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Astronomy. — *"Outlines of a new theory of Jupiter's satellites."* By Prof. W. DE SITTER.

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1. *Fundamental principles of the theory.*

The great difficulty in the theory of the four old satellites of Jupiter arises from the mutual commensurability of the mean motions of the three inner ones. The fourth satellite is not affected by this, and, so far as periodic inequalities are concerned, its theory does not give rise to particular difficulties, and is in many respects similar to the lunar theory, only much simpler, since the ratio of the month and the year, which for our moon is $^1/_{13}$, is for the fourth satellite only about $^1/_{260}$. The secular perturbations of the equations of the centre of the four satellites are however so intimately connected with each other, that it is not possible to keep the fourth satellite apart, but the theory of the *four* satellites must be treated as one whole.

I denote the satellites by the suffixes 1 to 4, and I put:

a_i the semi-axis major,
n_i the mean motion,
w_i the true orbit-longitude,
λ_i the mean longitude,
l_i the mean anomaly,
f_i the true anomaly,
r_i the radius-vector,
e_i the excentricity,
π_i the longitude of the perijove.

If now we put[1])

$$(n_2 - n_3)\, t = \tau\,,$$

then, if an appropriate zero of longitude and time is chosen, we have

$$\lambda_1 = (4 - \varkappa)\, \tau,$$
$$\lambda_2 = (2 - \varkappa)\, \tau + 180^\circ,$$
$$\lambda_3 = (1 - \varkappa)\, \tau,$$

1) This τ differs 180° from the angle which was called τ in my previous work. See these Proceedings Feb. and March 1908: "On the Masses and Elements of Jupiter's satellites, and the mass of the system." (Vol. X, pp. 653 and 710).

where $\varkappa$ is a small quantity whose value is approximately

$$\varkappa = \tfrac{1}{69}.$$

We will consider $\varkappa$ as a small quantity of the *first order*. The masses of the satellites then are of the *second* order (the largest is about $0{\cdot}8 \times 10^{-4}$).

Generally we put, for all satellites including the fourth

$$\lambda_i = (c_i - \varkappa)\,\tau + \lambda_{i0} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

We thus have, the value of c_4 being only given approximately,

$$c_1 = 4 \quad , \quad c_2 = 2 \quad , \quad c_3 = 1 \quad , \quad c_4 = 0{\cdot}437\,.$$

If we start with uniform motion in a circle as a first approximation, the inequalities can according to their periods be divided into four sharply separated groups.[1])

I. The inequalities of the first group have periods not exceeding 17 days. They can be subdivided into three sub-groups:

I*a*. The equations of the centre, which are

$$\delta w_i = 2 \sum \tau_{ij}\, \varepsilon_j \sin(\lambda_i - \varpi_j)\,,$$

$$\delta r_i = - a_i \sum \tau_{ij}\, \varepsilon_j \cos(\lambda_i - \varpi_j)\,,$$

where the sums are to be taken for the values of j from 1 to 4, and where ε_i and ϖ_i are the "own" excentricities and perijoves.

I*b*. The "great" inequalities. These are approximately:

$$\delta w_i = 2 e_i \sin c_i \tau$$

$$\delta r_i = - a_i e_i \cos c_i \tau$$

I*c*. Other inequalities of short periods.

II. The inequalities of the second group have periods between 400 and 500 days. Their expressions are

$$\delta w_i = \sum_j \varkappa_{ij} \sin \varphi_j,$$

with [2])

$$\varphi_i = \varkappa\tau + \varpi_i.$$

III. The libration has a period of about 7 years.

IV. The inequalities of the fourth group have periods of more than 12 years.

The inequalities I*b*, II and III arise out of the mutual commensurability of the mean motions. In a previous communication [3])

[1]) See also "Elements and Masses", p. 655, where however the libration (III) is left outside the groups and the group IV is numbered III.

[2]) These φ_i differ 180° from the angles so called in "Elements and Masses"; so the coefficients $\varkappa_{ij}$ here have the other sign.

[3]) "On the periodic solutions of a special case of the problem of four bodies", these Proceedings, Feb. 1909, Vol XI, p. 682.

I have pointed out that for the three inner satellites a periodic solution of the second kind (in which the own exentricities ε_i are zero, and the great inequalities Ib appear as excentricities) is a very good approximation to the true motion, in fact much better than the undisturbed Keplerian motion. The mean *anomalies* in the periodic solution are

$$l_i = c_i \tau. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The longitudes of the perijoves are given by

$$\pi_i = \lambda_i - l_i \, ,$$

and consequently their mean value is

$$\pi_i = - \varkappa \tau + \pi_{i_0} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The perijoves thus have a mean motion common to the three satellites. In the theory here outlined, the equations (1), (2), (3) are taken as a first approximation, or "intermediary orbit", also for the fourth satellite. The solution, considered as a whole, is then no longer periodic, since c_4 is mutually incommensurable with c_1, c_2, c_3. It is however not the periodicity which makes this solution such a good first approximation, but the moving perijove, combined with the circumstance that the "induced" equations of the centre Ib are, for the inner satellites, larger than the own, or "free" ones Ia. For IV the contrary is true. The own excentricity of IV is comparatively large ($^1/_{136}$) and the induced one is entirely negligible. For IV the Keplerian motion with a fixed perijove is indeed a better approximation. In the ordinary theory this approximation is also used for the three other satellites, where it is not appropriate. Here the method which is the best for I, II, III, is forced upon IV. This of course involves some drawbacks, but these are in my opinion not very serious and considerably smaller than those arising in the ordinary theory from the fact that the inequalities Ib appear as "perturbations", and must consequently be treated as quantities of the order of the masses (i.e. by our method of reckoning of the *second* order), while they actually are of the first order (the largest is about $^1/_{107}$).

Briefly the new theory may be stated thus: We start from an intermediary orbit in which the equations (2), (3) are rigorously satisfied. The radius-vector and the true anomaly are then computed from the mean anomaly and the excentricity (which is constant), by the ordinary formulas of Keplerian motion:

$$u_i - e_i \sin u_i = l_i$$

$$\tan \tfrac{1}{2} f_i = \sqrt{\frac{1+e_i}{1-e_i}} \tan \tfrac{1}{2} u_i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$r_i = \frac{a_i (1-e_i^2)}{1+e_i \cos f_i}$$

The true orbit-longitude then is

$$w_i = f_i + \pi_i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

This intermediary orbit is, also for the three inner satellites, not the complete periodic orbit, but only contains its leading terms. To get this intermediary orbit we must 1 restrict the perturbative function to a certain part of it (viz: the "secular" and the "critical" parts), and 2 we must take initial values, or constants of integration, which satisfy certain conditions. The complete solution is then derived by adding to the intermediary orbit:

1. "perturbations" which arise from the parts of the perturbative function that were at first neglected;

2. "variations" which are due to the fact that the actual constants of integration do not exactly satisfy the conditions for the intermediary orbit.

