We shall make an estimate of the value of θ by means of the following values, which have all been chosen very favourably. Density of magnetisation 1.000

$$I = 150, \dot{\varphi} = 100, t^2 = 10, B = 2, \frac{2m}{e} = 1,1 \,.\, 10^{-7}.$$

We then find

$$tg\,\theta = -\,0,00013,$$

from which we may conclude, that the deviation will be hardly perceptible.

Mathematics. — "*The circles that cut a plane curve perpendicularly*". II. By Prof. HENDRIK DE VRIES.

(Communicated in the meeting of February 26, 1916).

§ **7.** We found in the preceding § that through the point Z_∞ pass three different kinds of branches of the rest nodal curve, and in particular the branches of the first kind arose in groups of 4 at a time. If $S_{1\infty}$, $S_{2\infty}$ are two simple points of intersection of k^n with l_∞, then the lines of connection of these points with Z_∞ are double torsal lines of Ω (§ 2), and the 4 sheets passing through these torsal lines cut each other in 4 branches of the rest nodal curve, which of course all pass through Z_∞, and have only *one* tangent here, viz. the line of intersection of the two tangent planes along the torsal lines, i.e. the line of connection of Z_∞ with the intersection of the asymptotes of k^n in $S_{1\infty}$ and $S_{2\infty}$.

The branches of this first kind behave again differently according to their going towards the foci and the vertices, or to other points of k^n; the branches going to the foci and the vertices are their own images with regard to β, those to other points, as the nodes, the cusps, the intersections of the isotropical tangents, are each other's images. This difference has an influence on the nature of the tangents in Z_∞; for a branch that is its own image Z_∞ must be a point of inflexion, as on a straight line passing through this point and cutting the curve twice, the two points of intersection approach to Z_∞ from different sides; two branches on the contrary that are each other's images and pass through Z_∞, simply have the same tangent in this point. But whichever of the two cases may arise the projection of 4 branches belonging together produces a node in the intersection of the associated asymptotes of k^n. If namely a twisted curve is projected out of one of its points of inflection, the projection possesses a cusp in the intersection of the inflectional tangent; if therefore the curve is symmetrical with regard to β, and Z_∞ a point of inflection, the projection consists of a branch ending in the intersection of the asymptotes in $S_{1\infty}$ and $S_{2\infty}$, and which is described to and fro. This arises, however, 4 times, and the 4 branches ending thus in one and the same point run together into a curve with a node, for which the discontinuity is again cancelled. And if we have to do with branches which are each other's images and for which Z_∞ is consequently an ordinary point, these branches project themselves in pairs in one and the same branch, which passes, however, through the intersection of the asymptotes of k^μ, and in this way a node also arises in that case. So we have for all cases the following proposition: *the locus of the points out of which two equally long tangents may be drawn at k^μ has nodes in the* $\frac{1}{2}(\mu-2\varepsilon-2\sigma)(\mu-2\varepsilon-2\sigma-1)$ *intersections of the asymptotes of k^μ.*

For the general conic this phenomenon arises once: in fact the locus in question consists here of the two axes, and therefore has a node in the centre; the 4 branches passing through Z_∞ go to the foci here.

If two asymptotes of k^μ are chosen arbitrarily, and a hyperbola is constructed, which has these two lines for asymptotes, and e.g. in order to arrive at the greatest possible contact with k^μ, this curve is osculated in one of the two points at infinity in question, the difference between the tangents at the hyperbola and at the curve becomes practically imperceptible for some point or other in the immediate neighbourhood of the intersection of the asymptotes; from which we may conclude that our locus of points of equal tangents at k^μ passes the intersection of the two asymptotes in the same directions as the axes of the hyperbola, viz. in the directions of the bisectrices of the angles of the asymptotes. We may therefore complete the above found property of our curve by adding that *the two nodal tangents in an intersection of two asymptotes of k^μ bisect the angles of those asymptotes.*

The branches of the 2nd kind passing through Z_∞ arise from the points of contact of k^μ and l_∞, and appear in groups of 8 at a time (cf. § 6); they are in pairs each other's images with regard to β, or, if we subject Ω to a projective transformation, they are associated in pairs to each other in the involutory collineation of which Z_∞ is the centre, and β is the plane.

The 8 branches passing through Z_∞ have therefore in this point only 4 tangents (entirely lying in ε_∞), *and the intersections of these tangents (lying on l_∞) are simple points of the locus of the points* *of equal tangents at k^{μ}. The total number of these points amounts to* $2\sigma(\sigma-1)$, and to the same two points of contact $R_{1\infty}$, $R_{2\infty}$ of k^{μ} and l_{∞} belong 4 of those points. If they are considered as centres of circles twice cutting k^{μ} perpendicularly, the circle itself always coincides with l_{∞}, and though l_{∞} does not cut k^{μ} in $R_{1\infty}$, $R_{2\infty}$, but touches it, there can be no objection, for the line l_{∞} encloses with itself any arbitrary angle. Even planimetrically it is clear that to two points of contact with l_{∞} belong 4 simple infinite branches of the locus of equal tangents; if we viz. imagine 2 parabolic branches of k^{μ}, out of one point or another go 2 tangents at each of them, so that for the equality of 2 of those tangents, touching at different branches, there are 4 possibilities; in this way 4 simple infinite branches of the curve arise.

