c. The function $a(\alpha)$ is then in $\alpha = 0$ equal to c, and in the other points of the closed interval $(0,1)$ it is equal to $f(\alpha)$, so that, if $f(0) = c + p$. where p is a positive number, it has a saltus at $\alpha = 0$ on the right, the amount of which is p.

3rd. The function $a(\alpha)$ is continuous at the end-point on the left of the interval $(0, A)$, discontinuous at the other end-point.

a. $a(\alpha)$ is *finite* at A; *b.* $a(\alpha)$ is *infinite* at A.

Examples of the case $3a$. Take

$$a_m(x) = y^m + x^{m^2},$$

where $y = f(x)$ is a function of the kind considered satisfying the further condition that $f(1) < 1$; then $a(1) = 1$ and in the other points of the closed interval $(0,1)$ we have $a(\alpha) = f(\alpha)$, so that this function has a saltus at $\alpha = 1$ on the left of the amount θ, where

$$\theta = 1 - f(1).$$

If we take $y = 0$, we have the case that $f(\alpha)$ is in all points of the closed interval $(0,1)$ equal to zero, except at $\alpha = 1$, where $f(\alpha) = 1$.

Examples of the sub-case $3b$. Take

$$a_m(x) = y^m + m!\, x^{m^2},$$

where y is a function as regarded without any further restriction being necessary. If we take $y = 0$, $a(\alpha)$ is *zero* in the whole interval $(0,1)$ except at $\alpha = 1$, where $a(\alpha)$ is infinite.

4th. The function $a(\alpha)$ is discontinuous at both end-points of the interval $(0,A)$:

a. $a(\alpha)$ is finite at $\alpha = A$.

b. $a(\alpha)$ is infinite at $\alpha = A$.

Examples of $4a$. Take

$$a_m(x) = c^m + xy^m + x^{m^2},$$

where c is a positive constant less than 1, $y = f(x)$ a function of the kind already considered satisfying the further conditions $f(0) > c$

and $f(1) < 1$. We then have $a(0) = c$, $a(1) = 1$ and within the interval $(0, 1)$, $a(\alpha) = f(\alpha)$, so that, if

$$f(0) = c + p, \quad f(1) = 1 - q,$$

where p and q are positive numbers satisfying the inequality

$$c + p < 1 - q,$$

there is at $\alpha = 0$ on the right a saltus of $a(\alpha)$ of the amount p and at $\alpha = 1$ one of the amount q.

Examples of 4*b*. Take

$$a_m(x) = c^m + xy^m + m!\, x^{m2},$$

where $y = f(x)$ is a function as considered in the preceding example, without the restriction as to the value of $f(1)$ being necessary.

In all these cases we have imagined A to be essentially different from R, but matters do not alter substantially if we suppose $A = R$. We may construct examples of this case by taking in the preceding ones the function $y = f(x)$ such that its radius of convergence is unity, and then we may either choose $f(x)$ so as to be finite for $x = 1$ or infinite.

Returning to the functional operations we remark that, in consequence of the theorem of N°. 4, such an operation is complete only in domains (α) such that $\alpha < A$, or perhaps $\alpha = A$. We therefore have only to deal with domains of the latter kind and for those $a(\alpha)$ has now appeared to be a continuous function of α, except perhaps at $\alpha = 0$ and at $\alpha = A$. The case which we supposed to have been realized, in order to simplify our statements relating to the resultant of two complete transmutations, thus appears to be the only one possible. Moreover, as regards possible discontinuity at $\alpha = 0$, this is of no interest for the complete transmutation. If the function $a(\alpha)$ at $\alpha = 0$ has a saltus from the value b_0 to the value b (the limit of $a(\alpha)$ at $\alpha = 0$ on the right), then though the series P produces for all functions belonging to the circle (b_0) and not to (b) a transmuted at $x = 0$, it does not for all functions belonging to a circle (b') greater than (b_0) but smaller than (b) produce a transmuted in a certain *domain* of x_0, however small. We therefore shall assume as the domain (β) corresponding to $\alpha = 0$ the circle of radius b instead of the circle of radius b_0 and then the discontinuity of $a(\alpha)$ at $\alpha = 0$ has been removed.

A possible discontinuity on the left of $\alpha = A$ can, however, not be removed.

From formula (7) of N°. 4

$$\beta = \alpha + a(\alpha) \quad \ldots\ldots\ldots\ldots \quad (7)$$

it now follows that the number β corresponding to α is also continuous within the interval of completeness $(0,A)$. That β is, moreover, monotonously increasing together with α, since $a(\alpha)$ cannot decrease as α is increasing, has already been remarked in earlier parts of the present paper.

