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4. *The variational equations.*

The constants of integration of the intermediary orbit satisfy the conditions (18), (19), and the conditions of symmetry (16). The constants of integration of the actual orbit however do not exactly satisfy these conditions. We now put, instead of (17) and (20)

$$\left.\begin{array}{ll} \lambda_i = (c_i - \varkappa)\,\tau + \pi_{i0} + \omega_i, & \nu_i = \nu_i, \\ \pi_i = -\varkappa\tau + \pi_{i0} + g_i, & \eta_i = \overline{\eta}_i + \delta\eta_i \end{array}\right\} \quad . \quad (21)$$

Instead of g_i and $\delta\eta_i$ I introduce h_i and k_i by

$$\left.\begin{array}{l} \eta_i \cos g_i = \overline{\eta}_i + h_i \\ \eta_i \sin g_i = k_i \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

The equations then become

$$\frac{dh_i}{d\tau} = -\varkappa k_i - (1+\nu_i)^{\frac{1}{3}} \frac{\partial R_i}{\partial k_i} - \tfrac{1}{2}(1+\nu_i)^{\frac{1}{3}} (\overline{\eta}_i + h_i) \frac{\partial R_i}{\partial \lambda_i},$$

$$\frac{dk_i}{d\tau} = \varkappa(\overline{\eta}_i + h_i) + (1+\nu_i)^{\frac{1}{3}} \frac{\partial R_i}{\partial h_i} - \tfrac{1}{2}(1+\nu_i)^{\frac{1}{3}} k_i \frac{\partial R_i}{\partial \lambda_i},$$

$$\frac{d\omega_i}{d\tau} = (c_i - \varkappa)\,\nu_i - 2(1+\nu_i)^{\frac{1}{3}} a_i \frac{\partial R_i}{\partial a_i} + \tfrac{1}{2}(1+\nu_i)^{\frac{1}{3}} \eta_i \frac{\partial R_i}{\partial \eta_i},$$

$$\frac{d\nu_i}{d\tau} = -3(1+\nu_i)^{\frac{4}{3}} \frac{\partial R_i}{\partial \lambda_i}.$$

We still restrict ourselves to the non-periodic part $[R_i]$ of the perturbative function. Then, if we neglect the squares and products of h_i, k_i, ω_i, ν_i, these equations are of the form

$$\left.\begin{aligned}\frac{dh_i}{d\tau} &= \sum_j a_{ij}\,k_j + \sum_j b_{ij}\,\omega_j,\\ \frac{dk_i}{d\tau} &= -\sum_j a'_{ij}\,h_j - \sum_j b'_{ij}\,\nu_j,\\ \frac{d\nu_i}{d\tau} &= \sum_j d_{ij}\,k_j + \sum_j e_{ij}\,\omega_j,\\ \frac{d\omega_i}{d\tau} &= -\sum d'_{ij}\,h_j - \sum e'_{ij}\,\nu_j.\end{aligned}\right\} \quad . \quad . \quad . \quad (23)$$

The right-hand members have no constant term. For h_i and ν_i these terms are zero in consequence of the conditions of symmetry (16), since they contain only sines. For k_i and ω_i they are zero by the conditions (18) and (19).

The equations (23) are satisfied by

$$\left.\begin{aligned} h_i &= \sum_q c_{iq}\,\varepsilon_q \cos\varphi_q, & \nu_i &= \sum_q c'''_{iq}\,\varepsilon_q \cos\varphi_q,\\ k_i &= \sum_q c'_{iq}\,\varepsilon_q \sin\varphi_q, & \omega_i &= \sum_q c''_{iq}\,\varepsilon_q \sin\varphi_q.\\ &\varphi_q = \beta_q\tau + \varpi_{q0}. \end{aligned}\right\} \quad . \quad . \quad . \quad (24)$$