Of these the variations are the most important. To get these we must form the variational equations. These lead to a system of equations entirely similar to those which are used in the treatment of secular perturbations by the method of LAGRANGE. The resulting determinant has 5 roots $\beta_1 \ldots \beta_5$, corresponding to the four own perijoves ϖ_i, and the argument of the libration ϖ_5 respectively. The inequalities in longitude and radius-vector are then given by formulas which, if we restrict ourselves to the first order, assume the form

$$\left.\begin{aligned} \delta w_i &= \sum_j W_{ij}\, \varepsilon_j \sin(\lambda_i - \varpi_j) + \sum_j W'_{ij}\, \varepsilon_j \sin \varphi_j, \\ \delta r_i &= \sum_j R_{ij}\, \varepsilon_j \cos(\lambda_i - \varpi_j), \end{aligned}\right\} \quad . \quad . \quad (6)$$

where

$$\varphi_i = \beta_i \tau + \varphi_{i_0} = \varkappa \tau + \varpi_i$$

as above, and j assumes the values 1 to 5. These formulas include not only the free equations of the centre I*a*, *but also the inequalities of group* II (arguments φ_1—φ_4) *and the libration* III (argument φ_5).

As to the perturbations: by the introduction of c_i instead of n_i we have realised that *there are no small divisors.* In the ordinary theory small divisors appear in the inequalities I*b*, II and III. Of these I*b* is already included in the intermediary orbit; II and III appear as variations, which are treated by the method of LAGRANGE, together with Ia, which is also in the ordinary theory treated in the same way. We can say that all small divisors have been concentrated in the equations of condition for the constants of integration of the intermediary orbit. Once these equations have been solved, the small divisors have disappeared, and they cannot reappear in subsequent approximations.

2. *Formation of the differential equations.*

We take an arbitrary system of coordinate axes through the centre of Jupiter, and we put:

f = the Gaussian constant of attraction,
m_0 = the mass of Jupiter,
m_i = the mass of the body with index i, expressed in m_0 as unit,
s_i = the latitude of the body i referred to the plane of Jupiter's equator,
r_i = the distance of the body i from Jupiter,
Δ_{ij} = the distance between the bodies i and j,
V_{ij} = the angle between the radii-vectores r_i and r_j,
$180^\circ - \psi$ = the ascending node of Jupiter's equator on the plane of (xy),
π = the inclination of this equator on the same plane,
J, K = two constants connected with the compression of Jupiter,
b = the equatorial radius of Jupiter,

and further

$$\alpha = \sin\pi \sin\psi,$$
$$\beta = \sin\pi \cos\psi,$$
$$\gamma = \cos\pi.$$

Then the equations of motion are

$$\frac{d^2x_i}{dt^2} = \frac{\partial\Omega_i}{\partial x_i}, \quad \frac{d^2y_i}{dt^2} = \frac{\partial\Omega_i}{\partial y_i}, \quad \frac{d^2z_i}{dt^2} = \frac{\partial\Omega_i}{\partial z_i},$$

where

$$\left.\begin{aligned} \Omega_i = {} & fm_0(1+m_i)\left\{\frac{1}{r_i} + \tfrac{1}{3}\frac{Jb^2}{r_i^3}(1-3\sin^2 s_i) + \tfrac{1}{10}\frac{Kb^4}{r_i^5}(1-10\sin^2 s_i + \tfrac{35}{3}\sin^4 s_i) + \ldots\right\} \\ & + fm_0 \sum_j m_j \left\{\frac{1}{\Delta_{ij}} - \frac{r_i}{r_j^2}\cos V_{ij}\left[1 + \frac{Jb^2}{r_j^2}(1-5\sin^2 s_j) + \frac{Kb^4}{r_j^4}(\tfrac{1}{2} - 7\sin^2 s_j + \tfrac{21}{2}\sin^4 s_j) + \ldots\right]\right. \\ & \left. - 2\,\frac{\alpha x_i + \beta y_i + \gamma z_i}{r_j^2}\sin s_j\left[\frac{Jb^2}{r_j^2} + \frac{Kb^4}{r_j^4}(1 - \tfrac{7}{3}\sin^2 s_j) + \ldots\right]\right\}. \end{aligned}\right\} \quad (7)$$

The sums are to be taken over the values 1, 2, 3, 4 of j, with the exception of $j=i$, and further over the indices j which refer to the sun, Saturn, etc.

If the plane of (xy) is chosen near the mean position of the equator, then α and β are very small, and γ is very nearly equal to unity. The latitudes s_i of the satellites are very small (the largest does not exceed 0°.7), and also for bodies outside the system of Jupiter s_j is small (e.g. for the sun it never exceeds 3°.1).

Regarding the value (7) of Ω_i we may remark, that the terms multiplied with J and K in the complementary part of the perturbative function (second line of the formula) are here given for the first time. These terms are neglected by LAPLACE, and all subsequent investigators adopt LAPLACE's perturbative function without any criticism. LAPLACE was perfectly right, for these terms are beyond the limit of accuracy which he had set himself; but SOUILLART, who includes other terms of the same, and higher orders, ought also to have included these terms.

If now we put

$$x_i' = \frac{dx_i}{dt}, \quad y_i' = \frac{dy_i}{dt}, \quad z_i' = \frac{dz_i}{dt},$$

$$T_i = \tfrac{1}{2}(x_i'^2 + y_i'^2 + z_i'^2),$$

$$F_i = T_i - \Omega_i,$$

then the equations become

$$\frac{dx_i}{dt} = \frac{\partial F_i}{\partial x_i'}, \quad \frac{dx_i'}{dt} = -\frac{\partial F_i}{\partial x_i},$$

and similarly for the other two coordinates.