The branches of the 3rd kind finally arise from the combination of a simple intersection S_{∞} of k^{μ} and l_{∞} with a point of contact R_{∞} (§ 6); they arise in groups of 4, are associated to each other in pairs in the involutory collineation $Z_{\infty}\beta$, and have all 4 only *one* tangent in Z_{∞}, viz. apparently the line $Z_{\infty}S_{\infty}$ itself; the consequence of this is that through S_{∞} pass 2 branches of the locus of equal tangents, that is to say 2 branches for each point R_{∞}, consequently 2σ together. So: *the $\mu-2\varepsilon-2\sigma$ simple intersections of k^{μ} and l_{∞} are for the locus of the points of equal tangents 2σ-fold points.*

This result as well allows of being verified planimetrically.

Let us imagine a hyperbolic and a parabolic branch of k^{μ}, out of a point of β near the asymptote passes then *one* tangent lying very close to that asymptote, while two tangents touch at the parabolic branch; so there are 2 possibilities for the equality of those tangents, and consequently 2 branches of the locus of the points of equal tangents go in the direction of the asymptote towards infinity. Even, as $Z_{\infty}S_{\infty}$ is a torsal line of Ω the plane of osculation in Z_{∞} will coincide with the torsal plane, and consequently the tangent in S_{∞} at each branch of the locus of the points of equal tangents with the asymptote of k^{μ}. We may therefore add to the preceding: *all branches of the locus of the points of equal tangents passing through a point S_{∞} of k^{μ}, have in this point the asymptote of k^{μ} as a tangent.*

§ 8. In § 6 we have determined the number of intersections that an arbitrary plane passing through Z_{∞} has apart from Z_{∞} in common with the rest nodal curve of Ω, and we have determined from it the order d^{k} of the locus of the points of equal tangents at k^{μ}; for the plane ε_{∞}, however, the calculation is somewhat different, as several branches passing through Z_{∞} touch the plane ε_{∞}. The branches of the first kind (§ 6) *cut* ε_∞ in Z_∞; the number of intersections, arising from them, amounts therefore to:

$$2(\mu-2\varepsilon-2\sigma)(\mu-2\varepsilon-2\sigma-1).$$

The $4\sigma(\sigma-1)$ branches of the second kind, and the $4\sigma(\mu-2\varepsilon-2\sigma)$ of the third *touch* ε_∞ on the contrary in Z_∞ and so give respectively $8\sigma(\sigma-1)$ and $8\sigma(\mu-2\varepsilon-2\sigma)$-intersections; the sum total of these three numbers is: $2\mu^2-8\mu\varepsilon-2\mu+8\varepsilon^2+4\varepsilon-4\sigma$.

If this number is subtracted from the order of the rest nodal curve as given in § 6, the number of points ε_∞ has in common with the rest nodal curve apart from Z_∞ is found; this number amounts to

$$\varrho=4\mu\nu-4\mu\sigma-8\nu\varepsilon+8\varepsilon\sigma-13\nu+8\varepsilon+9\sigma+5\mu+\nu^2+\sigma^2-2\nu\sigma+3\iota.$$

These points lie in pairs harmonically with regard to Z_∞ and β, either on k^2_∞ or on the generatrices of Ω lying in ε_∞ and passing through Z_∞: their projections on l_∞ are points at infinity of the locus of the points of equal tangents at k^ν. To these points at infinity however belong also the projections of those points lying infinitely near to Z_∞, and which we have already determined in the preceding §, viz. $2\sigma(\sigma-1)$ simple, and $(\mu-2\varepsilon-2\sigma)$ 2σ-fold ones (the latter lying in the simple intersections of k^ν and l_∞). If these numbers are doubled, are then added to the given number ϱ mentioned above, and the result is divided by 2, the order d of § 6 is exactly found back.

If a branch of the rest nodal curve of Ω gets into k^2_∞, and if we let a point P describe that branch, and that in the direction towards k^2_∞, the image circle of P, which twice cuts k^ν perpendicularly gets greater and greater; if P moves along a h y p e r b o l i c branch, that circle will have as limit a straight line containing the intersection of the asymptote with β, and this straight line will twice cut k^ν perpendicularly, and consequently be a *double normal.* If, however, P moves along a p a r a b o l i c branch, the circle will in the end disappear into infinity. In this way the number of double normals of k^ν is therefore not to be determined; this, however, is not necessary, as we already determined this number pretty nearly in our former paper. (*Anw. Cykl.* p. 21). The double normals of k^ν are namely apparently double tangents of the evolute of k^ν, and for the number of these double tangents we found ibid.:

$$\tfrac{1}{2}\{(\mu+\nu-2\varepsilon-\sigma)(\mu+\nu-2\varepsilon-\sigma-1)-(\iota+3\mu)\}.$$

In this number l_∞, however, is comprised a number of times. The evolute namely has cusps in each of the $\mu-2\varepsilon-2\sigma$ simple intersections of k^ν with l_∞, and that in such a way that the cuspidal tangent coincides with l_∞, and in each of the σ points of contact of k^ν and l_∞ it has points of inflection, while the inflectional tangent coincides with l_∞ again; it is therefore clear that in l_∞ a certain number of double tangents coincide. This number is easy to determine. Let us first imagine 2 of the σ points of inflection and let us observe that in each point of inflection 3 points of the curve lie on a straight line, then it is clear that a double inflectional tangent absorbs 4 double tangents; l_∞ is, however, inflectional tangent for σ points of inflection, consequently it absorbs $\frac{1}{2}\sigma(\sigma-1).4$ double tangents on account of this.