Substituting (24) in (23) we find for c_{iq}, c'_{iq}, c''_{iq}, c'''_{iq} and β_q the conditions

$$\left.\begin{aligned} c_{iq}\,\beta_q + \sum_j a_{ij}\,c'_{jq} + \sum_j b_{ij}\,c''_{jq} &= 0,\\ c'_{iq}\,\beta_q + \sum_j a'_{ij}\,c_{jq} + \sum_j b'_{ij}\,c'''_{jq} &= 0,\\ c''_{iq}\,\beta_q + \sum_j d'_{ij}\,c_{jq} + \sum_j e'_{ij}\,c'''_{jq} &= 0,\\ c'''_{iq}\,\beta_q + \sum_j d_{ij}\,c'_{jq} + \sum_j e_{ij}\,c''_{jq} &= 0, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (25)$$

The condition that it shall be possible to determine c_{iq}, c'_{iq} .... from these equations is that their determinant is zero. This gives an equation of the sixteenth degree in β_q. To each root β_q belongs a set c_{iq} .... There are however not 16 different values of β_q. To begin with it is evident that, if we change φ_q to $-\varphi_q$, and consequently β_q to $-\beta_q$, and if at the same time we replace c'_{iq} and c''_{iq} by $-c'_{iq}$ and $-c''_{iq}$, the equations (25) are still satisfied, and (24) is not affected at all. It follows that if β_q is a root, then also $-\beta_q$ is a root.

Further there are *six* roots $\beta = 0$. Each term in the equations (24), i. e. each root β, represents an oscillation of the true motion with respect to the intermediary orbit with the period $2\pi/\beta$. Each of these oscillations corresponds to a small change of the initial values, i. e. a small deviation of the constants of integration from those of the intermediary orbit. The term corresponding to a root $\beta = 0$ is not an oscillation, but a constant correction to one of the elements, which does not affect the character of the motion. Now there are six possible deviations, i. e. six constants of integration by a change in which the intermediary orbit is not essentially altered. These are:

1. A change of the zero of the longitudes and the time. This evidently does not affect the motion at all, and since two constants of integration are involved, it corresponds to two roots $\beta = 0$.

2. A change of $n_2 - n_3 = \dfrac{d\tau}{dt}$ and of $\varkappa$. The first is evidently only a change in the unit of time. The other does affect the motion of the three inner satellites, but only in so far as the intermediary orbit is replaced by another of entirely the same character.

3. A change of c_4, say to $c_4 + \delta c_4$. We can then call $c + \delta c_4$ again c_4 and nothing essential will be altered.

4. A change of ω_4. In the intermediary orbit we assumed $\omega_4 = 0$. In doing this we neglected a small quantity, and evidently the exact amount of the neglected quantity is of no importance. This corresponds to the fact that all coefficients b_{i4} and e_{i4} are zero, as is found when they are worked out.

It must therefore be possible to transform the equation of the 16th degree in β to an equation of the 5th degree in β^2. This is effected as follows.

By differentiating the second and fourth of (23) we find equations of the form:

$$\left.\begin{aligned} \frac{d^2 k_i}{d\tau^2} + \sum_j A_{ij}\, k_j + \sum_j B_{ij}\, \omega_j = 0, \\ \frac{d^2 \omega_i}{d\tau^2} + \sum_j C_{ij}\, k_j + \sum_j D_{ij}\, \omega_j = 0. \end{aligned}\right\} \quad . \; . \; . \; . \quad (26)$$

Hence we find for c'_{iq}, c''_{iq} and β_q the conditions

$$\left.\begin{aligned} c'_{iq}\, \beta^2_q - \sum_j A_{ij}\, c'_{jq} - \sum_j B_{ij}\, c''_{jq} = 0, \\ c''_{iq}\, \beta^2_q - \sum_j C_{ij}\, c'_{jq} - \sum_j D_{ij}\, c''_{jq} = 0. \end{aligned}\right\} \quad . \; . \; . \quad (27)$$

The determinant of these equations is

$$\Delta = \begin{vmatrix} A_{11}-\beta^2 & A_{12} \ldots\ldots & A_{14} & B_{11} \ldots\ldots & B_{14} \\ A_{21} & A_{22}-\beta^2 \ldots & A_{24} & B_{21} \ldots\ldots & B_{24} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{41} & A_{42} \ldots\ldots & A_{44}-\beta^2 & B_{41} \ldots\ldots & B_{44} \\ C_{11} & C_{12} \ldots\ldots & C_{14} & D_{11}-\beta^2 \ldots & D_{14} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ C_{41} & C_{42} \ldots\ldots & C_{44} & D_{41} \ldots\ldots & D_{44}-\beta^2 \end{vmatrix} \quad . \quad . \quad (28)$$