Let us finally consider the case that the functions $a_m(x)$ do not coincide with their natural majorants, and let us write the latter with the usual lines over them. We proved in N° 23 that the identity

$$a(\alpha)=\overline{a}(\alpha) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (95)$$

is valid under the following two conditions: 1st. If the uniformity supposition of N°. 4 is satisfied; 2nd if the quantity $\overline{a}(\alpha)$ is a *continuous* function of α within the interval $(0, A)$. The latter has now been proved to be always the case, so that we may infer that the equality (95) is only a consequence of the same uniformity supposition from which we derived in N°. 4 the extended theorem of Bourlet.

37. In connection with the latter considerations it may be convenient to observe that the uniformity supposition of N°. 4 can be replaced for either of the two purposes mentioned by one of somewhat wider compass. If the reasoning in N°. 23, leading to the identity (95) be carefully examined, it appears that another, viz.

$$A(\alpha)\equiv\overline{\lim_{m=\infty}}\,[A_m(\alpha)]^{\frac{1}{m}}=\overline{a}(\alpha)\,, \quad . \quad . \quad . \quad . \quad . \quad . \quad (96)$$

where $A_m(\alpha)$ is the maximummodulus of $a_m(x)$ on the circumference of (α) can be derived from the *continuity* of $\overline{a}(\alpha)$ only,[1]) and since the latter is always realized, the same holds for formula (96) so

[1]) We found namely that on the circumference of an *arbitrary* circle $(\alpha') < (\alpha)$ and concentric with (α)

$$\overline{a}_m(\alpha') < \frac{\alpha A_m(\alpha)}{\alpha-\alpha'},$$

thus

$$\overline{a}(\alpha') \leq A(\alpha)\,;$$

further

$$\overline{a}(\alpha) \geq A(\alpha)\,,$$

so that

$$\overline{a}(\alpha) = A(\alpha),$$

by the further assumption that $\overline{a}(\alpha)$ should be a *continuous* function of α. Meanwhile, now that this continuity has appeared to hold universally, the question arises if it is possible to show the latter identity in a *direct* manner, without having recourse to the continuity of $\overline{a}(\alpha)$. This may in fact be done as follows. Let us again suppose the quantity $\overline{a}_m(\alpha)$ to be divided in the manner exhibited that *for this* the uniformity supposition is superfluous. The latter however served us to prove further that also the identity

$$a(\alpha) = A(\alpha) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (97)$$

is valid. But now we may observe that for the latter the following uniformity-supposition, wider than that of N°. 4, is sufficient:

A. Corresponding to an arbitrarily chosen number ε, as small as we please, there is an integral number N_ε such that at all points x of the closed domain (α)

$$|a_m(x)| < (a(\alpha) + \varepsilon)^m \quad for \quad m \geq N_\varepsilon \quad . \quad . \quad . \quad . \quad (98)$$

We shall not explain any further that this supposition is sufficient in order to deduce the equality (97): it is easy to see. From (97) however and the identity (96), which has appeared to be valid independent of any particular hypothesis, the equality (95) may be derived, so that the function $a(\alpha)$ is equal to $\overline{a}(\alpha)$ and thus continuous within the interval $(0, A)$ under the single condition denoted by *A*.

But it cannot yet be inferred from that condition that we have $\beta = \overline{\beta}$. If the latter is to be true as well we must have

$$\beta = \overline{\beta} = \alpha + \overline{a}(\alpha) = \alpha + a(\alpha),$$

that is the number β corresponding to α must be determined by formula (7), copied at the end of the preceding paragraph. This formula was obtained in N°. 4 and based upon the uniformity-supposition of that paragraph. But on examining the proof of the completeness-theorem we gave there it will appear that the formula is a consequence of the following supposition only:

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by (94) and, for a certain value of α, the integral number k so chosen that we again have (writing in this case $\overline{P}$ instead of P)

$$\overline{a}(\alpha) = \overline{\lim_{m=\infty}} \, [\overline{P}_m(\alpha)]^{\frac{1}{m}}$$