We now introduce the canonical elements of DELANNAY

$$l_i, \quad g_i, \quad \vartheta_i, \quad L_i, \quad G_i, \quad \Theta_i,$$

where

$$L_i = \beta_i \sqrt{a_i}, \quad G_i = L_i \sqrt{1-e_i^2}, \quad \Theta_i = G_i \cos i_i,$$

I now put[1])

$$S_i' = \Omega_i - \frac{\beta_i^2}{r_i} \quad \dots\dots\dots \quad (8)$$

Then we have

$$F_i = -\frac{\beta_i^4}{2L_i^2} - S_i'$$

and the equations become

[1]) See: *On Canonical elements*, these Proceedings Sept. 1913, Vol. XVI pages 285 and 287.

$$\frac{dl_i}{dt} = \frac{\beta_i^4}{L_i^3} - \frac{\partial S_i'}{\partial L_i},$$

$$\frac{dg_i}{dt} = -\frac{\partial S_i'}{\partial G_i}, \text{ etc.}$$

It is usual to take $\beta_i^2 = fm_0(1 + m_i)$. By keeping β_i indeterminate we have a parameter at our disposal, which can afterwards be so chosen that the intermediary orbit assumes the desired form.

I now introduce instead of l_i, g_i, ϑ_i, L_i, G_i, Θ_i the canonical set

$$\lambda_i = l_i + g_i + \vartheta_i \quad , \quad l_i \quad , \quad \psi_i = -\vartheta_i \ ,$$
$$G_i \quad , \quad H_i = L_i - G_i \quad , \quad \Psi_i = G_i - \Theta_i$$

Then we have

$$\frac{\partial S_i'}{\partial G_i} = \left(\frac{\partial S_i'}{\partial G_i}\right) - \tfrac{1}{2}\frac{\gamma_i}{G_i}\frac{\partial S_i'}{\partial \gamma_i},$$

where

$$\gamma_i = 2 \sin \tfrac{1}{2} i_i = \sqrt{\frac{2\Psi_i}{G_i}}$$

The second term is of the second degree in γ_i, and consequently very small. If now we put

$$\lambda_i = \lambda_i' + \sigma_i,$$

and if we determine σ_i by

$$\frac{d\sigma_i}{dt} = \tfrac{1}{2}\frac{\gamma_i}{G_i}\frac{\partial S_i'}{\partial \gamma_i}, \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (9)$$

then we have

$$\frac{d\lambda_i'}{dt} = \frac{\beta_i^4}{L_i^3} - \left(\frac{\partial S_i'}{\partial G_i}\right),$$

where the parentheses denote that S'_i must *not* be differentiated with respect to G_i so far as it depends on G_i through the inclinations γ_i. This being agreed upon, we can omit the accent of λ'_i and the parentheses of $\left(\frac{\partial S'_i}{\partial G_i}\right)$, if we apply to the value of λ_i so determined the correction

$$\delta\lambda_i = \sigma_i \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (10)$$

where σ_i is determined from (9).

In the theory of the inclinations the approximate commensurability does not give rise to particular difficulties. For this theory the most important point is to choose the plane of (xy) so that the inclinations of the satellites and of the equator always remain small. I will not enter upon this problem here, and I will further exclusively consider the four elements

$$\lambda_i \quad , \quad l_i \quad , \quad G_i, \qquad H_i.$$

We can choose a unit of time so that $t = \tau$. This unit is about 9/8 of a day.

Then we have

$$\frac{dl_i}{d\tau} = \frac{\beta_i^4}{L_i^3} - \frac{\partial S_i'}{\partial H_i} \quad , \quad \frac{d\lambda_i}{d\tau} = \frac{\beta_i^4}{L_i^3} - \frac{\partial S_i'}{\partial G_i}.$$

As a general rule I will denote by $[X]$ the non-periodic part of a function X. We must have

$$\left[\frac{dl_i}{d\tau}\right] = c_i \quad , \quad \left[\frac{d\lambda_i}{dt}\right] = c_i - \varkappa.$$

This can be realised in two ways. We can take

$$\left[\frac{\beta_i^4}{L_i^3}\right] = c_i \quad , \quad \left[\frac{\partial S_i'}{\partial H_i}\right] = 0 \quad , \quad \left[\frac{\partial S_i'}{\partial G_i}\right] = \varkappa. \quad . \quad . \quad . \quad (A)$$

or

$$\left[\frac{\beta_i^4}{L_i^3}\right] = c_i - \varkappa \quad , \quad \left[\frac{\partial S_i'}{\partial H_i}\right] = -\varkappa \quad , \quad \left[\frac{\partial S_i'}{\partial G_i}\right] = 0 \quad . \qquad (B)$$

Now the perturbative function is given as a development (e.g. by NEWCOMB's method) in terms of λ_i, l_i, a_i and e_i, and we have

$$\frac{\partial S_i'}{\partial H_i} = \frac{2}{\beta \sqrt{a_i}} a_i \frac{\partial S_i'}{\partial a_i} + \frac{\cos^2 \varphi_i}{\beta \sqrt{a_i}} \frac{1}{e_i} \frac{\partial S_i'}{\partial e_i},$$

$$\frac{\partial S_i'}{\partial G_i} = \frac{2}{\beta \sqrt{a_i}} a_i \frac{\partial S_i'}{\partial a_i} - \frac{\cos \varphi_i \tan \frac{1}{2} \varphi_i}{\beta \sqrt{a_i}} \frac{\partial S_i'}{\partial e_i},$$

where we have put $e_i = \sin \varphi_i$.

In the case (A) we thus find that $a_i \frac{\partial S_i'}{\partial a_i}$ is of the order of $\varkappa$, in the case (B) it is of the order of $\varkappa . e_i^2$, in both cases $\frac{\partial S_i'}{\partial e_i}$ is of the order of $\varkappa . e_i$ It thus appears that the method (B) is preferable.

Instead of $H_i = 2 L_i \sin^2 \frac{1}{2}\varphi_i$ I now introduce

$$\eta_i = 2 \sin \tfrac{1}{2} \varphi_i.$$

We have

$$\frac{da_i}{a_i} = \frac{2}{\beta_i \sqrt{a_i}} (dH_i + dG_i),$$

$$d\eta_i = \frac{1}{\beta_i \sqrt{a_i}} \left(\frac{\cos \varphi}{\eta_i} dH_i - \tfrac{1}{2} \eta_i \, dG_i \right).$$

We find everywhere the denominator $\beta_i \sqrt{a_i}$. We can thus simplify our formulas by putting

$$R_i = \frac{S_i}{\beta_i \sqrt{\mathrm{a}_i}},$$

where a_i is a constant, which, in accordance with (B), is determined by

$$\left[\frac{\beta_i^4}{L_i^3}\right] = \frac{\beta_i}{\mathrm{a}_i^{3/2}} = c_i - \varkappa.$$