Now it can be shown that

$$\left.\begin{aligned} \overline{\eta}_1 A_{i1} + \overline{\eta}_2 A_{i2} + \overline{\eta}_3 A_{i3} + \overline{\eta}_4 A_{i4} + B_{i1} + B_{i2} + B_{i3} + B_{i4} &= 0, \\ 4 B_{i1} + 2 B_{i2} + B_{i3} &= 0, \\ B_{i4} &= 0, \end{aligned}\right\} \quad (29)$$

and the same equations remain true if A_{ij} is replaced by C_{ij}, and B_{ij} by D_{ij}. It follows that the equation $\Delta = 0$, which is of the 8th degree in β^2, has three roots $\beta^2 = 0$, and can therefore be reduced to an equation of the fifth degree. To prove (29) it would be necessary to develop the coefficients A_{ij}, B_{ij} ...., which cannot be done here. The proof will be given in a more detailed publication that will soon appear in the Annals of the Observatory at Leiden (Vol. XII, Part I).

There are thus 5 different values of β^2_q. To each of these belongs a set of values of c'_{iq} and c''_{iq}, which are found from (27), and of c_{iq} and c'''_{iq}, which are then found from the first and last of (25).

The first four elements of the diagonal of the determinant Δ are approximately

$$A_{ii} = \varkappa^2.$$

All other elements are at least of the third order. It follows that four of the roots β_q are very nearly equal to $\varkappa$, the fifth being much smaller. If we neglect the masses of the satellites and the compression of the planet, then this fifth root becomes zero, and the four others are rigorously equal to $\varkappa$. The motion $-\varkappa$ of the perijoves in the intermediary orbit is then exactly cancelled by the variations, and since in that case also $\overline{\eta}_i = 0$, and the intermediary orbit is a circle, the varied orbit consists of four Keplerian ellipses with the excentricities ε_i and the fixed perijoves ϖ, as it evidently must be.

If we consider the constants of integration ε_i as quantities of the first order, like $\overline{\eta}_i$[1]), and if we put

$$\varphi_i = \beta_i \tau + \varpi_{i0} = \varkappa \tau + \varpi_i,$$

[1]) If follows from (18) that $\varkappa . \overline{\eta}_i$ is of the second order, and consequently $\overline{\eta}_i$ of the first.

then the effect of the variations on the radius-vector and the longitude is found to be, to the first order

$$\left.\begin{aligned} \delta r_i &= -\tfrac{2}{3} a_i \sum_q c'''_{iq} \varepsilon_q \cos \varphi_q - a_i \sum_q \{\tfrac{1}{2}(c_{iq} + c'_{iq}) \varepsilon_q \cos(\lambda_i - \varpi_q) + \\ &\qquad + \tfrac{1}{2}(c_{iq} - c'_{iq}) \varepsilon_q \cos(c_i \tau + \varphi_q)\}, \\ \delta w_i &= \sum_q c''_{iq} \varepsilon_q \sin \varphi_q + \sum_q \{(c_{iq} + c'_{iq}) \varepsilon_q \sin(\lambda_i - \varpi_q) + \\ &\qquad + (c_{iq} - c'_{iq}) \varepsilon_q \sin(c_i \tau + \varphi_q)\}. \end{aligned}\right\} . \quad (29)$$

As a first approximation we have $a_{ii} = -\varkappa$, a_{ij} and b_{ij} being of the second order. Also with the same approximation, for $q = 1 \ldots . 4$, $\beta_q = \varkappa$, and consequently from (25) $c_{iq} = c'_{iq}$ approximately. The difference $c_{iq} - c'_{iq}$ is thus of a higher order, and the last term of (29) can be omitted. Further also d_{ij} and e_{ij} are at least of the second order, and consequently by the last of (25) c'''_{iq} is of a higher order than c'_{iq} and c''_{iq}. It follows that the first term of δr_i can also be omitted in the first approximation. The equations (29) then have entirely the form (6). At the same time we see the reason why the inequalities II and III are so much smaller in the radius-vector than in the longitude.

