substance. Also from a physico-chemical point of view it will be desirable to start researches on this subject.

SUMMARY.

It was set forth that it is of importance:

1. To possess by the side of the macro- and microgravimetric analysis, another method which enables us to make quantitative determinations of very slight quantities of a substance in a simpler and more accurate way.

2. The method detailed here is based on the principle that after the reagent has been added, the precipitate formed is centrifugated in a calibrated capillary tube until the volume remains constant and can be read off. When the volume of $BaSO_4$ corresponding to a SO_4-sol. of a known concentration has been determined, then it is possible to determine by means of this result the SO_4-concentration of an unknown sulphate-solution.

3. In order to make a quantitative determination of SO_4 we always add $2^1/_2$ cc of HCl 1 : 1 (concentrated NCl diluted with an equal volume of water) to 5 cc of the SO_4 fluid, and to this mixture 5 cc of a $BaCl_2$ 2 aq-solution of 2.44 $^0/_0$, containing 3 to 5 drops of acetone. The precipitate formed is centrifugated until the volume remains the same.

Whether the 5 cc of fluid contains much sulphate or only a little and whether these 5 cc of fluid contain Na, K, Ca, Mg, Cl and PO_4 makes no difference whatever, as regards the results: an n-fold quantity of SO_4 gives an n-fold volume of $BaSO_4$ and the presence of the above-mentioned admixtures does not affect the volume of the precipitate.

One division = 0.0004 cc. of the $BaSO_4$-solution corresponds to 0.000294 grammes of SO_4. Mistakes greater than 0.000294 grammes of SO_4 are not made if the method described sub 3 is carefully followed.

Groningen, April 1916. *Physiological Laboratory of the University.*

Physics. — *"Direct optical measurement of the velocity at the axis in the apparatus for* FIZEAU's *experiment".* By Prof. P. ZEEMAN.

(Communicated in the meeting of May 27, 1916).

For the comparison with theory of the *absolute* values of the shifts of the interference fringes, which I determined for light of different colours in FIZEAU's experiment, the magnitude of the velocity at the axis of the tubes conveying the water must be known. This velocity at the axis was deduced from the mean velocity by means of a numerical coefficient φ, which represents the ratio between the mean velocity and the velocity at the axis in a cylindrical tube for turbulent motion. [1])

At first I adopted for φ the value 0,84 as determined by American engineers. Afterwards I devised an optical method for measuring the mentioned coefficient. In a model of part of the apparatus for measuring FRESNEL's coefficient, the value $\varphi = 0{,}843$ was found. On that occasion (Communication IV) I suggested that it would be preferable, though rather difficult, to measure φ in the very apparatus used in my repetition of FIZEAU's experiment. Only lately have I succeeded in performing the necessary measurements with the original apparatus. The velocity at the axis, which is of primary importance, is now measured *directly*. The value of φ is of minor importance, but may of course be calculated from the measured mean velocity. It should be noticed that for the measurement of the total volume a verification of the water meter is necessary, so that a fault in this verification affects also φ. By the use of the method now under review one is quite independent of any verification of watermeters.

For the application of our optical method — rotating mirror; air bubbles in the running water; intense, narrow beam of light at the axis — it is necessary to have a small window in the wall of the brass tube. For this purpose an aperture of two centimeters length, one centimeter width made in the thin walled tube was closed with a cylindrical piece of glass of a mean curvature equal to that of the wall of the tube Between the brass and the glass a thin layer of rubber was interposed to make the apparatus watertight; in order to withstand the considerable pressure the window was pressed against the tube by means of adequately constructed springs.

[1]) For easy references my communications relating to FIZEAU's experiment are referred to as Communications I, II, III, and IV:

I. The convection coefficient of FRESNEL for light of different colour (I). These Proceedings **17**, 445, 1914.

II. The convection coefficient of FRESNEL for light of different colours (II) These Proceedings **18**, 398, 1915.

III. On a possible influence of the FRESNEL coefficient on solar phenomena. These Proceedings **18**, 711, 1915.

IV. An optical method for determining the ratio between the mean and axial velocities in the turbulent motion of fluids in a cylindrical tube. Contribution to the experiment of FIZEAU. These Proceedings **18**, 1240, 1916.

The window was arranged for in the inferior tube of Fig. 2 B (Plate of Communication I), at the left side of the drawing near the prism and at a distance of about 36 cm. of the plane parallel plates of glass or about 25 cm. reckoned from the beginning of the moving water column.