Further we put

$$\frac{\beta_i^4}{L_i^3} = (c_i - \varkappa)(1 + \nu_i).$$

Consequently ν_i is purely periodic, and we have

$$\mathrm{a}_i^{3/2} = a_i^{3/2}(1 + \nu_i).$$

Therefore

$$d\nu_i = -\tfrac{3}{2}(1 + \nu_i)\frac{da_i}{a_i}.$$

If then we introduce again

$$\pi_i = \lambda_i - l_i,$$

where of course π_i requires the same correction

$$d\pi_i = \sigma_i \quad \ldots\ldots\ldots\ldots \quad (11)$$

as λ_i, and if we suppose the perturbative function R_i expressed in the variables λ_i, π_i, a_i and η_i, then the equations become

$$\left.\begin{aligned}
\frac{d\lambda_i}{d\tau} &= (c_i-\varkappa)(1+\nu_i) - 2(1+\nu_i)^{\frac{1}{3}} a_i \frac{\partial R_i}{\partial a_i} + \tfrac{1}{2}(1+\nu_i)^{\frac{1}{3}} \eta_i \frac{\partial R_i}{\partial \eta_i}, \\
\frac{dl_i}{d\tau} &= (c_i-\varkappa)(1+\nu_i) - 2(1+\nu_i)^{\frac{1}{3}} a_i \frac{\partial R_i}{\partial a_i} - (1+\nu_i)^{\frac{1}{3}} \frac{\cos\varphi_i}{\eta_i} \frac{\partial R_i}{\partial \eta_i}, \\
\frac{d\pi_i}{d\tau} &= (1+\nu_i)^{\frac{1}{3}} \frac{1}{\eta_i} \frac{\partial R_i}{\partial \eta_i}, \\
\frac{d\nu_i}{d\tau} &= -3(1+\nu_i)^{\frac{4}{3}} \frac{\partial R_i}{\partial \lambda_i}, \\
\frac{d\eta_i}{d\tau} &= -(1+\nu_i)^{\frac{1}{3}} \frac{1}{\eta_i} \frac{\partial R_i}{\partial \pi_i} - \tfrac{1}{2}(1+\nu_i)^{\frac{1}{3}} \eta_i \frac{\partial R_i}{\partial \lambda_i}.
\end{aligned}\right\} \quad \ldots \quad (12)$$

Of the first three equations we can arbitrarily choose two for use in the computations. The simplest formulas are found if we use λ_i and π_i.

Instead of β_i (a_i) I now introduce the constant μ_i, which is determined by

$$\mu_i = 1 - \frac{fm_0(1+m_i)}{\mathrm{a}_i^3 (c_i-\varkappa)^2} \quad \ldots\ldots\ldots \quad (13)$$

Further we put

$$\varrho_i = \frac{r_i}{\mathrm{a}_i} = (1+\nu_i)^{-\frac{2}{3}} \frac{r_i}{a_i} . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Then the terms of R_i which are independent of the latitudes become

$$\left. \begin{aligned} R_i = & - \frac{(c_i - \varkappa)\,\mu_i}{\varrho_i} + (c_i - \varkappa)(1-\mu_i)\left[\tfrac{1}{3}\frac{J_i}{\varrho_i^3} + \tfrac{1}{10}\frac{K_i}{\varrho_i^5}\right] + \\ & + \frac{(c_i-\varkappa)(1-\mu_i)}{1+m_i} \sum_j m_j \left\{ \frac{\mathrm{a}_i}{\Delta_{ij}} - \frac{\mathrm{a}_i^2}{\mathrm{a}_j^{2}} \frac{\varrho_i}{\varrho_j^2} \cos V_{ij} \left[1 + \frac{J_j}{\varrho_j^2} + \tfrac{1}{2}\frac{K_j}{\varrho_j^4} \right] \right\}, \end{aligned} \right\} \quad . \quad (15)$$

where we have put

$$J_i = \frac{Jb^2}{\mathrm{a}_i^2} \quad , \quad K_i = \frac{Kb^4}{\mathrm{a}_i^4} .$$

3. *The intermediary orbit.*

The perturbative function R_i consists of a series of terms of the form

$$K \cos D,$$
$$D = p\lambda_j - p\lambda_i + ql_i + q'l'_j$$

To get the intermediary orbit we take

$$R_i = [R_i].$$

Hence the argument D must satisfy the condition

$$(p + q')\, c_j + (q - p)\, c_i = 0.$$

The function $[R_i]$ includes the "secular" part of R_i, for which $q = p$, $q' = -p$, and the "critical" part, which becomes non-periodic as a consequence of the commensurability of c_i and c_j.

Since $\frac{\partial R_i}{\partial \lambda_i}$ and $\frac{\partial R_i}{\partial \pi_i}$ contain only sines, we shall have

$$\frac{d\nu_i}{d\tau} = 0 \quad , \quad \frac{d\eta_i}{d\tau} = 0$$

if

$$\lambda_{j_0} - \lambda_{i_0} = k \times 180^\circ$$

or

$$\pi_{i_0} = k \times 180^\circ, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

k being any integer number.

If we count the time from the epoch of an opposition of II and III, and the longitudes from the longitude of III at that epoch, then we have

$$\pi_{10} = 0 \quad , \quad \pi_{20} = 180^\circ \quad , \quad \pi_{30} = 0.$$

As to the fourth satellite, the condition $\lambda_{40} = 0$ or 180° is generally not satisfied, since there is no relation between the longitudes of the fourth satellite and of the others, as there is in the case of the three inner satellites. It is however easy to choose as origin an opposition of II and III at which the condition is very nearly satisfied. [1])

Then for all satellites η_i and ν_i are constant in the intermediary orbit. We take

$$\nu_i = 0, \quad \text{and consequently} \quad a_i = a_i \quad ; \quad \eta_i = \bar{\eta}_i \quad . \quad . \quad . \quad (17)$$

Then also $a_i \dfrac{\partial [R_i]}{\partial a_i}$ and $\dfrac{\partial [R_i]}{\partial \eta_i}$ are constants. These must be determined so that

$$\frac{\partial [R_i]}{\partial \bar{\eta}_i} = -\varkappa \bar{\eta}_i, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

$$a_i \frac{\partial [R_i]}{\partial a_i} = -\tfrac{1}{4} \varkappa \bar{\eta}_i^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Then we have

$$l_i = c_i \tau \quad , \quad \pi_i = -\varkappa\tau + \pi_{i_0} \quad , \quad \lambda_i = (c_i - \varkappa)\, \tau + \pi_{i_0} \quad . \quad (20)$$

The radius-vector r_i and the true orbit-longitude w_i are now determined by (4) and (5), and we thus see that the intermediary orbit is a Keplerian ellipse with the constant semi-axis a_i, the constant excentricity $e_i = \sin \bar{\varphi}_i$, determined by $2 \sin \frac{1}{2} \bar{\varphi}_i = \bar{\eta}_i$, and the mean anomaly $l_i = c_i \tau$, and these ellipses rotate in their plane with the angular velocity $-\varkappa$, common to all satellites.