Further is l_∞ $(\mu-2\varepsilon-2\sigma)$-times cuspidal tangent; as such it contains therefore $\frac{1}{2}(\mu-2\varepsilon-2\sigma)(\mu-2\varepsilon-2\sigma-1)$ further double tangents; and finally each cusp may be combined with each point of inflection, which produces 2 double tangents every time, in consequence of the 3 points which lie infinitely near to each other in the point of inflection; to the two preceding numbers $2\sigma(\mu-2\varepsilon-2\sigma)$ must still be added. If the sum of these three numbers is subtracted from the number given higher up, *we find for the number of double normals* of k^ν:

$$\tfrac{1}{2}(2\mu\nu-2\mu\sigma-3\mu+\nu^2-4\nu\varepsilon-2\nu\sigma-\nu+4\varepsilon\sigma+\sigma^2+3\sigma-\iota).$$

To each of these two points of k^2_∞ harmonical with regard to Z_∞ and β are associated, and if we now subtract the double number of double normals from the number of points that ε_∞ outside Z_∞ has in common with the rest nodal curve, as was given at the beginning of this §, we find that

$$2\mu\nu-2\mu\sigma-4\nu\varepsilon+4\varepsilon\sigma-12\nu+8\varepsilon+6\sigma+8\mu+4\iota$$

remain.

Now we know already from the example of the parabola (§ 5) that points of this kind exist in fact; there we found 2, viz. the two points at infinity of the nodal curve of Ω, and the number arrived at here really gives 2 for the parabola. These points lie on the line connecting Z_∞ with the point of contact of the parabola and l_∞, and form a necessary completion of a few numbers found in § 7; where we namely in § 7 considered tangents of equal length at *different* parabolical branches or at parabolical and hyperbolical branches, there we have of course also to consider tangents of equal length at one and the same parabolical branch, and this we find here now; the total number of these points amounts to 2σ, so that the remaining ones indicate parabolical branches of the rest nodal curve.

§ 9. The order of the rest nodal curve of Ω may moreover be

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determined in quite another way than was done in § 6, viz. by making use of the first polar surface of Z_∞. This surface, of which the order is one unit lower than the one of Ω, and therefore (cf. § 2) amounts to $2\mu + 2\nu - 4\varepsilon - 2\sigma - 1$, produces in the first place the "contour apparent" of Ω, seen from point Z_∞; but it is easy to see that there can hardly be question of a real "contour apparent". If namely a straight line passing through Z_∞ touches Ω (in a point outside β we will suppose), consequently intersects it in two points lying infinitely near to each other, it must be possible to describe round the foot of that perpendicular two circles cutting k^ρ perpendicularly whose rays only differ infinitely little, and this is not impossible, for it holds good for all the points of the inflectional tangents of Ω, but then exclusively for them. We found in fact before that the generatrices passing through the points of inflection of k^ρ were torsal lines of Ω with vertical tangent planes; these torsal lines, to the number of 2ι, belong therefore to the intersection of Ω with the first polar surface of Z_∞, and in reality form the only "contour" of Ω for Z_∞, barring of course k^ρ itself, which, as a matter of course and as is clear from the simple example of the hyperboloid of revolution, also belongs to the "contour apparent."

Let us imagine a line passing through Z_∞ drawn to a point P of k^ρ. We know that along k^ρ 2 sheets of Ω osculate each other, and now ask how many points the line $Z_\infty P$ in P has in common with Ω. This number will amount to 4, just as when the sheets simply touched each other along k^ρ, for otherwise the two branches would possess points of inflection in P in an intersection with a plane passing through $Z_\infty P$, which is not the case. Besides, round the foot of an *arbitrary* line passing through Z_∞ ν circles are to be described cutting k^ρ perpendicularly, and round P $\nu-2$; 4 points of Ω have consequently coincided on $Z_\infty P$ in P. Of the first polar surface have therefore on $Z_\infty P$ 3 points coincided in P, and the question is now what is the shape of that polar surface, as it can only touch the two sheets of Ω passing through k^ρ. The fact is that the first polar surface breaks up into a surface and the plane β, and consequently has k^ρ as a nodal curve. A line $Z_\infty P$ now contains the point P of the nodal curve, and moreover a neighbouring point, and therefore 3 indeed.

That β is a part of the first polar surface of Z_∞ ensues already from the symmetry of Ω with regard to β. An arbitrary straight line passing through Z_∞ cuts Ω only in 2ν points not coinciding with Z_∞, and consequently in $2\mu - 4\varepsilon - 2\sigma$ points that do coincide with Z_∞, the first polar surface therefore cuts that straight line as well in $2\mu - 4\varepsilon - 2\sigma$ points coinciding with Z_∞, and further in the $2\nu-1$ harmonic poles of Z_∞ with regard to the 2ν points of Ω not coinciding with Z_∞. Now the first polar surface of Z_∞ must of course be symmetrical with regard to β, as Ω is so too; but the number $2\nu - 1$ is odd, while of these points not one can lie in infinity; consequently *one* must lie in β, and this holds good for any straight line passing through Z_∞.