In order to enable us to describe the results the four cocks for regulating the water supply to the tube system (see the Plate of Comm. I) are supposed to be lettered from right to left; A, B, C, D. When the cocks A and C were open. a determination of the velocity at the axis was made by means of the optical method, the result came out in the neighbourhood of 500 cm/sec. This is a very unexpected result, for on a former occasion (Communication II) the velocity at the axis deduced by means of the mean velocity was found 553,6 cm/sec. At first the possibility of some serious error of the optical method was thought of. The deviation was, however, entirely beyond the experimental errors. The result now obtained undoubtedly ought to be of an accuracy superior to the determination in Communication IV, for the effective distance ($l = 46$ cm.) from the axis of the tube to the rotation axis of the mirror exceeds the one formerly used ($l = 32$ cm.). After reversal of the direction of the water current (cocks B and D opened) the velocity appeared to be 580 cm/sec. This value exceeds the adopted value. These observations tended to show, first that the seemingly obvious supposition that with reversal of the water current the velocity is only changed in direction and not in magnitude was wrong and further that the velocity distribution along the axis of the FIZEAU tubes was much more complicated than supposed in the beginning. Nothing short of a measurement of the velocity at a number of points situated along the axis of the tubes became necessary. It seemed at first to suffice to investigate the distribution for only one of the tubes. In the course of the observations it became clear, however, that the measurement of the velocity at the axis had to be extended to the two tubes and to both directions of the water current.

As it was unpracticable to arrange for windows (as described above) in the brass tubes at a number of different points and as it was yet desirable to include not too few points in the survey, use was made of a PITOT tube, *verified by the optical method.* This tube facing the current at the axis can be temporarily placed at a number of points; after removal the small aperture necessary for the adaptation of the PITOT tube can be closed again. If a PITOT tube is placed in a stationary current with the velocity v, we may suppose that the velocity at the aperture facing the current is zero, and hence that $p = \frac{1}{2} \varrho v^2$ or at any rate $p = \frac{k^2}{2} \varrho v^2$. Here ϱ represents the density of the fluid, k^2 is a constant which is, as is shown by the optical method, very nearly equal to unity [1]). The pressure p may be measured independently of the static pressure in the tube, by observing the difference of pressure between that in the small PITOT tube and that in a small hole in the wall of the tube. The small hole in the wall of the tube was made in the horizontal plane passing through the aperture of the PITOT tube and at the same time in the vertical cross section through that aperture. The difference of pressure was read upon a water manometer. The pressures varied from 100 to 180 cm. of water. Part of the connexion between the hole in the wall of the tube and the manometer consists of a short length of rubber tubing, so that by means of a binding screw the variations of pressure, corresponding to variations of the velocity can be damped. The height of the manometer is a time integral [2]).

Fig. 1 represents to scale the two tubes with the windows V_A, V_B, V_C and indicating the points, where the small PITOT tubes were successively introduced. The apertures in the walls are not shown, the dotted lines represent the virtual ends of the *moving* fluid column, the whole length of which is 2×302 cm.

Fig. 1

The arrows indicate the direction of the current when the cocks A and C (see above) are open. The right angled totally reflecting prism is to be figured to the left, the interferometer to the right of the drawing.

The results of all the determinations of the velocity are given in the Table; the velocities are reduced to the formerly adopted mean initial pressure of 2,14 kg./cm². [3])

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1) This point will be considered in a separate paper.

2) The mean velocity at a point is defined as the mean of all the velocities to be found at that point of the tube during a certain, not too short, time interval. The component of this mean velocity in the direction of the axis determines the volume of the fluid, passing per second through a cross section. As the indications of PITOT's tube are rather insensible to changes of direction of the current it seems possible that under special conditions the apparent total flow of fluid surpasses the real flow.

3) The principal cocks in the supply tubes as well as some of the places for the PITOT tubes are indicated by the same letters A, B, C, D. From the text the meaning will always be sufficiently clear.

TUBE I.

Aperture	Distance from beginning of current	v A and C open	v B and D open
A_0	9	549	509
A	24	580	530
B	54	578	510
D	100.3	590	566
C	146.6	567	581
E	219.0	536	573
F	292.2	498	568

TUBE II.

