Mathematics. — "*A simply infinite system of hyperelliptical twisted curves of order five.*" By Prof. JAN DE VRIES.

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§ **1.** By the equations

$$\frac{\alpha a^2_x + \beta b^2_x}{c^2_x} = \frac{\alpha a'_x + \beta b'_x}{c'_x} = \frac{\alpha a''_x + \beta b''_x}{c''_x} \quad . \; . \; . \; . \; . \quad (1)$$

a simply infinite system of twisted curves is determined, which are each the partial intersection of a cubic and a quadratic surface. For, if, for the sake of brevity, the equations (1) are replaced by

$$\frac{d^2_x}{c^2_x} = \frac{d'_x}{c'_x} = \frac{d''_x}{c''_x} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

it appears that the surfaces $d^2_x c'_x = c^2_x d'_x$ and $d'_x c''_x = c'_x d''_x$ have in common the straight line t, which is represented by $c'_x = 0$, $d'_x = 0$. A plane passing through t intersects the two surfaces moreover along a conic and a straight line; from this it ensues that t is a trisecant of the twisted curve ϱ^5 which is determined by the surfaces mentioned.

As (2) may be replaced by

$$\frac{d^2_x}{c^2_x} = \frac{d'_x + \lambda d''_x}{c'_x + \lambda c''_x} = \frac{d''_x}{c''_x} \quad . \; . \; . \; . \; . \; . \quad (3)$$

the trisecants of ϱ^5 may be represented by

$$d'_x + \lambda d''_x = 0 \quad , \quad c'_x + \lambda c''_x = 0 \quad . \; . \; . \; . \; . \; . \quad (4)$$

They form one of the systems of generatrices on the hyperboloid $d'_x c''_x = c'_x d''_x$; the second system of generatrices consists of bisecants of ϱ^5.

The *trisecants* of the curves ϱ^5 determined by (1) are therefore indicated by

$$\alpha a'_x + \beta b'_x + \lambda (\alpha a''_x + \beta b''_x) = 0 \quad . \quad c'_x + \lambda c''_x = 0 \quad . \quad (5)$$

They lie on the hyperboloids of the pencil

$$\alpha (a'_x c''_x - a''_x c'_x) + \beta (b'_x c''_x - b''_x c'_x) = 0 \quad . \; . \; . \; . \quad (6)$$

The base of this pencil consists of the straight line c, represented by $c'_x = 0$, $c''_x = 0$, and a cubic γ^3, of which c is a chord: γ^3 is indicated by

$$\begin{vmatrix} a'_x & b'_x & c'_x \\ a''_x & b''_x & c''_x \end{vmatrix} = 0 \quad . \; . \; . \; . \; . \; . \; . \quad (7)$$

All the *trisecants* t intersect the straight line c and the base-curve γ^3; they form therefore the *congruence* (1,3), which has c and γ^3 as directrices.

Through each point of γ^3 passes a plane pencil of trisecants; this appears moreover from (5): the plane pencil in the plane (λ) has as vertex the intersection of the planes $a'_x + \lambda a''_x = 0$, $b'_x + \lambda b''_x = 0$, $c'_x + \lambda c''_x = 0$.

§ **2.** As the system (1) may be replaced by the system

$$\left.\begin{aligned} \alpha a^2_x + \beta b^2_x + \gamma c^2_x &= 0 \\ \alpha a'_x + \beta b'_x + \gamma c'_x &= 0 \\ \alpha a''_x + \beta b''_x + \gamma c''_x &= 0, \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \quad (8)$$

all the curves ϱ^5 lie on the *quartic surface* Φ^4, represented by

$$\begin{vmatrix} a^2_x & b^2_x & c^2_x \\ a'_x & b'_x & c'_x \\ a''_x & b''_x & c''_x \end{vmatrix} = 0 \; . \quad . \; . \; . \; . \quad (9)$$

Through a point of Φ^4 passes, in general, *one* curve ϱ^5; we shall therefore call the system (ϱ^5) a *pencil*.

An arbitrary straight line is therefore cut by *four* curves ϱ^5. On Φ^4 lies also the curve γ^3, any trisecant t intersects Φ^4 on γ^3 and in the three points, in which it meets the corresponding curve ϱ^5.

All the ϱ^5 pass through the points C_1 and C_2 indicated by $c_x^2 = 0$, $c'_x = 0$, $c''_x = 0$. These points are therefore *singular points* of (ϱ^5).

From (1) it appears that the surface Φ^4 may be produced by combining the pencil

$$\alpha\,(a'_x c''_x - c'_x a''_x) + \beta\,(b'_x c''_x - c'_x b''_x) = 0 \quad . \; . \; . \; . \quad (6)$$

with one of the pencils

$$\left.\begin{aligned} \alpha\,(a^2_x c'_x - c^2_x a'_x) + \beta\,(b^2_x c'_x - c^2_x b'_x) &= 0, \\ \alpha\,(a^2_x c''_x - c^2_x a''_x) + \beta\,(b^2_x c''_x - c^2_x b''_x) &= 0 \end{aligned}\right\} \quad . \; . \; . \; . \quad (10)$$

As product of two projective pencils we find then besides Φ^4 the plane $c'_x = 0$ or the plane $c''_x = 0$.