Now, for every value of n, $A_m(\alpha) > \overline{c}_{m,n}\,\alpha^n$, if $\overline{c}_{m,n}$ be the modulus of the coefficient $c_{m,n}$ in the power-series of $a_m(x)$. Let $\overline{c}_{m,p}\,\alpha^p$ be the term of maximum value in $\overline{P}_m(\alpha)$; the number p will in general vary with m, but we always have $\overline{c}_{m,p}\,\alpha^p > \overline{P}_m(\alpha) : km$ and thus also $A^m(\alpha) > \overline{P}_m(\alpha) : km$. From this it follows, since $\lim_{m=\infty} m^{\frac{1}{m}} = 1$,

$$A(\alpha) \equiv \overline{\lim_{m=\infty}} [A_m(\alpha)]^{\frac{1}{m}} \geq \overline{\lim_{m=\infty}} \, [\overline{P}_m(\alpha)]^{\frac{1}{m}} = \overline{a}(\alpha),$$

and thus, since $A(\alpha)$ cannot be greater than $\overline{a}(\alpha)$, we must have $A(\alpha) = \overline{a}(\alpha)$.

B. The maximum value $a(\alpha)$ of a_x *for the whole domain* (α) is equal to that *for the circumference.*

That the series

$$Pu = \sum_0^\infty {}_m \frac{a_m u^{(m)}}{m!}$$

converges for *all* functions belonging to the circle with radius $\alpha + a(\alpha)$ and that this circle is the *minimum* circle for which this property holds, therefore follows only from the hypothesis *B*. This latter however is not sufficient to derive from it the *uniform* convergence of the above series. But this at once follows if the hypothesis *A* is added to *B*; it is not necessary to explain this further since it may easily be derived from the proof given in N°. 4. The uniformity of the convergence is, however, of interest since in this case we may be certain that the transmuted of a function which is regular in (β) is itself a *regular* function, the domain of regularity being at least (α); in other words the *transmutation* in question is then always a *regular* one for the F. F. of functions belonging to (β) and the N. F. O. (α). For this reason we shall retain the hypothesis *A* for the extended theorem of Bourlet, treated in N°. 4, and thus *substitute for the uniformity-supposition of that paragraph the two suppositions A and B, which are independent of one another.*

That the uniformity-supposition of N°. 4 is narrower than the two suppositions *A* and *B* together is proved by the following example in which both *A* and *B* are satisfied, but not the former supposition. Let

$$a_m(x) = x^n - 1 \quad , \quad n = 2^{\mathscr{E}\left(\frac{\log m}{\log 2}\right)}$$

so that n depends on m in such a way as to be equal to the highest power of 2 that is contained in the number m; thus n passes through all integral powers of 2, but after every change of its value it remains constant for a certain number of m-values. For $\alpha < 1$, $a(\alpha) = 1$, for $\alpha > 1$, $a(\alpha) = \alpha$, since n is never greater than m but is equal to m for an infinite number of m-values. The quantity *A* is therefore, as the quantity *R* (see above) *infinite*, so that the series having the above quantities $a_m(x)$ as its coefficients is complete in any domain (α), however large. If we imagine $\alpha > 1$ the circle of radius unity lies wholly in the domain (α); at points on the circumference of this circle having as their argument

$$\frac{2\pi}{2^s},$$

where s is some integral number, all the coefficients $a_m(x)$ are zero from and after the value $m = 2^s$ and thus a_x is also zero at those points. But we can always find among these points such as satisfy the condition that, corresponding to a number Q, chosen arbitrarily great, there exists a value of $m > Q$ for which the quantity

$$|a^m(x)|^{\frac{1}{m}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (99)$$

is more than a certain fixed amount $\varkappa$ greater than the limit a_x of that quantity, which is zero: we need only choose the number s greater than Q and $m = 2^{s-1}$, and then $a_m(x)$ is equal to -2 in the points corresponding to those s-values, so that the quantity (99) has a value which is greater than unity. Thus we cannot assign a number m_ε independent of x such that in the *whole* domain (a)

$$|a_m(x)| < (a_x + \varepsilon)^m \quad , \quad \text{for } m \geq m_\varepsilon\,,$$

and this was the very uniformity-supposition of N°. 4. The supposition under A however is satisfied, because $|x^n - 1|$ is at most equal to 2 for points of the circular domain of radius unity and thus less than $(1 + \varepsilon)^m$, that is less than

$$[a(1) + \varepsilon]^m$$

where ε may be prescribed arbitrarily small, if only m be chosen large enough. Again the supposition B is satisfied: in the first place we immediately see that at all points of a circle arbitrarily little smaller than that of radius unity the quantity a_x is equal to 1 so that there are in an arbitrarily small neighbourhood of the circumference of the latter circle points where a_x is equal to the upper limit of that quantity for the closed domain of that circle: this, though not exactly the same as the supposition B, agrees with it as to its consequences, viz. the validity of formula (7), and it might therefore be substituted for the supposition B. But also for the circumference of the circle (1) itself the upper limit of a_x is equal to 1. For there corresponds to any arbitrarily chosen number ε a prime number p such that at a point on the circumference of that circle having the argument

$$\varphi = \frac{\pi}{p}$$

the argument of

$$x^{2^k} = e^{i\frac{2k\pi}{p}}\,, \text{ i. e. } \frac{2k\pi}{p}\,,$$

that is $\frac{2^k\pi}{p}$, differs from π by less than ε for an infinite number of k-values.

