Physics. — *"On the use of Third Degree Terms in the Energy of a Deformed Elastic Body."* By J. TRESLING. (Communicated by Prof. H. A. LORENTZ).

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§ 1. We shall indicate the deviations to which a point x, y, z of an elastic body is subjected in a deformation by ξ, η, ζ.

It is easily shown [1]) that this change of form can be obtained by making the dimensions in 3 definite directions normal to each other resp. $\sigma_1, \sigma_2, \sigma_3$ times smaller, and by then rotating the body. If we write S_i for $\frac{1}{\sigma_i^2}$, the three values of S_i are determined as the three roots of the equation:

$$S^3 - (3+2J_1)S^2 + (3+4J_1+4J_2)S - (1+2J_1+4J_2+8J_3) = 0$$

in which:

$$J_1 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$$
$$J_2 = \varepsilon_2\varepsilon_3 + \varepsilon_3\varepsilon_1 + \varepsilon_1\varepsilon_2 - \tfrac{1}{4}(\gamma_1^2 + \gamma_2^2 + \gamma_3^2)$$
$$J_3 = \varepsilon_1\varepsilon_2\varepsilon_3 + \tfrac{1}{4}(\gamma_1\gamma_2\gamma_3 - \varepsilon_1\gamma_1^2 - \varepsilon_2\gamma_2^2 - \varepsilon_3\gamma_3^2)$$

and the $\varepsilon_1 \ldots \gamma_3$ are the following functions of the 9 differential quotients $\frac{\partial \xi}{\partial x} \ldots$ etc.:

$$\varepsilon_1 = \frac{\partial \xi}{\partial x} + \tfrac{1}{2}\left[\left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial x}\right)^2 + \left(\frac{\partial \zeta}{\partial x}\right)^2\right]$$
$$\varepsilon_2 = \frac{\partial \eta}{\partial y} + \tfrac{1}{2}\left[\left(\frac{\partial \xi}{\partial y}\right)^2 + \left(\frac{\partial \eta}{\partial y}\right)^2 + \left(\frac{\partial \zeta}{\partial y}\right)^2\right]$$
$$\varepsilon_3 = \frac{\partial \zeta}{\partial z} + \tfrac{1}{2}\left[\left(\frac{\partial \xi}{\partial z}\right)^2 + \left(\frac{\partial \eta}{\partial z}\right)^2 + \left(\frac{\partial \zeta}{\partial z}\right)^2\right]$$
$$\gamma_1 = \frac{\partial \eta}{\partial z} + \frac{\partial \zeta}{\partial y} + \frac{\partial \xi}{\partial y}\frac{\partial \xi}{\partial z} + \frac{\partial \eta}{\partial y}\frac{\partial \eta}{\partial z} + \frac{\partial \zeta}{\partial y}\frac{\partial \zeta}{\partial z}$$
$$\gamma_2 = \frac{\partial \zeta}{\partial x} + \frac{\partial \xi}{\partial z} + \frac{\partial \xi}{\partial z}\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial z}\frac{\partial \eta}{\partial x} + \frac{\partial \zeta}{\partial z}\frac{\partial \zeta}{\partial x}$$
$$\gamma_3 = \frac{\partial \xi}{\partial y} + \frac{\partial \eta}{\partial x} + \frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + \frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial \zeta}{\partial x}\frac{\partial \zeta}{\partial y}.$$

The J_1, J_2, J_3 are invariant in case of axial rotation. The free energy of an elastic isotropic body will be a symmetric function of $\sigma_1, \sigma_2, \sigma_3$.

If we confine ourselves to terms which are of the 2nd and 3rd

[1]) DUHEM, Recherches sur l'élasticité, 1906.

degree with respect to the 9 differential quotients, we may write for that free energy :

$$\varphi\, d\tau = [(\tfrac{1}{2}\lambda + \mu) J_1^2 - 2\mu J_2 + CJ_1^3 + DJ_1J_2 + EJ_3]\, d\tau ,$$

in which $d\tau$ is the volume element of the undeformed body. In strained condition it is $d\tau'$.

The variation of energy after a virtual transformation from a strained condition amounts to :

$$\frac{\partial \varphi}{\partial \varepsilon_1} d\varepsilon_1 + \ldots . + \frac{\partial \varphi}{\partial \gamma_3} d\gamma_3 .$$

The virtual transformation is determined by its 6 strain components $D_1, D_2, D_3, G_1, G_2, G_3$.

The $d\varepsilon_1 \ldots d\gamma_3$ can be expressed linearly in these. The variation of energy amounts to :

$$(X_x D_1 + Y_y D_2 + Z_z D_3 + 2\, Y_z G_1 + 2\, Z_x G_2 + 2\, X_y G_3)\, d\tau'$$

in which

$$-(1+J_1) X_x = \left(1 + \frac{\partial \xi}{\partial x}\right)^2 \frac{\partial \varphi}{\partial \varepsilon_1} + \left(\frac{\partial \xi}{\partial y}\right)^2 \frac{\partial \varphi}{\partial \varepsilon_2} + \left(\frac{\partial \xi}{\partial z}\right)^2 \frac{\partial \varphi}{\partial \varepsilon_3} +$$
$$+ 2 \frac{\partial \xi}{\partial y}\frac{\partial \xi}{\partial z}\frac{\partial \varphi}{\partial \gamma_1} + 2 \frac{\partial \xi}{\partial z}\left(1 + \frac{\partial \xi}{\partial x}\right)\frac{\partial \varphi}{\partial \gamma_2} + 2\left(1 + \frac{\partial \xi}{\partial x}\right)\frac{\partial \xi}{\partial y}\frac{\partial \varphi}{\partial \gamma_3}$$

with two analogons for Y_y and Z_z.