Analytically too it is easy to see. The 2ν points referred to higher up may be represented by an equation of the form.

$$(z^2 - a^2)(z^2 - b^2)(z^2 - c^2)\ldots = 0,$$

and the harmonical centres for the pole Z_∞ are found from it by differentiation with regard to z; it appears then at once that each term contains the factor z.

The first polar surface of Z_∞ consequently breaks up into the plane β and a surface π_1, which only reaches the order $2\mu+2\nu-4\varepsilon-2\sigma-2$, and only contains k^μ as a simple curve. By the way we will observe that Ω contains still other torsal lines with vertical tangent planes, viz. the tangents out of the two absolute circle-points at k^μ (§ 2), and that these lines therefore also belong to the intersection of Ω with the first polar surface; as they lie, however, in β, and are only to be counted once, they do not lie on π_1.

The intersection of Ω with π_1 now consists of the following parts:

1. The curve k^μ. We saw already that π_1 touches the two sheets of Ω passing through k^μ in k^μ itself, but those two sheets osculate each other, that is to say, in each intersection with a plane passing through Z_∞ they have not 2 but 3 points in common; π_1 must also pass through this third point, that is to say, osculates each of those two sheets, and consequently has the curve k^μ six times in common with Ω.

2. The curve k^2_∞. It is for Ω $(\nu-\sigma)$-fold (§ 2), consequently for π_1 $(\nu-\sigma-1)$-fold; in the intersection of Ω and π_1 this conic counts therefore $(\nu-\sigma)(\nu-\sigma-1)$ times.

3. The $\mu-2\varepsilon-2\sigma$ double torsal lines of Ω arising from the simple intersections of k^μ with l_∞; these lines show for π_1 the same character as for Ω, that is to say, through each of these lines, which are to be considered twice as torsal lines, pass 2 sheets of Ω and 2 of π_1, so that such a line counts 8 times in the intersection.

4. The σ double torsal lines of Ω, arising from the points of contact of k^μ with l_∞; for them the same holds good as for those of the preceding group, each of these σ lines therefore counts 8 times in the intersection.

5. The $2\varkappa$ cuspidal edges of Ω, the 45°-lines passing through the cusps of k'' (§ 3). The cuspidal tangent planes of these cuspidal edges always pass through Z_∞, so that a line $Z_\infty P$ connecting Z_∞ with a point P of such a cuspidal edge has in P with Ω 3, and consequently with π_1 2 points in common. It is, however, easy to see that in the intersection of Ω with π_1, each cuspidal edge is to be counted 4 times, and this is only to be brought into conformity with the rest if we accept that a cuspidal edge of Ω is a nodal edge of π_1.

That this must be so in fact is most easily seen in the example of the plane cubic with cusp. It is of class 3, and the polar conic of an arbitrary pole P passes through the cusp, touches at the cuspidal tangent here and consequently has 3 points in common here with the curve, the remaining 3 intersections are the points of contact of the 3 tangents out of P. If, however, P lies on the cuspidal tangent there touch at the curve but 2 tangents besides this one, the polar conic of P must therefore have now in the cusp 4 points in common with the curve, but the cuspidal tangent has in the cusp only 3 points in common with the curve, and consequently only 2 points in common with the polar conic. These various conditions are only satisfied at the same time if the polar conic degenerates into a pair of lines whose node lies in the cusp. By applying this argument to the cuspidal edges of Ω we easily find that they are nodal edges of π_1, and consequently count 4 times in the intersection with Ω, and as through each cusp of k'' pass *two* of those cuspidal edges, the share contributed by all those cuspidal edges to the intersection is of order $8\varkappa$.

6. The 2ι torsal lines of Ω passing through the points of inflection of k'' (vide supra).

7. The rest nodal curve. It lies on π_1 as a simple curve, and consequently is counted twice in the intersection. The calculation of order d of the nodal curve in this way is as follows. The surfaces Ω and π_1 are respectively of the orders $2\mu+2\nu-4\varepsilon-2\sigma$ and $2\mu+2\nu-4\varepsilon-2\sigma-2$, the order of their complete intersection is therefore the product of these two numbers. In order to find $2d$ now this product is to be diminished by 6μ, $2(\nu-\sigma)(\nu-\sigma-1)$, $8(\mu-2\varepsilon-2\sigma)$, 8σ, $8\varkappa$, 2ι.

§ 10. The second polar surface of Z_∞ with regard to Ω, which we shall call π_2, is of order $2\mu+2\nu-4\varepsilon-2\sigma-2$, consequently as Ω of even order, and therefore need not break up as the former. In fact, if we represent the intersections of a line passing through Z_∞

with the complete 1st polar surface by:

$$z(z^2 - a^2)(z^2 - b^2)\ldots = 0$$

it appears at once by means of differentiation that $z = 0$ satisfies no more. π_2 contains the curve k'', for the latter is a nodal curve for the complete first polar surface, as we saw in the preceding §. It further follows from the symmetry with regard to β that the tangent planes at π_2 are vertical in all the points of k'', while this is also to be deduced from the fact that a straight line $Z_\infty P$, connecting Z_∞ with a point P of k'', touches the surface π_1 in P, and so has here 3 points in common with the complete first polar surface $\pi_1 + \beta$.