For this purpose, we need only choose p so great that $p\varepsilon > \pi$. The congruence

$$2^k \equiv 1 \ (\text{mod.}\ p)$$

can be satisfied according to a theorem of FERMAT for all k-values being a multiple of $p-1$; that is, for all those values we have

$$2^k = (2l+1)p + 1$$

or

$$\frac{2^k \pi}{p} = (2l+1)\pi + \frac{\pi}{p},$$

so that in connection with the above choice of the number p the required condition is satisfied. Thus there are always points on the circumference of the circle (1) for which the quantity

$$|x^{2^k} - 1|$$

differs from the value 2 by less than an arbitrarily small amount so that it is for instance greater than 1. Since in $a_m(x) = x^{n-1}$ the number n assumes *all* integral powers of 2 as a value, $a_m(x)$ is in the just-mentioned points for an infinite number of m-values greater than 1, so that $a_x = 1$ in all those points.

We have thus constructed an example in which the conditions A and B are satisfied, but the uniformity-supposition of N° 4 is not satisfied. As, now, regards the two former suppositions, we should like to have an example in which either one of them or both were not realized. But we have not succeeded as yet in constructing any of the kind, nor, on the contrary, in proving that this would be impossible, in which last case A and B would hold *universally*. If there be a point on the circumference of the circle (a) such that at that point the quantity $|a_m|^{\frac{1}{m}}$ be for an infinite number of m-values, $m_1, m_2, \ldots m_n, \ldots$ equal to the maximum

$$A_m^{\frac{1}{m}}(a)$$

of the same quantity on the circumference of (a), and if at the same time the upper limit of the *partial* sequence

$$A_{m_1}^{\frac{1}{m_1}}(a)\ ,\quad A_{m_2}^{\frac{1}{m_2}}(a)\,,\ldots,\ A_{m_n}^{\frac{1}{m_n}}(a)\,,\ldots$$

be equal to that of the *complete* one, then at the point mentioned $a_x = A(a)$, and thus $a(a) = A(a)$ so that both A and B are satisfied. In constructing a pathological example as mentioned we should therefore take care that there is no such point on the circumference of (a). But this is by no means sufficient. For it may be possible that, corresponding to an arbitrarily chosen number ε there exists a point in the domain (α), where the quantity $|a_m|^{\frac{1}{m}}$ for an *infinite* number of m-values is greater than $A(\alpha)-\varepsilon$, and also in that case the upper limit $a(\alpha)$ of a_x is equal to $A(\alpha)$. The condition A is satisfied then and it will be quite an ordinary thing if the condition B is also realized, though we do no longer see the necessity of it. If, on the contrary, there can be assigned an amount $\varkappa$ such that in no point of the domain (α) the quantity $|a_m|^{\frac{1}{m}}$ is greater than $A(\alpha)-\varkappa$ for an *infinite* number of m-values, then $a(\alpha)$ is certainly not greater than $A(\alpha)-\varkappa$. The condition A is not satisfied now; for if this were the case, then, since $A_m(\alpha)$ is the modulus of $a_m(x)$ for at least one point at the circumference of (α), there would, corresponding to any arbitrarily small quantity ε, be an integer N_ε such that, from and after $m = N_\varepsilon$ we should have $A_m(\alpha) < [a(\alpha) + \varepsilon]^m$, or

$$A_m^{\frac{1}{m}}(\alpha) < a(\alpha) + \varepsilon \leq A(\alpha) - \varkappa + \varepsilon ,$$

and this is impossible, if ε be chosen less than $\varkappa$, since $A(\alpha)$ is the upper limit for $m = \infty$, of $A_m^{\frac{1}{m}}(\alpha)$. As to the condition B it might or might not be satisfied.

We have made the preceding observations in order to elucidate a few more cases as considered here. Meanwhile such cases, if they are possible at all, may undoubtedly be regarded as pathological ones rarely occurring in practice: the observations made may be able to make this even clearer.

With this our considerations on complete transmutation have come to an end.