5. *The perturbations.*

We must now take into account the part of the perturbative function

$$R - [R_i],$$

which contains terms whose argument D varies with the time, thus $D = E\tau$. We will only give the theory in its broad outlines. For details we refer to the publication in the Leiden Annals. We put for abbreviation

$$h_i = x_i, \qquad k_i = y_i, \qquad \nu_i = x_{i+4}, \qquad \omega_i = y_{i+4}.$$

The differential equations then assume the form

$$\left.\begin{aligned} \frac{dx_i}{d\tau} &= \sum_E \alpha_{i,E} \sin E\tau + \sum_j \sum_E f_{i,j,E} \sin E\tau \, x_j + \sum_j \sum_E g_{i,j,E} \cos E\tau \, y_j, \\ \frac{dy_i}{d\tau} &= -\sum_E \alpha'_{i,E} \cos E\tau - \sum_j \sum_E f'_{i,j,E} \cos E\tau \, x_j - \sum_j \sum_E g'_{i,j,E} \sin E\tau \, y_j, \end{aligned}\right\} , \quad (30)$$

where i and j take the values from 1 to 8. The arguments are of the form

$$D = E\tau = k\tau + k' c_4 \tau,$$

k and k' being any integers, positive, negative or zero. If we take only $k = k' = 0$, the equations (30) are reduced to (23). Thus we have, e.g.

$$f_{i,j,0} = 0, \qquad g_{i,j,0} = a_{i,j}, \qquad g_{i,j+4,0} = b_{ij} \quad \text{etc.}$$

The equations (30) can be satisfied by

$$\left.\begin{aligned} x_i &= \sum_E A_{i,E} \cos E\tau + \sum_j \sum_E M_{i,j,E}\, \varepsilon_j \cos(\varphi_j + E\tau), \\ y_i &= \sum_E A'_{i,F} \sin E\tau + \sum_j \sum_E M'_{i,j,E}\, \varepsilon_j \sin(\varphi_j + E\tau), \end{aligned}\right\} \quad . \quad . \quad (31)$$

where

$$\varphi_i = \beta_i \tau + \varpi_{i0}.$$

Substituting these in (30) we find again equations of condition for β_q, $M_{i,q,E}$ and $M'_{i,q,E}$. There is an infinite number of these equations. Hence the condition for β_q is an infinite determinant put equal to zero. It is evident however that if

$$\beta = \beta_q$$

is a root, then all numbers of the form

$$\beta' = \pm \beta_q \pm k \pm k' c_4 \qquad (k, k' = -\infty \ldots + \infty)$$

are also roots, since changing β to β' does not affect x_i and y_i beyond a change in the notation by which the different coefficients are distinguished.

It is not difficult to get an infinite determinant for β^2 instead of β. If we put

$$\begin{aligned} P_{i,j,E} &= \tfrac{1}{2}(M_{i,j,E} + M_{i,j,-E}), \\ P'_{i,j,E} &= \tfrac{1}{2}(M_{i,j,E} - M_{i,j,-E}), \\ Q'_{i,j,E} &= \tfrac{1}{2}(M'_{i,j,E} + M'_{i,j,-E}), \\ Q_{i,j,E} &= \tfrac{1}{2}(M'_{i,j,E} - M'_{i,j,-E}), \end{aligned}$$

Then the equations become

$$\left.\begin{aligned} &\beta P_{i,E} + EP'_{i,E} + \tfrac{1}{2}\sum_j \sum_F \{(g_{i,j,F-E} + g_{i,j,F+E})\, Q'_{j,F} - \\ &\qquad\qquad - (f_{i,j,F-E} + f_{i,j,F+E})\, P'_{j,F}\} = 0, \\ &\beta P'_{i,E} + EP_{i,E} + \tfrac{1}{2}\sum_j \sum_F \{(g_{i,j,F-E} - g_{i,j,F+E})\, Q_{j,F} - \\ &\qquad\qquad - (f_{i,j,F-E} - f_{i,j,F+E})\, P_{j,F}\} = 0, \\ &\beta Q'_{i,E} + EQ_{i,E} + \tfrac{1}{2}\sum_j \sum_F \{(g'_{i,j,F-E} + g'_{i,j,F+E})\, Q_{j,F} + \\ &\qquad\qquad + (f'_{i,j,F-E} + f'_{i,j,F+E})\, P_{j,F}\} = 0, \\ &\beta Q_{i,E} + EQ'_{i,E} + \tfrac{1}{2}\sum_j \sum_F \{(g'_{i,j,F-E} - g'_{i,j,F+E})\, Q'_{j,F} + \\ &\qquad\qquad + (f'_{i,j,F-E} - f'_{i,j,F+E})\, P'_{j,F}\} = 0. \end{aligned}\right\} \quad . \quad (32)$$