This observation is important to us as we want to intersect π_2 with the rest nodal curve of Ω, all the points of the rest nodal curve namely that are lying in β and at the same time on k'' will as a matter of course belong to those intersections. But also the intersections of the rest nodal curve and β not lying on k'', viz. the foci of k'', lie on π_2. The first polar surface $\pi_1 + \beta$ namely must contain the complete nodal curve, π_1 therefore contains the rest nodal curve, consequently also the foci of k'', and the vertical lines passing through these points have in those points three points in common with $\pi_1 + \beta$ and consequently two points with π_2. *So we state that π_2 and the rest nodal curve have in common.* 1. $2(\nu - 2\varepsilon - \tau)^2$ *points, lying in the foci of* k''; 2°) $2(5\mu - 3\nu + 3\iota - 8\varepsilon - 3\sigma)$ *points lying in the vertices of* k''

The rest nodal curve intersects β further in the $2(\mu - \varepsilon - 2)(\nu - 2\varepsilon - \sigma)$ points P, which the tangents out of the two isotropical points have moreover in common with k'' (cf § 6); through each of these points pass 2 branches of the rest nodal curve, which both touch at the same vertical line, and 3 sheets of Ω, which touch at the line $Z_\infty P$ as well; from the latter we deduce that $Z_\infty P$ has in P in common with Ω 6 points, consequently with $\pi_1 + \beta$ 5 points and with π_2 4 points. If we now move the line $Z_\infty P$ a little, and do so parallel to itself, those 4 coinciding points diverge and arrange themselves into 2 pairs which are each other's image with regard to β; from this it ensues that through P pass 2 sheets of π_2, which both touch at $Z_\infty P$ in P. Each branch of the rest nodal curve touches at those two sheets and consequently has with them together 4 coinciding points in common; the two branches consequently 8 points, from which it ensues that *the rest nodal curve and π_2* 3°) *have in common* $16(\mu - \varepsilon - 2)(\nu - 2\varepsilon - \sigma)$ *points, lying in the intersections of the tangents out of the isotropical points at k'' with k''.*

Let us consider a node D of k''. According to § 6 there pass through this point 4 branches of the rest nodal curve, and 4 sheets of Ω, which all touch at the line $Z_\infty D$ in D; $Z_\infty D$ has therefore in D with Ω 8 points in common, consequently 6 points with π_2, and by applying again, as above, the proceeding of the parallel shifting, we find that through D 3 sheets of π_2 pass, which all touch at the line $Z_\infty D$. Each of the 4 branches of the rest nodal curve which pass through D, touches at each of these 3 sheets; this procures in total 24 coinciding points, so that we can say: *the rest nodal curve and π_2 have* 4°) 24δ *points in common, lying in the nodes of k''.*

Let us consider a cusp K of k''. Through an arbitrary point of k'' pass 2 tangents less than through a point that does not lie on k'', and as with each tangent 2 generatrices of Ω correspond, the vertical passing through an arbitrary point of k'' has 4 coinciding points in common with Ω, and passing through a node 8 (vide supra). Through a cusp pass three tangents less than through an arbitrary point of the plane; therefore the vertical $Z_\infty K$ has in K 6 points in common with Ω, and consequently 4 points with π_2. Through K pass 2 cuspidal edges of Ω, and we know (cf. § 9) that they are nodal edges for π_1 and consequently simple edges for π_2. Let us therefore imagine a vertical in the neighbourhood of $Z_\infty K$, cutting the two cuspidal edges, the latter has then on those cuspidal edges already 2 points in common with π_2, and consequently quite close to them 2 more other points, which are each other's image with regard to β. From this we infer that through K too, pass 3 sheets of π_2, viz 2 through the two cuspidal edges and still a third formed by those two other points and touching at the line $Z_\infty K$, because it must have 4 points in common with π_2; and as through K pass 6 branches of the rest nodal curve and that without vertical tangents, there lie in K 18 intersections of the rest nodal curve and π_2 united. Consequently *the rest nodal curve and π_2 have* 5°) $18\,\varkappa$ *points in common, lying in the cusps of k''.*

With this the intersections of the rest nodal curve and π_2, as far as they lie in β, have been summed up.

The rest nodal curve and π_2 have also points at infinity in common. In the first place Z_∞ has as a point of π_2 the same character as as a point of π_1, and as a point of Ω, so that all the points of the rest nodal curve, of which we calculated in § 8 that they lay in Z_∞ or infinitely near to it, also belong to π_2. This number amounted to (§ 8) $2\mu^2 - 8\mu\varepsilon - 2\mu + 8\varepsilon^2 + 4\varepsilon - 4\sigma$, and this would therefore be the number that we have in view here, if π_2, as ε_∞, had a simple point in Z_∞. This, however, is by no means the case, as we observed just now, and the consequence of it is that a far greater number of intersections of the rest nodal curve and π_2 have coincided in Z_∞ than we mentioned just now.