In connection with this we consider the curve φ^4, in which Φ^4 is intersected by the arbitrary plane φ, as product of a cubic pencil with a quadratic pencil. The first, (φ^3), has two base-points F_1, F_2 on the intersection f of φ with $c'_x = 0$ and seven base-points F_k ($k = 3$ to 9) on φ^4. The second pencil, (φ^2), has a base-point G_1 on f, the remaining three, G_k ($k = 2, 3, 4$) on φ^4. The projectivity has been arranged in such a way that two homologous curves intersect in a point Q of f.

The two pencils determine on φ^4 the same involution I^5; each group Q_x ($k = 1$ to 5) consists of the intersections of φ with one of the curves ϱ^5.

We shall now determine the class of the curve enveloped by the straight line $Q_k Q_l$; it is at the same time the order of the line-complex formed by the bisecants of the curves ϱ^5.

To this purpose we make use of the following general proposition[1]: If a curve φ^n is intersected by a pencil (φ^p) in the groups of an involution I^s, the lines connecting the pairs envelop a curve of class $\frac{1}{2}(n-1)(2s-n)$.

From this it appears that *the bisecants of the curves* ϱ^5 *form a complex of order nine.*

§ **3.** We arrive at the same result by paying attention to the pairs of lines of the pencil (φ^2). The straight line G_2G_3 determines by its intersection Q with f, a φ^2, which meets the straight line G_1G_4 in three points of a curve ϱ^5, hence G_1G_4 is a trisecant t. In the same way G_1G_2 and G_1G_3 appear to be trisecants These three straight lines evidently replace nine bisecants. No bisecant can belong to the plane pencil (G, φ) as it would have to lie then on a pair of lines of (φ^2). From this it ensues that the *complex* of the bisecants is of order *nine.*

In a plane passing through the straight line c, lies, as appeared above, a *plane pencil of trisecants.* As all the ϱ^5 pass through the points C_1, C_2, any ray passing through one of these points, is bisecant for three different curves ϱ^5, which are indicated by the intersections of that ray with Φ^4. In any plane passing through c the complex curve degenerates therefore into *three plane pencils,* which must each be counted thrice.

The *complex cone* of an arbitrary point P has *three triple edges.* One of them is the ray, which the congruence (1,3) of the trisecants sends through P, the other two connect P with the *cardinal points* C_1, C_2.

For a point of the straight line c the complex cone is replaced by the *rational cubic cone,* which projects the curve γ^3, this cone consists completely of *trisecants* and is therefore to be counted thrice.

If P is taken on Φ^4, the complex cone degenerates into the *cone of order four* $\mathfrak{K}^4$, which projects the curve ϱ^5 indicated by P, consequently has a *nodal edge,* and a *cone of order five* $\mathfrak{K}^5$, which is the locus of the sets of four bisecants which the curves ϱ^5 send through P.

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[1]) Cf. my paper "Quadruple involutions on biquadratic curves". (Proceedings and Communications of the Royal Ac. of Sc. section Physics, series III, volume 4, p. 312 French translation in Archives Neerlandaises, Vol. 23, page 93).

If P lies on the curve γ^4, $\mathfrak{K}$ degenerates moreover into the *plane pencil of the trisecants,* lying in the plane (Pc) and a *quadratic cone* $\mathfrak{K}^2$.

This cone contains the bisecants belonging to the second system of generatrices of the hyperboloids (6). For they are indicated (cf. § 1) by

$$\alpha a'_x + \beta b'_x + \gamma c'_x = 0 \quad , \quad \alpha a''_x + \beta b''_x + \gamma c''_x = 0,$$

they are therefore the bisecants of the curve γ^3, which according to (7) is determined by

$$\begin{vmatrix} a'_x & b'_x & c'_x \\ a''_x & b''_x & c''_x \end{vmatrix} = 0.$$

The edges of $\mathfrak{K}^2$ *therefore project* γ^3 *out of* P *as centre.*

§ **4.** An involution I^s, which is produced by the intersection of a pencil of curves on a curve of genus g, has $2(g+s-1)$ groups with a double point [1]). In an arbitrary plane lie therefore 14 touching bisecants; in other words the *tangents* r of the curves ϱ^5 form a *congruence* of *class fourteen.*

If the plane φ passes through c, *ten* of those tangents belong to the plane pencil of trisecants lying in it; they are the tangents at φ^4 from the vertex of the plane pencil. The tangents in C_1 and C_2 at φ^4 are therefore to be counted *twice*

In order to be able to determine the order of the congruence $[r]$, we consider the twisted curve containing the ends Q, Q' of the chords lying on the complex cone of a point P. As this cone is of order 9, the order of the curve in question amounts to 18; this φ^{18} has evidently *nodes* in the ends of the triple chord, lying on that cone (§ 3) and triple points in C_1 and C_2.

The planes connecting Q and Q' with the arbitrary straight line l agree in an involutorial correspondence (18, 18). of which the plane (Pl) represents an 18-fold coincidence. The remaining 18 arise from pairs $Q' \equiv Q$, consequently from tangents r; hence the order of $[r]$ is *eighteen.*

The points C_1 and C_2 are *singular points* of *order one.*

1) Ibid. p. 322.