The conditions (18) and (19) serve to determine the two parameters μ_i and $\bar{\eta}_i$. For the inner satellites this intermediary orbit is, as has already been pointed out, a very good approximation, better than the fixed Keplerian ellipse. For IV the excentricity as determined from (18) is extremely small, and the intermediary orbit consequently differs very little from a circle described with the uniform velocity $c_i - \varkappa$.

(*To be continued next page.*)

[1]) Thus e. g. on 1899 June 28, $11^h\ 47^m\ 35^s$ G. M. T. the longitudes counted from the first point of Aries are:

$\lambda_1 = 193°{,}64 \quad , \quad \lambda_2 = 13°.64 \quad , \quad \lambda_3 = 193.64 \quad , \quad \lambda_4 = 192°.75.$

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By Prof. W. DE SITTER. (Continued).

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4. *The variational equations.*

The constants of integration of the intermediary orbit satisfy the conditions (18), (19), and the conditions of symmetry (16). The constants of integration of the actual orbit however do not exactly satisfy these conditions. We now put, instead of (17) and (20)

$$\left.\begin{array}{ll} \lambda_i = (c_i - \varkappa)\,\tau + \pi_{i0} + \omega_i, & \nu_i = \nu_i, \\ \pi_i = -\varkappa\tau + \pi_{i0} + g_i, & \eta_i = \overline{\eta_i} + \delta\eta_i \end{array}\right\} . \quad (21)$$

Instead of g_i and $\delta\eta_i$ I introduce h_i and k_i by

$$\left.\begin{array}{l} \eta_i \cos g_i = \overline{\eta_i} + h_i \\ \eta_i \sin g_i = k_i \end{array}\right\} \quad . \; . \; . \; . \; . \; . \quad (22)$$

The equations then become

$$\frac{dh_i}{d\tau} = -\varkappa k_i - (1+\nu_i)^{\frac{1}{3}} \frac{\partial R_i}{\partial k_i} - \tfrac{1}{2}(1+\nu_i)^{\frac{1}{3}} (\overline{\eta_i} + h_i) \frac{\partial R_i}{\partial \lambda_i},$$

$$\frac{dk_i}{d\tau} = \varkappa(\overline{\eta_i} + h_i) + (1+\nu_i)^{\frac{1}{3}} \frac{\partial R_i}{\partial h_i} - \tfrac{1}{2}(1+\nu_i)^{\frac{1}{3}} k_i \frac{\partial R_i}{\partial \lambda_i},$$

$$\frac{d\omega_i}{d\tau} = (c_i - \varkappa)\,\nu_i - 2(1+\nu_i)^{\frac{1}{3}} a_i \frac{\partial R_i}{\partial a_i} + \tfrac{1}{2}(1+\nu_i)^{\frac{1}{3}} \eta_i \frac{\partial R_i}{\partial \eta_i},$$

$$\frac{d\nu_i}{d\tau} = -3(1+\nu_i)^{\frac{1}{3}} \frac{\partial R_i}{\partial \lambda_i}.$$

We still restrict ourselves to the non-periodic part $[R_i]$ of the perturbative function. Then, if we neglect the squares and products of h_i, k_i, ω_i, ν_i, these equations are of the form

$$\left.\begin{aligned}
\frac{dh_i}{d\tau} &= \sum_j a_{ij}\,k_j + \sum_j b_{ij}\,\omega_j, \\
\frac{dk_i}{d\tau} &= -\sum_j a'_{ij}\,h_j - \sum_j b'_{ij}\,\nu_j, \\
\frac{d\nu_i}{d\tau} &= \sum_j d_{ij}\,k_j + \sum_j e_{ij}\,\omega_j, \\
\frac{d\omega_i}{d\tau} &= -\sum d'_{ij}\,h_j - \sum e'_{ij}\,\nu_j.
\end{aligned}\right\} \quad . \quad . \quad . \quad (23)$$

The right-hand members have no constant term. For h_i and ν_i these terms are zero in consequence of the conditions of symmetry (16), since they contain only sines. For k_i and ω_i they are zero by the conditions (18) and (19).

The equations (23) are satisfied by

$$\left.\begin{aligned}
h_i &= \sum_q c_{iq}\,\varepsilon_q \cos \varphi_q, & \nu_i &= \sum_q c'''_{iq}\,\varepsilon_q \cos \varphi_q, \\
k_i &= \sum_q c'_{iq}\,\varepsilon_q \sin \varphi_q, & \omega_i &= \sum_q c''_{iq}\,\varepsilon_q \sin \varphi_q. \\
& & \varphi_q = \beta_q \tau + \varpi_{q0}. &
\end{aligned}\right\} \quad . \quad . \quad . \quad (24)$$

Substituting (24) in (23) we find for c_{iq}, c'_{iq}, c''_{iq}, c'''_{iq} and β_q the conditions

$$\left.\begin{aligned}
c_{iq}\,\beta_q + \sum_j a_{ij}\,c'_{jq} + \sum_j b_{ij}\,c''_{jq} &= 0, \\
c'_{iq}\,\beta_q + \sum_j a'_{ij}\,c_{jq} + \sum_j b'_{ij}\,c'''_{jq} &= 0, \\
c''_{iq}\,\beta_q + \sum_j d'_{ij}\,c_{jq} + \sum_j e'_{ij}\,c'''_{jq} &= 0, \\
c'''_{iq}\,\beta_q + \sum_j d_{ij}\,c'_{jq} + \sum_j e_{ij}\,c''_{jq} &= 0,
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (25)$$

The condition that it shall be possible to determine c_{iq}, $c'_{iq}\ldots.$ from these equations is that their determinant is zero. This gives an equation of the sixteenth degree in β_q. To each root β_q belongs a set $c_{iq}\ldots.$ There are however not 16 different values of β_q. To begin with it is evident that, if we change φ_q to $-\varphi_q$, and consequently β_q to $-\beta_q$, and if at the same time we replace c'_{iq} and c''_{iq} by $-c'_{iq}$ and $-c''_{iq}$, the equations (25) are still satisfied, and (24) is not affected at all. It follows that if β_q is a root, then also $-\beta_q$ is a root.