where we have omitted the index q in β_q, $P_{i,q,E}$, $P'_{i,q,E}$ etc. It is only necessary to consider these equations for positive values of E. The sums however include *all* values of F. We have

$$\begin{aligned} P_{i,F} &= P_{i,-F}, \qquad & Q_{i,F} &= -Q_{i,-F}, \\ P'_{i,F} &= -P'_{i,-F}, \qquad & Q'_{i,F} &= Q'_{i,-F}. \end{aligned}$$

Multiplying the second and third of (30) by β, and then substituting in them the values of βP_{iE} and βQ_{iE} derived from the first and last we find equations which contain only P'_{iE} and Q'_{iE}. These have the form

$$\left.\begin{aligned}(\beta^2-E^2)\,Q'_{i,E}+\sum_j\sum_F G_{i,j,E,F}\,Q'_{j,F}+\sum_j\sum_F H_{i,j,E,F}\,P'_{j,F}.\\(\beta^2-E^2)\,P'_{i,E}+\sum_j\sum_F G'_{i,j,E,F}\,Q'_{j,F}+\sum_j\sum_F H'_{i,j,E,F}\,P'_{j,F},\end{aligned}\right\}\quad(33)$$

the coefficients G, H, G', H' being all at least of the second order. We have

$$P_{i,0}=c_i,\qquad P_{i+4,0}=c'''_i,\qquad Q'_{i,0}=c'_i,\qquad Q'_{i+4,0}=c''_i$$
$$P'_{i,0}=0,\qquad Q_{i0}=0.$$

The infinite determinant resulting from the elimination of P' and Q' from (33) has, for each argument E, 16 rows and columns, corresponding to the 16 unknowns $P'_{i,E}$ and $Q'_{i,E}$. For $E=0$ there are only 8 unknowns, and also the first of (33) becomes an identity for $E=0$, so that there are only 8 columns and rows. The determinant formed by the elements common to these 8 columns and rows may be called the central determinant.

All elements of the determinant outside the diagonal are of the second order [1]). The elements of the diagonal have the form $G+E^2-\beta^2$, where G is of the second order at least. In the central square we have $E=0$, outside the central square E has a finite value, and therefore $G+E^2$ is of the order zero. The manner in which the determinant is reduced to its central square will be explained by a simple example, in which I take for each argument E only 2 instead of 16 rows and columns, and of the rest of the determinant also only 2 rows and columns are written. This is sufficient to illustrate the principle. We then have the transformation

$$\begin{vmatrix}a_{11}-\beta^2 & a_{12} & a_{13} & a_{14}\\ a_{21} & a_{22}-\beta^2 & a_{23} & a_{24}\\ a_{31} & a_{32} & a_{33}+E^2-\beta^2 & a_{34}\\ a_{41} & a_{42} & a_{43} & a_{44}+E^2-\beta^2\end{vmatrix}=$$

[1]) This is not correct. There are elements outside the diagonal of the orders zero and one. The conclusions reached in the text are however not affected and remain correct. For a more thorough discussion see Leiden Annals XII. 1. (Note added in the English translation).