F'	292.2	567	550
E'	219.0	559	533
C'	154.8	554	540
D'	100.3	565	552
B'	57.4	520	588
A'	9	457	573

All numbers are expressed in cm.

It seems worth while to graph the results. We begin with the case that the cocks A and C are open. Fig. 2 refers to this case.

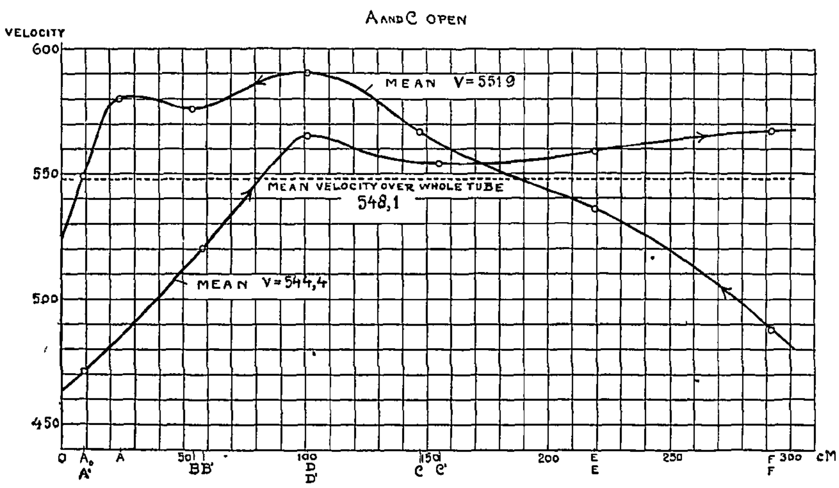

Fig. 2.

The curve with the arrows pointing to the left is travelled over first.

We now proceed to give some details as to the current *at the axis of the tube.* The velocity of the water is rather small immediately after traversing the O-tubes. From F onwards the velocity increases continuously, reaches a maximum at D and then decreases, after passing a smaller maximum, to about 530 cm./sec. After the passage of the horizontal O-tube, connecting the two tubes, the current reaches A' with low velocity. This increases to D', decreases somewhat and again increases to F'. From this representation the mean velocity at the axis in the tube first traversed is 551,9 cm./sec. In the tube with the windows which is passed next, the mean velocity is 544,4 cm./sec. [1]) Hence the mean velocity over the whole length of 604 cm., becomes 548,1 cm. (cocks A and C open).

If the cocks B and D are open, the water first passes by F'. The velocity changes as indicated in Fig. 3. Just as in the case

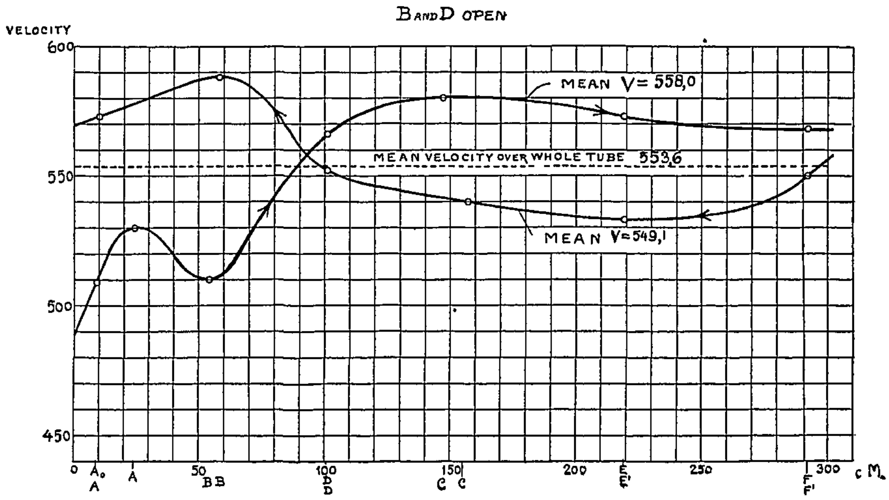

Fig. 3.

first considered the curve with arrows pointing to the left is first travelled over, but in order to apply Fig. 1 to the present case the arrows in it must of course be reversed. Near F' the velocity is rather great, it decreases to E', increases to B' and again decreases, the water after traversing the horizontal O-tube arriving in A_0 with low velocity. After attaining to a secondary maximum in A, the velocity increases to C and then somewhat decreases. The mean velocity in

[1]) [It need scarcely be pointed out that the difference between these two numbers extremely probably is real.] (Note to the translation)

the tube with the windows now appears to be 549.1. In the other tube it is 558.0 cm/sec. The mean value for the whole tube becomes 553,6 (cocks B and D open). The result that in the two cases, when the cocks A and C are open, and, when the cocks B and D are open, the velocities differ, may appear less startling when it is considered that there is a small dissymmetry in the supply tubes of the apparatus and that with A open the water before discharge undergoes a greater change of direction than with D open.