Further there are *six* roots $\beta = 0$. Each term in the equations (24), i. e. each root β, represents an oscillation of the true motion with respect to the intermediary orbit with the period $2\pi/\beta$. Each of these oscillations corresponds to a small change of the initial values, i. e. a small deviation of the constants of integration from those of the intermediary orbit. The term corresponding to a root $\beta = 0$ is not an oscillation, but a constant correction to one of the elements, which does not affect the character of the motion. Now there are six possible deviations, i. e. six constants of integration by a change in which the intermediary orbit is not essentially altered. These are:

1. A change of the zero of the longitudes and the time. This evidently does not affect the motion at all, and since two constants of integration are involved, it corresponds to two roots $\beta = 0$.

2. A change of $n_2 - n_1 = \frac{d\tau}{dt}$ and of $\varkappa$. The first is evidently only a change in the unit of time. The other does affect the motion of the three inner satellites, but only in so far as the intermediary orbit is replaced by another of entirely the same character.

3. A change of c_4, say to $c_4 + \delta c_4$. We can then call $c + \delta c_4$ again c_4 and nothing essential will be altered.

4. A change of ω_4. In the intermediary orbit we assumed $\omega_4 = 0$. In doing this we neglected a small quantity, and evidently the exact amount of the neglected quantity is of no importance. This corresponds to the fact that all coefficients b_{i4} and e_{i4} are zero, as is found when they are worked out.

It must therefore be possible to transform the equation of the 16th degree in β to an equation of the 5th degree in β^2. This is effected as follows.

By differentiating the second and fourth of (23) we find equations of the form:

$$\left.\begin{aligned} \frac{d^2 k_i}{d\tau^2} + \sum_j A_{ij}\, k_j + \sum_j B_{ij}\, \omega_j = 0, \\ \frac{d^2 \omega_i}{d\tau^2} + \sum_j C_{ij}\, k_j + \sum_j D_{ij}\, \omega_j = 0. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (26)$$

Hence we find for c'_{iq}, c''_{iq} and β_q the conditions

$$\left.\begin{aligned} c'_{iq}\, \beta^2_q - \sum_j A_{ij}\, c'_{jq} - \sum_j B_{ij}\, c''_{jq} = 0, \\ c''_{iq}\, \beta^2_q - \sum_j C_{ij}\, c'_{jq} - \sum_j D_{ij}\, c''_{jq} = 0. \end{aligned}\right\} \quad . \quad . \quad . \quad (27)$$

The determinant of these equations is

$$\Delta = \begin{vmatrix} A_{11}-\beta^2 & A_{12} \dots\dots & A_{14} & B_{11} \dots\dots & B_{14} \\ A_{21} & A_{22}-\beta^2 \dots & A_{24} & B_{21} \dots\dots & B_{24} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{41} & A_{42} \dots\dots & A_{44}-\beta^2 & B_{41} \dots\dots & B_{44} \\ C_{11} & C_{12} \dots\dots & C_{14} & D_{11}-\beta^2 \dots & D_{14} \\ \vdots & \vdots & & \vdots & \vdots \\ C_{41} & C_{42} \dots\dots & C_{44} & D_{41} \dots\dots & D_{44}-\beta^2 \end{vmatrix} \quad . \quad . \quad (28)$$

Now it can be shown that

$$\left.\begin{array}{c} \overline{\eta}_1 A_{i1} + \overline{\eta}_2 A_{i2} + \overline{\eta}_3 A_{i3} + \overline{\eta}_4 A_{i4} + B_{i1} + B_{i2} + B_{i3} + B_{i4} = 0, \\ 4 B_{i1} + 2 B_{i2} + B_{i3} = 0, \\ B_{i4} = 0, \end{array}\right\} \quad (29)$$

and the same equations remain true if A_{ij} is replaced by C_{ij}, and B_{ij} by D_{ij}. It follows that the equation $\Delta = 0$, which is of the 8th degree in β^2, has three roots $\beta^2 = 0$, and can therefore be reduced to an equation of the fifth degree. To prove (29) it would be necessary to develop the coefficients A_{ij}, B_{ij} ...., which cannot be done here. The proof will be given in a more detailed publication that will soon appear in the Annals of the Observatory at Leiden (Vol. XII, Part I).

There are thus 5 different values of β^2_q. To each of these belongs a set of values of c'_{iq} and c''_{iq}, which are found from (27), and of c_{iq} and c'''_{iq}, which are then found from the first and last of (25).

The first four elements of the diagonal of the determinant Δ are approximately

$$A_{ii} = \varkappa^2.$$

All other elements are at least of the third order. It follows that four of the roots β_q are very nearly equal to $\varkappa$, the fifth being much smaller. If we neglect the masses of the satellites and the compression of the planet, then this fifth root becomes zero, and the four others are rigorously equal to $\varkappa$. The motion $-\varkappa$ of the perijoves in the intermediary orbit is then exactly cancelled by the variations, and since in that case also $\overline{\eta}_i = 0$, and the intermediary orbit is a circle, the varied orbit consists of four Keplerian ellipses with the excentricities ε_i and the fixed perijoves ϖ, as it evidently must be.

If we consider the constants of integration ε_i as quantities of the first order, like $\overline{\eta}_i$[1]), and if we put

$$\varphi_i = \beta_i \tau + \varpi_{i0} = \varkappa \tau + \varpi_i,$$

[1]) If follows from (18) that $\varkappa . \overline{\eta}_i$ is of the second order, and consequently $\overline{\eta}_i$ of the first.

then the effect of the variations on the radius-vector and the longitude is found to be, to the first order

$$\left.\begin{aligned} \delta r_i = & -\tfrac{2}{3} a_i \sum_q c'''_{iq} \varepsilon_q \cos\varphi_q - a_i \sum_q \{\tfrac{1}{2}(c_{iq} + c'_{iq})\varepsilon_q \cos(\lambda_i - \varpi_q) + \\ & \qquad + \tfrac{1}{2}(c_{iq} - c'_{iq})\varepsilon_q \cos(c_i\tau + \varphi_q)\}, \\ \delta w_i = & \sum_q c''_{iq}\varepsilon_q \sin\varphi_q + \sum_q \{(c_{iq} + c'_{iq})\varepsilon_q \sin(\lambda_i - \varpi_q) + \\ & \qquad + (c_{iq} - c'_{iq})\varepsilon_q \sin(c_i\tau + \varphi_q)\}. \end{aligned}\right\} \quad (29)$$

As a first approximation we have $a_{ii} = -\varkappa$, a_{ij} and b_{ij} being of the second order. Also with the same approximation, for $q = 1 \ldots 4$, $\beta_q = \varkappa$, and consequently from (25) $c_{iq} = c'_{iq}$ approximately. The difference $c_{iq} - c'_{iq}$ is thus of a higher order, and the last term of (29) can be omitted. Further also d_{ij} and e_{ij} are at least of the second order, and consequently by the last of (25) c'''_{iq} is of a higher order than c'_{iq} and c''_{iq}. It follows that the first term of δr_i can also be omitted in the first approximation. The equations (29) then have entirely the form (6). At the same time we see the reason why the inequalities II and III are so much smaller in the radius-vector than in the longitude.