We have to go back to the 3 different kinds of branches of the rest nodal curve passing through Z_∞, which we summed up in § 6. The branches of the first kind, arising from the sheets of Ω passing through the $\mu-2\varepsilon-2\sigma$ double generatrices going to the intersections S_∞ of k'' and l_∞, appear in groups of 4 and touch in Z_∞ at the straight lines going to the intersections of the asymptotes of k''; such a branch touches at the 4 sheets of π_2 passing through the same double generatrices as the sheets of Ω that produce the group of 4, and has therefore in consequence of this already 8 points in common with π_2. It further cuts the $2(\mu-2\varepsilon-2\sigma-2)$ remaining sheets of π_2 of the same kind one by one simply and also the 2σ sheets of π_2 passing through the σ double generatrices of Ω that go to the points of contact R_∞ of k'' and l_∞, so that it has totally in Z_∞ $8+2(\mu-2\varepsilon-2\sigma-2)+2\sigma$ points in common with π_2. The group now consists of 4 of those branches and the number of groups amounts to $\frac{1}{2}(\mu-2\varepsilon-2\sigma)(\mu-2\varepsilon-2\sigma-1)$, we find therefore a number of points:

$$a=4\{8+2(\mu-2\varepsilon-2\sigma-2)+2\sigma\}\cdot\tfrac{1}{2}(\mu-2\varepsilon-2\sigma)(\mu-2\varepsilon-2\sigma-1).$$

The second group arose from the sheets of Ω passing through the lines $Z_\infty R_\infty$, which sheets all touch at ε_∞, the branches meant here appear in groups of 8. One branch out of such a group touches at all the 2σ sheets of π_2 passing through the lines $Z_\infty R_\infty$ and has alone from this source therefore already 4σ points in common with π_2. It cuts on the contrary the $2(\mu-2\varepsilon-2\sigma)$ sheets of π_2 passing through the lines $Z_\infty S_\infty$, and the number of groups amounts to $\frac{1}{2}\sigma(\sigma-1)$; as a second contribution to the number looked for by us we find therefore:

$$b=8\{4\sigma+2(\mu-2\varepsilon-2\sigma)\}\cdot\tfrac{1}{2}\sigma(\sigma-1).$$

Finally we have the branches of the 3rd group procured by the intersection of the sheets of Ω passing through the $\mu-2\varepsilon-2\sigma$ straight lines $Z\ S_\infty$ and the σ straight lines $Z_\infty R_\infty$; these branches appear in groups of 4, and all touch at ε_∞. Such a branch now touches in the first place at the 2σ sheets of π_2 passing through the lines $Z_\infty R_\infty$, which therefore produces 4σ points, touches then moreover at the 2 sheets of π_2 passing through the line $Z_\infty S_\infty$, which so to say belongs to the group, which procures 4 further points, and cuts the $2(\mu-2\varepsilon-2\sigma-1)$ sheets of π_2 passing through the remaining lines $Z_\infty S_\infty$; the number of groups is moreover $(\mu-2\varepsilon-2\sigma)\sigma$, so that the third and last contribution to the numbers wanted is:

$$c = 4\{4\sigma + 4 + 2(\mu - 2\varepsilon - 2\sigma - 1)\}\ (\mu - 2\varepsilon - 2\sigma)\sigma.$$

The number of intersections of the rest nodal curve with π_2 coinciding in Z_∞ is therefore represented by the sum of the numbers a, b and c. At infinity there lie, however, other intersections yet, viz. among others on k^2_∞. This conic is for Ω a $(\nu-\sigma)$-fold one and consequently for π_2 a $(\nu-\sigma-2)$-fold curve, and contains according to § 8 $\varrho-2\sigma$ simple points of the rest nodal curve; the total amount of intersections on k^2_∞ amounts therefore to $(\varrho-2\sigma)\ (\nu-\sigma-2)$.

As to the 2σ points that are withdrawn from ϱ they lie (cf. the example of the parabola in § 5) on the straight lines $Z_\infty R_\infty$, and are intersections of the rest nodal curve with ε_∞. As, however, two sheets of π_2 pass through each line $Z_\infty R_\infty$, the number of intersections from this source becomes 4σ.

§ 11. We have now calculated how many intersections the rest nodal curve and π_2 have in β, and how many they have in ε_∞, if there are more yet, they lie consequently neither in β nor in ε_∞, and it is the nature of these points we really want to find out. As the rest nodal curve lies both on Ω and π_1, an intersection of the rest nodal curve and π_2 lies on Ω, π_1, and π_2, and from this it ensues that the line connecting this point with Z_∞ has in this point 3 coinciding points in common with Ω. This may in general happen as one of the 2 principal tangents of that point passes through Z_∞, but such a thing is excluded for the surface Ω as we saw; if namely a vertical line should contain 3 infinitely near points of Ω, 3 circles cutting k'' perpendicularly might be drawn whose rays differ infinitely little, and this would only be possible if k'' possessed a tangent having contact in 4 points, which we have not supposed. Another possibility remains now, viz. that the points in question are triple points of Ω. Such a point arises when 3 different generatrices of Ω pass through the same point; it is in that case a triple point for the rest nodal curve, and its cyclographic image-circle will therefore cut k'' thrice perpendicularly.