$$
= \begin{vmatrix} b_{11}-\beta^2 & b_{12} & 0 & 0 \\ b_{21} & b_{22}-\beta^2 & 0 & 0 \\ a_{31} & a_{32} & a'_{33}+E^2-\beta^2 & a'_{34} \\ a_{41} & a_{42} & a'_{43} & a'_{44}+E^2-\beta^2 \end{vmatrix},
$$

where

$$
\begin{aligned}
b_{11} &= a_{11} - x_1 a_{31} - y_1 a_{41}, & b_{12} &= a_{12} - x_1 a_{32} - y_1 a_{42}, \\
b_{21} &= a_{21} - x_2 a_{31} - y_2 a_{41}, & b_{22} &= a_{22} - x_2 a_{32} - y_2 a_{42}, \\
a'_{33} &= a_{33} + x_1 a_{31} + x_2 a_{32}, & a'_{34} &= a_{34} + y_1 a_{31} + y_2 a_{32}, \\
a'_{43} &= a_{43} + x_1 a_{41} + x_2 a_{42}, & a'_{44} &= a_{44} + y_1 a_{41} + y_2 a_{42},
\end{aligned}
$$

and the multipliers x_1, x_2, y_1, y_2 are determined by

$$
\left.\begin{aligned}
a_{13} + x_1 a_{11} + x_2 a_{12} - x_1 (a'_{33} + E^2) - y_1 a'_{43} &= 0 \\
a_{23} + x_1 a_{21} + x_2 a_{22} - x_2 (a'_{33} + E^2) - y_2 a'_{43} &= 0 \\
a_{14} + y_1 a_{11} + y_2 a_{12} - y_1 (a'_{44} + E^2) - x_1 a'_{34} &= 0 \\
a_{24} + y_1 a_{21} + y_2 a_{22} - y_2 (a'_{44} + E^2) - x_2 a'_{34} &= 0
\end{aligned}\right\} \quad . \quad . \quad (34)
$$

The determinant is thus reduced to the product of two determinants. In our case we will in this way "peel off" 16 rows and columns at a time, instead of two. It follows from (34) that x_i and y_i are of the second order at least. The corrections

$$b_{ij} - a_{ij}$$

to be applied to the inner terms are thus of the fourth order. If now we proceed to remove the columns and rows of another argument F, the effect of these corrections on the central determinant will be of the *sixth* order. Consequently, if we agree to neglect quantities of the sixth order in β^2, and therefore, since β itself is of the first order, quantities of the fifth order (i.e. of the order of 10^{-10}) in β, then the rows and columns of each argument can be removed *separately*, independently of all other arguments. The determinant finally is reduced to a product of an infinite number of determinants, of which the central one has 8 columns and rows, and all others 16. Each of these corresponds to one argument $\pm E$. As has been pointed out above, to each root β_q belongs a root $\beta_q + E$ and a root $\beta_q - E$. It is thus evidently only necessary to determine the 8 roots of the corrected central determinant. The corrections which have been applied to the elements of the central square are at least of the fourth order. These 8 roots will therefore differ very little from those of the uncorrected determinant Δ, of which three are zero. For the corrected determinant the relations (29) do not hold, and also the a priori reasoning by which we showed that there must be six roots $\beta = 0$, do not apply here, since $x_i = y_i = 0$ is not a particular solution of the equations (30). The three roots of the corrected determinant corresponding to the three zero roots of $\Delta = 0$ may therefore differ slightly from zero, but they will in any case be extremely small, and for all practical purposes the other five roots are the only important ones.

It remains to determine the coefficients A_{iE} and A'_{iE}. We have the equations

$$\left.\begin{aligned} EA_{i,E} + \tfrac{1}{2} \sum_j \sum_F \{f_{i,j,F}(A_{j,E-F} - A_{j,E+F}) + \\ + g_{i,j,F}(A'_{j,E-F} + A'_{j,E+F})\} + a_{i,E} = 0, \\ EA'_{i,E} + \tfrac{1}{2} \sum_j \sum_F \{f'_{i,j,F}(A_{j,E-F} + A_{j,E+F}) - \\ - g'_{i,j,F}(A'_{E-F} - A'_{E+F})\} + a'_{i,E} = 0, \end{aligned}\right\} \tag{36}$$

from which the coefficients can be solved by successive approximation. Nearly always the first approximation

$$A_{i,E} = -\frac{a_{i,E}}{E}, \qquad A'_{i,E} = -\frac{a'_{i,E}}{E}$$

will be sufficient.