5. *The perturbations.*

We must now take into account the part of the perturbative function

$$R - [R_i],$$

which contains terms whose argument D varies with the time, thus $D = E\tau$. We will only give the theory in its broad outlines. For details we refer to the publication in the Leiden Annals. We put for abbreviation

$$h_i = x_i, \qquad k_i = y_i, \qquad \nu_i = x_{i+4}, \quad \omega_i = y_{i+4}.$$

The differential equations then assume the form

$$\left.\begin{aligned} \frac{dx_i}{d\tau} &= \sum_E \alpha_{i,E} \sin E\tau + \sum_j \sum_E f_{i,j,E} \sin E\tau\, x_j + \sum_j \sum_E g_{i,j,E} \cos E\tau\, y_j, \\ \frac{dy_i}{d\tau} &= -\sum_E \alpha'_{i,E} \cos E\tau - \sum_j \sum_E f'_{i,j,E} \cos E\tau\, x_j - \sum_j \sum_E g'_{i,j,E} \sin E\tau\, y_j, \end{aligned}\right\} \quad (30)$$

where i and j take the values from 1 to 8. The arguments are of the form

$$D = E\tau = k\tau + k' c_4 \tau,$$

k and k' being any integers, positive, negative or zero. If we take only $k = k' = 0$, the equations (30) are reduced to (23). Thus we have, e.g.

$$f_{i,j,0} = 0, \qquad g_{i,j,0} = a_{i,j}, \qquad g_{i,j+4,0} = b_{ij} \quad \text{etc.}$$

The equations (30) can be satisfied by

$$\left.\begin{aligned} x_i &= \sum_E A_{i,E} \cos E\tau + \sum_j \sum_E M_{i,j,E}\, \varepsilon_j \cos(\varphi_j + E\tau), \\ y_i &= \sum_E A'_{i,F} \sin E\tau + \sum_j \sum_E M'_{i,j,E}\, \varepsilon_j \sin(\varphi_j + E\tau), \end{aligned}\right\} \quad . \quad . \quad (31)$$

where

$$\varphi_i = \beta_i \tau + \varpi_{i0}.$$

Substituting these in (30) we find again equations of condition for β_q, $M_{i,q,E}$ and $M'_{i,q,E}$. There is an infinite number of these equations. Hence the condition for β_q is an infinite determinant put equal to zero. It is evident however that if

$$\beta = \beta_q$$

is a root, then all numbers of the form

$$\beta' = \pm \beta_q \pm k \pm k' c_4 \qquad (k, k' = -\infty \ldots +\infty)$$

are also roots, since changing β to β' does not affect x_i and y_i beyond a change in the notation by which the different coefficients are distinguished.

It is not difficult to get an infinite determinant for β^2 instead of β. If we put

$$\begin{aligned} P_{i,j,E} &= \tfrac{1}{2}(M_{i,j,E} + M_{i,j,-E}), \\ P'_{i,j,E} &= \tfrac{1}{2}(M_{i,j,E} - M_{i,j,-E}), \\ Q'_{i,j,E} &= \tfrac{1}{2}(M'_{i,j,E} + M'_{i,j,-E}), \\ Q_{i,j,E} &= \tfrac{1}{2}(M'_{i,j,E} - M'_{i,j,-E}), \end{aligned}$$

Then the equations become

$$\left.\begin{aligned} &\beta P_{i,E} + EP'_{i,E} + \tfrac{1}{2} \sum_j \sum_F \{(g_{i,j,F-E} + g_{i,j,F+E})\, Q'_{j,F} - \\ &\qquad\qquad - (f_{i,j,F-E} + f_{i,j,F+E})\, P'_{j,F}\} = 0, \\ &\beta P'_{i,E} + EP_{i,E} + \tfrac{1}{2} \sum_j \sum_F \{(g_{i,j,F-E} - g_{i,j,F+E})\, Q_{j,F} - \\ &\qquad\qquad - (f_{i,j,F-E} - f_{i,j,F+E})\, P_{j,F}\} = 0, \\ &\beta Q'_{i,E} + EQ_{i,E} + \tfrac{1}{2} \sum_j \sum_F \{(g'_{i,j,F-E} + g'_{i,j,F+E})\, Q_{j,F} + \\ &\qquad\qquad + (f'_{i,j,F-E} + f'_{i,j,F+E})\, P_{j,F}\} = 0, \\ &\beta Q_{i,E} + EQ'_{i,E} + \tfrac{1}{2} \sum_j \sum_F \{(g'_{i,j,F-E} - g'_{i,j,F+E})\, Q'_{j,F} + \\ &\qquad\qquad + (f'_{i,j,F-E} - f'_{i,j,F+E})\, P'_{j,F}\} = 0. \end{aligned}\right\} \quad . \quad (32)$$

where we have omitted the index q in β_q, $P_{i,q,E}$, $P'_{i,q,E}$ etc. It is only necessary to consider these equations for positive values of E. The sums however include *all* values of F. We have

$$\begin{aligned} P_{i,F} &= P_{i,-F}, & Q_{i,F} &= -Q_{i,-F}, \\ P'_{i,F} &= -P'_{i,-F}, & Q'_{i,F} &= Q'_{i,-F}. \end{aligned}$$