There is, however, a third possibility yet, and this is concerned with cuspidal edges of Ω. Through each cusp of k'' pass two cuspidal edges of Ω, and according to § 9 each suchlike cuspidal edge is a nodal edge of π_1, and consequently also a simple straight line of π_2, while the 3rd polar surface does not contain it any more and so cuts in a certain number of points. Let us consider such an intersection P. This point lies on Ω and on the 1st, 2d and 3d polar surface, from which it ensues that the line $Z_\infty P$ has in P 4 coinciding points in common with Ω. Now has $Z_\infty P$ in P 3 points in common with Ω, if P is an a r b i t r a r y point of the cuspidal edge, a 4^{th} point can therefore only arise as the cuspidal edge is cut by an ordinary generatrix of Ω, the number of these points we find therefore by cutting the cuspidal edge with the 3^{rd} polar surface. This third polar surface now is of order $2\mu + 2\nu - 4\varepsilon - 2\sigma - 3$, and as this order is odd, the surface will have to contain again this plane β with a view to the symmetry with regard to β; what remains is of order $2\mu + 2\nu - 4\varepsilon - 2\sigma - 4$, and will be called π_3. Of the intersections of this surface π_3 with the cuspidal edge, one more point, however, lies in β, viz. in the cusp K, so that only $2\mu + {} + 2\nu - 4\varepsilon - 2\sigma - 5$ remain that do n o t lie in β. According to the preceding § the line $Z_\infty K$ has namely in K 4 points in common with π_2, consequently 3 with $\pi_3 + \beta$, of these o n e belongs to β, so that 2 remain for π_3, and with a view to the symmetry of π_3 with regard to β, they can only lie on a tangent of π_3, π_3 therefore passes through K with o n e sheet. *Each cuspidal edge of Ω is therefore cut by $2\mu + 2\nu - 4\varepsilon - 2\sigma - 5$ ordinary generatrices, and these points are cusps of the rest nodal curve, while their image circles cut k'' twice perpendicularly, of which once in the associated cusp.*

The latter is a matter of course, that, however, the points in question are cusps of the rest nodal curve follows at once from the consideration of the generatrices of Ω, which lie close to the one that cuts the cuspidal edge; they form namely with each other a certain sheet of Ω, and this of course cuts the two sheets meeting in the cuspidal edge in a cusp, the cuspidal tangent lies then in the cuspidal tangent plane of Ω, that is to say in the vertical plane passing through the cuspidal edge. *In the projection we find therefore for the locus of the points of equal tangents at k'' a cusp, lying on a cuspidal tangent; and the number of these points on one and the same cuspidal tangent of k'' amounts to $2\mu + 2\nu - 4\varepsilon - 2\sigma - 5$.*

We saw above that in a cusp P of the rest nodal curve, lying on a cuspidal edge of Ω, the line $Z_\infty P$ has 4 points in common with Ω, and consequently with π_2, which also contains the cuspidal edge, 2; the tangent plane in P at π_2 passes therefore through the cuspidal edge and is vertical. The cuspidal tangent in P lies now, according to the above mentioned fact, in this vertical plane, from which it ensues that the nodal curve and π_2 have 3 coinciding points in common in each point P.

The complete number of intersections of the rest nodal curve and π_2 amounts to $(2\mu + 2\nu - 4\varepsilon - 2\sigma - 2)\,d$; if we now put apart from this all the groups of points summed up in this and the preceding §, the triple points of the rest nodal curve remain, or, more clearly stated, a number of points remain which must of necessity coincide in groups of 3. If we therefore call this number x, the number of triple points of the rest nodal curve is $\frac{1}{3}x$. These points lie in pairs symmetrical with regard to β, *the number of points in β therefore, which are centres of circles that cut k^μ thrice perpendicularly, is $\frac{1}{6}x$, and these points are triple points for the locus of the points of equal tangents.*

We find the following formula for x:

$$x = (2\mu+2\nu-4\varepsilon-2\sigma-2)\,d-2\,(\nu-2\varepsilon-\sigma)^2-2\,(5\mu-3\nu+3\iota-8\varepsilon-3\sigma)-$$
$$-16(\mu-\varepsilon-2)(\nu-2\varepsilon-\sigma)-24\delta-18\varkappa-a-b-c-4o-(\varrho-2\sigma)(\nu-\sigma-2)-$$
$$-6\varkappa(2\mu+2\nu-4\varepsilon-2\sigma-5)\,.$$

It is of course possible to express x exclusively in the fundamental characters chosen by us, viz. $\mu, \nu, \varepsilon, \sigma, \iota$; the formula in that case, however, becomes very intricate, so we prefer to leave it in the form given here, a form which is not more circumstantial for the calculation, and has the advantage that of the parts that must be subtracted, the meaning is easily recognized.

If it is applied either to the general conic or the parabola, it gives $x = 0$, which is correct, as with the conic no circles can appear that cut the curve thrice perpendicularly; for the c_3^3 the calculation is as follows; $\mu = \nu = 3$, $\varepsilon = \sigma = 0$, $\delta = 0$, $\varkappa = 1$, $\iota = 1$; $a = 120$, $b = c = 0$; $d = 36$, $\varrho = 24$, consequently $x = 10.36-18-18-18-48-120-24-42 = 72$; there are therefore 12 points that are centres of circles that cut c_3^3 thrice perpendicularly and are therefore triple points of the locus of equal tangents at k^μ; this locus has moreover 7 cusps on the cuspidal tangent of k^μ, and with that line itself as tangent; they are the centres of the circles that cut c_3^3 perpendicularly in the cusp and moreover somewhere else. For c_4^3 we have: $\mu = 3$, $\nu = 4$, $\varepsilon = o = 0$, $\iota = 3$, $\delta = 1$, $\varkappa = 0$, $d = 48$, $a = 120$, $b = c = 0$, $\varrho = 36$, and so:

$$x = 12.48 - 32 - 24 - 64 - 24 - 120 - 72 = 240\,;$$

the locus of the points of equal tangents has therefore 40 triple points.

To wind up this § we will now sum up what we have found of the locus of the points of equal tangents at k^μ.

