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Mathematics. — *"On linear inner limiting sets"* [1]). By Prof. L. E. J. Brouwer.

(Communicated in the meeting of April 27, 1917).

We consider an inner limiting set I, determined inside the unit interval as the intersection (greatest common divisor) of the sets of (non-overlapping) intervals $i_1, i_2, \ldots$, each point of $i_{\nu+1}$ being also a point of i_ν. Then the complementary set C of I with regard to the closed unit interval is the union (common measure) of the closed sets $a_1, a_2, \ldots$, each $a_{\nu+1}$ containing a_ν. *We shall suppose that I as well as C is uncountable in each sub-interval of the unit interval*; then we may assume that each a_ν contains as its nucleus a perfect set p_ν. The difference of a_ν and p_ν will be indicated by v_ν, the complementary set of p_ν, considered as a set of intervals, by u_ν, and the inner limiting set determined as the intersection of $u_1, u_2, \ldots$ by U. Then the points of each u_ν lie everywhere dense, and each u_ν is a set of order-type η of intervals, whose length does not exceed a certain value ε_ν having the limit zero for indefinitely increasing ν.

Let us assume that we dispose of such a set j_ν of order-type η of intervals each being an element of one of the sets $u_\nu, u_{\nu+1}, u_{\nu+2}, \ldots$, that j_ν contains no point of v_ν, but does contain all points of U not belonging to v_ν. We shall indicate a method leading from j_ν to such a set $j_{\nu+1}$ of order-type η of intervals each lying inside an interval of j_ν, and being an element of one of the sets $u_{\nu+1}, u_{\nu+2}, u_{\nu+3}, \ldots$, that $j_{\nu+1}$ contains no point of $v_{\nu+1}$, but does contain all points of U not belonging to $v_{\nu+1}$, each interval of j_ν containing a subset of $j_{\nu+1}$ of order-type η.

Let AB be an arbitrary element of j_ν being at the same time an element of u_μ ($\mu \geq \nu$), let ρ be the subset of $u_{\mu+1}$ lying inside AB,

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[1]) To the last footnote of my former communication on inner limiting sets (these *Proceedings* XVIII, p. 49) must be added that the changed form in which Schoenflies has referred to my reasoning (applying it to a special case only, and deducing the general theorem from this special case) is irrelevant. The error is contained in the sentence (Entwickelung der Mengenlehre I, p. 359, line 5—8 from the top): "Ist namlich P irgend eine abzählbare Menge, die nicht dicht in bezug auf eine perfekte Menge ist, und geht man durch Hinzufügung samtlicher Grenzpunkte zu einer abgeschlossenen Menge Q über, so kann diese keinen perfekten Bestandteil enthalten ist also ebenfalls abzählbar".

and w the set of intervals which is left from r after destroying all points of $v_{\nu+1}$ contained in r. Let PQ be an arbitrary element of w, s_ρ the set of intervals determined as the intersection of u_ρ and PQ, t_ρ the set of intervals which is left from s_ρ after destroying its first and its last element, *in so far those elements exist*, γ the set of intervals determined as the union of $t_1, t_2, t_3, \ldots$, and φ the set of intervals which is generated by constructing in each element of w a set of intervals in the same way as γ has been constructed in PQ. Then the required set of intervals $j_{\nu+1}$ is generated by constructing in each element of j_ν a set of intervals in the same way as φ has been constructed in AB.

If we understand by u_0 as well as by j_0 the unit interval itself, then we arrive from j_0 at j_1 by the same process which has led us from j_ν to $j_{\nu+1}$.

The inner limiting set determined as the intersection of $j_1, j_2, \ldots$ contains all points of U belonging to none of the sets v_ν, so a fortiori all points of U belonging to none of the sets a_ν, so also all points of the unit interval belonging to none of the sets a_ν. As, on the other hand, this inner limiting set can neither contain a point of a v_ν, nor (as a subset of U) a point of a p_ν, it finally cannot contain a point of a a_ν either. *So it is identical to the complementary set of C, i.e. to I.*

If we construct a ternal scale on the unit interval, and if (designing by ϱ_ν an arbitrary finite series of digits 0, 1 or 2, among which ν digits 1 occur) we understand by $d_{\nu+1}$ the set of the intervals l_{ρ_ν} whose end-points have the coordinates $\cdot\varrho_\nu 1$ and $\cdot\varrho_\nu 2$, then we can first represent the set of intervals j_1, biuniformly and with invariant relations of order, on the set of intervals d_1; thereupon we can in each interval of j_1 represent the subset of j_2 contained in it, biuniformly and with invariant relations of order, on the subset of d_2 contained in the corresponding interval of d_1; and so on. In this way we determine a continuous one-one transformation of the unit interval in itself by which I passes into the set τ_2 of the points expressible in the ternal scale by means of a sequence of digits containing an infinite number of digits 1, whilst C passes into the set τ_1 of the points expressible in the ternal scale by means of a sequence of digits containing only a finite number of digits 1. Thus, indicating the *geometric types*[1]) of τ_1 and τ_2 by $\bar{\mu}$ and $\bar{\nu}$ respectively, we have proved the following

THEOREM 1. *Each inner limiting set contained in a linear interval,*

1) Comp. these *Proceedings* XV, p. 1262.

and, as well as its complementary set, uncountable in each sub-interval, possesses the geometric type $\bar{\nu}$*, and its complementary set possesses the geometric type* $\bar{\mu}$*.*

Let H and K be two arbitrary points of τ_1, we can choose ν in such a way that neither H nor K is an endpoint of an interval of d_ν. Let us indicate the set of points which is the complementary set of d_ν, by e_ν, and the set whose elements are the intervals of d_ν, and the points of e_ν, by r_ν. Then we can construct a one-one transformation of d_ν and e_ν each in itself, by which the relations of order between the elements of r_ν remain invariant, and H passes into K. This transformation can be extended to a continuous one-one transformation of the unit interval in itself, for which the subsets of τ_1 and τ_2 contained in corresponding intervals of d_ν, correspond to each other. We thus have generated a continuous one-one transformation of the unit interval in itself, by which τ_1 passes into itself, and the point H chosen arbitrarily in τ_1, passes into the point K chosen likewise arbitrarily in τ_1, *so that* τ_1 *is a homogeneous set of points.*

Let H and K be two arbitrary points of τ_2 contained in the intervals h_ν and k_ν of d_ν respectively, we can construct a one-one transformation of the set of intervals d_1 in itself leaving invariant the relations of order, by which h_1 passes into k_1; this transformation can be extended to a one-one transformation of the set of intervals d_2 in itself leaving invariant the relations of order, by which h_2 passes into k_2; continuing indefinitely in this way, we generate a continuous one-one transformation of the unit interval in itself, by which each d_ν, so also τ_2 passes into itself, and the point H chosen arbitrarily in τ_2, passes into the point K chosen likewise arbitrarily in τ_2, *so that* τ_2 *too is a homogeneous set of points*, and we have proved the following

THEOREM 2. *Each inner limiting set contained in a linear interval, and, as well as its complementary set, uncountable in each sub-interval, is homogeneous, and its complementary set is likewise homogeneous.*