Multiplying the second and third of (30) by β, and then substituting in them the values of βP_{iE} and βQ_{iE} derived from the first and last we find equations which contain only P'_{iE} and Q'_{iE}. These have the form

$$\left.\begin{array}{l}(\beta^2-E^2)\,Q'_{i,E}+\sum\limits_{j}\sum\limits_{F} G_{i,j,E,F}\,Q'_{j,F}+\sum\limits_{j}\sum\limits_{F} H_{i,j,E,F}\,P'_{j,F}\cdot \\ (\beta^2-E^2)\,P'_{i,E}+\sum\limits_{j}\sum\limits_{F} G'_{i,j,E,F}\,Q'_{j,F}+\sum\limits_{j}\sum\limits_{F} H'_{i,j,E,F}\,P'_{j,F},\end{array}\right\} \quad (33)$$

the coefficients G, H, G', H' being all at least of the second order. We have

$$P_{i,0}=c_i, \qquad P_{i+4,0}=c'''_i, \qquad Q'_{i,0}=c'_i, \qquad Q'_{i+4,0}=c''_i$$
$$P'_{i,0}=0, \qquad Q_{i0}=0.$$

The infinite determinant resulting from the elimination of P' and Q' from (33) has, for each argument E, 16 rows and columns, corresponding to the 16 unknowns $P'_{i,E}$ and $Q'_{i,E}$. For $E=0$ there are only 8 unknowns, and also the first of (33) becomes an identity for $E=0$, so that there are only 8 columns and rows. The determinant formed by the elements common to these 8 columns and rows may be called the central determinant.

All elements of the determinant outside the diagonal are of the second order [1]). The elements of the diagonal have the form $G+E^2-\beta^2$, where G is of the second order at least. In the central square we have $E=0$, outside the central square E has a finite value, and therefore $G+E^2$ is of the order zero. The manner in which the determinant is reduced to its central square will be explained by a simple example, in which I take for each argument E only 2 instead of 16 rows and columns, and of the rest of the determinant also only 2 rows and columns are written. This is sufficient to illustrate the principle. We then have the transformation

$$\begin{vmatrix} a_{11}-\beta^2 & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22}-\beta^2 & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33}+E^2-\beta^2 & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44}+E^2-\beta^2 \end{vmatrix} =$$

[1]) This is not correct. There are elements outside the diagonal of the orders zero and one. The conclusions reached in the text are however not affected and remain correct. For a more thorough discussion see Leiden Annals XII. 1. (Note added in the English translation).

$$= \begin{vmatrix} b_{11}-\beta^2 & b_{12} & 0 & 0 \\ b_{21} & b_{22}-\beta^2 & 0 & 0 \\ a_{31} & a_{32} & a'_{33}+E^2-\beta^2 & a'_{34} \\ a_{41} & a_{42} & a'_{43} & a'_{44}+E^2-\beta^2 \end{vmatrix},$$

where

$$\begin{aligned} b_{11} &= a_{11} - x_1 a_{31} - y_1 a_{41}, & b_{12} &= a_{12} - x_1 a_{32} - y_1 a_{42}, \\ b_{21} &= a_{21} - x_2 a_{31} - y_2 a_{41}, & b_{22} &= a_{22} - x_2 a_{32} - y_2 a_{42}, \\ a'_{33} &= a_{33} + x_1 a_{31} + x_2 a_{32}, & a'_{34} &= a_{34} + y_1 a_{31} + y_2 a_{32}, \\ a'_{43} &= a_{43} + x_1 a_{41} + x_2 a_{42}, & a'_{44} &= a_{44} + y_1 a_{41} + y_2 a_{42}, \end{aligned}$$

and the multipliers x_1, x_2, y_1, y_2 are determined by

$$\left.\begin{aligned} a_{13} + x_1 a_{11} + x_2 a_{12} - x_1 (a'_{33} + E^2) - y_1 a'_{43} &= 0 \\ a_{23} + x_1 a_{21} + x_2 a_{22} - x_2 (a'_{33} + E^2) - y_2 a'_{43} &= 0 \\ a_{14} + y_1 a_{11} + y_2 a_{12} - y_1 (a'_{44} + E^2) - x_1 a'_{34} &= 0 \\ a_{24} + y_1 a_{21} + y_2 a_{22} - y_2 (a'_{44} + E^2) - x_2 a'_{34} &= 0 \end{aligned}\right\} \quad . \quad . \quad (34)$$

The determinant is thus reduced to the product of two determinants. In our case we will in this way "peel off" 16 rows and columns at a time, instead of two. It follows from (34) that x_i and y_i are of the second order at least. The corrections

$$b_{ij} - a_{ij}$$

to be applied to the inner terms are thus of the fourth order. If now we proceed to remove the columns and rows of another argument F, the effect of these corrections on the central determinant will be of the *sixth* order. Consequently, if we agree to neglect quantities of the sixth order in β^2, and therefore, since β itself is of the first order, quantities of the fifth order (i.e. of the order of 10^{-10}) in β, then the rows and columns of each argument can be removed *separately*, independently of all other arguments. The determinant finally is reduced to a product of an infinite number of determinants, of which the central one has 8 columns and rows, and all others 16. Each of these corresponds to one argument $\pm E$. As has been pointed out above, to each root β_q belongs a root $\beta_q + E$ and a root $\beta_q - E$. It is thus evidently only necessary to determine the 8 roots of the corrected central determinant. The corrections which have been applied to the elements of the central square are at least of the fourth order. These 8 roots will therefore differ very little from those of the uncorrected determinant Δ, of which three are zero. For the corrected determinant the relations (29) do not hold, and also the a priori reasoning by which we showed that there must be six roots $\beta = 0$, do not apply here, since $x_i = y_i = 0$ is not a particular solution of the equations (30). The three roots of the corrected determinant corresponding to the three zero roots of $\Delta = 0$ may therefore differ slightly from zero, but they will in any case be extremely small, and for all practical purposes the other five roots are the only important ones.

It remains to determine the coefficients A_{iE} and A'_{iE}. We have the equations

$$\left.\begin{aligned} EA_{i,E} + \tfrac{1}{2} \sum_j \sum_F \{ f_{i,j,F} (A_{j,E-F} - A_{j,E+F}) + \\ + g_{i,j,F} (A'_{j,E-F} + A'_{j,E+F}) \} + a_{i,E} = 0, \\ EA'_{i,E} + \tfrac{1}{2} \sum_j \sum_F \{ f'_{i,j,F} (A_{j,E-F} + A_{j,E+F}) - \\ - g'_{i,j,F} (A'_{E-F} - A'_{E+F}) \} + a'_{i,E} = 0, \end{aligned}\right\} \quad (36)$$

from which the coefficients can be solved by successive approximation. Nearly always the first approximation

$$A_{i,E} = -\frac{a_{i,E}}{E}, \qquad A'_{i,E} = -\frac{a'_{i,E}}{E}$$

will be sufficient.