A new proof for the change of velocity at the axis of the tube is given by ascertaining the *velocity distribution over the cross section* of the tube. In A_0 the velocity distribution is entirely different from that encountered for example in B', that is to say before and after the stream traversed the horizontal part.

When the cocks A and C are open the curve traced in A_0 is of a parabolic character, in B' the central part of the curve is of smaller curvature, corresponding to a smaller velocity at the axis. The velocity distribution in a vertical line passing through A_0 is represented in Fig. 4. The curve is of a parabolic character, and nearly, but not quite, symmetrical.

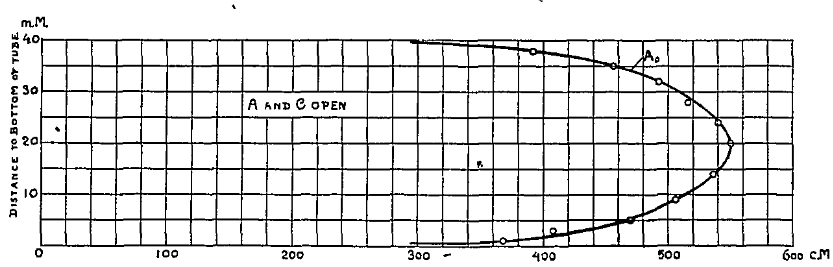

Fig. 4.

Taking the mean abscissae for points at equal distances above and below the axis and constructing a curve with these points we may determine the volume enclosed by the surface of revolution, originating when the constructed curve revolves about the axis. This enables us to determine the mean velocity, which is found to be 468 cm./sec., whereas on a former occasion (Comm. II) using the total quantity of water passing in a given time we obtained 465 cm./sec.

It is worth while to state that the distribution of velocity would according to Poiseuille's law give a parabola of far smaller width than the curve of Fig. 4, if the maximum velocity should be that in the figure.

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When the cocks A and C are open the mean velocity becomes 548.1, with B and D open 553.6. The halved sum of these numbers is 550.8. In observing the interference fringes we did not measure the displacement from a zero position, but determined the total displacement with reversal of the current, which therefore must be proportional to 550.8.

In my former paper I accepted $v_{max} = 553.6$ cm./sec. We now obtain a value for the velocity v_{max} differing only by $^1/_2$ percent from the value used in Communication II and giving excellent agreement between the experiments and the formula with the dispersion term. The difference between the two expressions under consideration amounts for the wavelength 4500 Å. U. to quite 5 percent. We therefore conclude that also as to the *absolute* phase-difference the results of Communication II remain largely in favour of the LORENTZ expression.

The comparison between theory and observation now has become very simple, v_{max} being measured directly. A separate determination of $\varphi = v/v_{max}$ is avoided. Finally however, the value of this ratio may be calculated from the results obtained at the pressure of 2.14 K.g./cm², $v_0 = 465$ and $v_{max} = 550.8$. This gives for the ratio 0.844. This mean number is not, however, a general physical constant but a constant of the apparatus. The course of the curves in Fig. 2 and Fig. 3, suggests for a long tube a final value of φ perhaps 1 or 2 percent lower than 0.844. [1]

The formula for the displacement of the interference fringes must henceforth be written with a factor $\int_0^l v_{max} \, . \, dl$ instead of the simple product $v_{max} \, . \, l$; v_{max} being a function of the distance to the origin of the moving fluid column.

[1] [Only after finishing my investigation, I became acquainted with the important memoir on fluid motion in pipes by Drs. STANTON and PANNELL. (Phil. Trans. vol. 214. 1914). From their data the often mentioned ratio appears to be 0 82 for my case. There is no conflict between the two cases, as their observations were made after the passage of a length of pipe varying from 90 to 140 diameters. This length is sufficient to enable any irregularities in the distribution of velocities to die away. In my apparatus this ideal is largely departed from]. (Note to the translation).