This curve is of order d^* (§ 6); in each node of k^ν it has 2 cusps, while through each cusp of k^μ pass 3 branches, which all touch at the cuspidal tangent. Further it passes through each vertex and through each focus of k^ν and it possesses nodes in the intersections of the asymptotes of k^μ, while the nodal tangents bisect the asymptotal angles. Its points at infinity are: 1. $2\sigma(\sigma-1)$ *simple points* (§ 7), 2. $\mu-2\varepsilon-2\sigma$ 2σ*-fold points lying in the intersections of k^μ with l_∞,* *while here all the branches have the asymptotes of k^μ as tangents* (§ 7); 3. $\frac{1}{2}\varrho-\sigma$ *simple points* (§ 8); 4 σ *simple points lying in the points of contact of* k^ν *and* l_∞. *And finally it possesses* $\frac{1}{6}x$ *triple points, and on each cuspidal tangent of* k^ν $2\mu+2\nu-4\varepsilon-2\sigma-5$ *cusps whose tangents all coincide with the cuspidal tangent.*

§ 12. As, by the preceding investigations, the cyclographic surface Ω has been completely inquired into, we must be able now to give an answer to any questions that may arise concerning the circles that cut k^ν perpendicularly. Let us therefore in the first place inquire after the curve that arises if we measure off on each tangent of k^ν from the point of contact a piece of prescribed length on either side; it is clear that we have simply to cut Ω with a plane that runs parallel with β, and that we have to project the intersection on β. We find therefore a curve of order $2(\mu+\nu-2\varepsilon-\sigma)$, which has nodes anywhere where it meets the rest nodal curve, and cusps where it meets the cuspidal edges of Ω. It further passes $(\nu-\sigma)$ times through the absolute points of β, while it passes with 2 branches, which each have the asymptote of k^ν as an asymptote, through each of the $\mu-2\varepsilon-2\sigma$ simple intersections of k^ν and l_∞, and likewise passes with 2 branches through each of the σ points of contact of k^ν and l_∞, while those two branches touch here at k^ν as well.

For the ellipse we find in that way a curve of order 8, consisting of two completely separated and closed ovals. The curve does not possess cusps, but does possess 8 nodes, 4 on the major axis and 4 on the minor one. Of the 4 on the major axis 2 are real nodes, and they of course lie outside the ellipse, while the two others are isolated and lie between the foci; of the 4 on the minor axis 2 are likewise real nodes, and they of course lie again outside the ellipse, while the two others are imaginary in this case. Real points at infinity the curve does not possess at all, they appear with the hyperbola where every time 2 branches also have as asymptotes the asymptotes of the hyperbola. For this, however, the nodes in one axis viz. the non-intersecting one, become all 4 imaginary, while in the chachacter of the 4 on the other axis no change arises; the 2 nodes lie now only between the vertices, and the two isolated nodes outside the foci.

For the parabola the curve is of order 6; it possesses 2 real nodes, both lying on the axis of the parabola; one, a real node, lies outside the parabola, the other, an isolated node, between the focus and the point at infinity, moreover 2 parabolical branches touch at l_∞ in the point at infinity of the parabola. Further are, in the case of the parabola, the circle points, simple points, in the case of the two other conics, nodes.

If the plane of intersection is placed in an oblique position so that it gets an intersection d in common with β, the circles are found that cut k^{v} perpendicularly and d under an angle of constant cosine, which cosine may very well be > 1 (viz. if the angle of the plane with β is $< 45°$), if the angle is exactly 45° the circles cutting k^{v} perpendicularly and touching at d are found.

The circles cutting k^{v} perpendicularly and passing through a given point P of β (which point may or may not lie on k^{v}) are found by cutting the surface Ω with the equilateral cone of revolution with vertical axis, whose vertex lies in P; the circles touching at a given circle by cutting Ω with one of the two equilateral cones of revolution with vertical axis which have the given circle as base-circle; on the other hand we find the circles that cut, besides k^{v}, also an arbitrarily given circle perpendicularly. by cutting Ω with the equilateral hyperboloid of revolution for which that circle is the throat circle.

But instead of the simple figures, point, straight line, and circle, a second arbitrary curve k' may be considered, of order μ', etc. and we may inquire after the circles cutting both these curves perpendicularly at a time, especially *one* twice, the other *once*; it is clear that the answer to any questions that may be put here will be obtained by cutting the surfaces Ω and Ω' with each other. And if one goes a step farther in this direction and combines the surfaces Ω, Ω', Ω'', all the circles that cut 3 given curves at a time perpendicularly are found.

Finally, the cyclographic surfaces, belonging to touching circles and perpendicularly cutting circles may be combined together, and so e.g. investigate the circles that cut *one* of two given curves perpendicularly and touch the other, with the peculiarities consequent on this, as e.g. the circles of curvature, or the twice touching circles of *one* curve, which cut the other perpendicularly, or the circles that cut *one* curve twice perpendicularly and touch at the other, etc., and if one imagines only *one* curve as given, but for this one constructs both the surface belonging to the touching circles and the one belonging to perpendicularly cutting circles, one finds by their intersection the circles that touch a given curve and cut it perpendicularly at a time, with all the peculiarities that may arise here, and without other difficulties having to be overcome with it but those comprised in the tracing of the unreal, and therefore to be separated, solutions.