$$-(1+J_1) Y_z = \frac{\partial \eta}{\partial x}\frac{\partial \zeta}{\partial x}\frac{\partial \varphi}{\partial \varepsilon_1} + \left(1 + \frac{\partial \eta}{\partial y}\right)\frac{\partial \zeta}{\partial y}\frac{\partial \varphi}{\partial \varepsilon_2} + \frac{\partial \eta}{\partial z}\left(1 + \frac{\partial \zeta}{\partial z}\right)\frac{\partial \varphi}{\partial \varepsilon_3} +$$
$$+ \left\{\left(1 + \frac{\partial \eta}{\partial y}\right)\left(1 + \frac{\partial \zeta}{\partial z}\right) + \frac{\partial \eta}{\partial z}\frac{\partial \zeta}{\partial y}\right\}\frac{\partial \varphi}{\partial \gamma_1} + \left\{\frac{\partial \eta}{\partial z}\frac{\partial \zeta}{d x} + \frac{\partial \eta}{\partial x}\left(1 + \frac{\partial \zeta}{\partial z}\right)\right\}\frac{\partial \varphi}{d \gamma_3} +$$
$$+ \left\{\frac{\partial \eta}{\partial x}\frac{\partial \zeta}{\partial y} + \left(1 + \frac{\partial \eta}{\partial y}\right)\frac{\partial \zeta}{\partial x}\right\}\frac{\partial \varphi}{\partial \gamma_3}$$

with two analogons for Z_x and X_y . .

They give the stress components as sum of differential quotients of the φ with respect to the strain components.

§ 2. In § 1 we have placed side by side the formulae of DUHEM, which we shall require for the comparison of two papers [1]) on the changes which take place in the dimensions when a strained steel wire is twisted. In this § we shall give the results of their application in some special cases.

Let us give to a body a dilatation α, β, γ, resp. in the x, y, and z

[1]) H. A. LORENTZ. The expansion of Solid Bodies by Heat. Verslag Kon. Ak. Oct. 1915 p. 671. POYNTING. On the changes in the Dimensions of a Steel Wire when Twisted and on the Pressure of Distortional Waves in Steel. Proc. Royal Soc. (A) 86, 1912, p. 534.

direction, which causes the point x, y, z to get at the place x', y', z'; $x' = x(1+\alpha)$, $y' = y(1+\beta)$, $z' = z(1+\gamma)$. Then we may inquire into the tensions which occur in a new deformation, and also into the increase of energy that takes place then.

I have carried out this calculation for two special cases.

If the said dilatations are followed by a shear

$$x'' = x' + \varepsilon z' \quad y'' = y' \quad z'' = z', \quad . \quad . \quad . \quad . \quad . \quad (1)$$

then:

$$X_z(1 + \alpha + \beta + \gamma) = \frac{\partial \varphi}{\partial \varepsilon}$$

$$\varphi\, d\tau = \varphi_0\, d\tau + \frac{\varepsilon^2}{2}\left[\mu\left(1 - \frac{\alpha}{2} - \beta + \frac{3}{2}\gamma\right) + \right.$$

$$\left. + \left(\lambda + \frac{5}{2}\mu - \frac{D}{2}\right)(\alpha + \beta + \gamma) - \frac{E + 3\mu}{2}\beta\right] d\tau \quad . \quad . \quad (2)$$

In φ_0 the terms are comprised which are independent of ε. We may also write:

$$X_z = \varepsilon\left[\mu\left(1 - \frac{3}{2}\alpha - 2\beta + \frac{1}{2}\gamma\right) + \right.$$

$$\left. + \left(\lambda + \frac{5}{2}\mu - \frac{D}{2}\right)(\alpha + \beta + \gamma) - \frac{E + 3\mu}{2}\beta\right] \quad . \quad . \quad . \quad (3)$$

If, however, after the same dilatations α, β, γ we apply the shear ε in such a way that the new change of position of the particles is expressed by:

$$x'' = x' + \frac{\varepsilon}{2} z' \quad y'' = y' \quad z'' = \frac{\varepsilon}{2} x' + z' \quad . \quad . \quad . \quad . \quad (4)$$

we find other values, namely:

$$\varphi d\tau = \varphi_0\, d\tau + \frac{\varepsilon^2}{2}\left[\mu + \left(\frac{\lambda}{2} + \frac{5}{2}\mu - \frac{D}{2}\right)(\alpha + \gamma) - \left(-\frac{\lambda}{2} + \frac{D}{2} + \frac{E}{2}\right)\beta\right] d\tau \; . \; (5)$$

$$X_z = \varepsilon\left[\mu\left(1 - \frac{\alpha}{2} - 2\beta - \frac{\gamma}{2}\right) + \left(\lambda + \frac{5}{2}\mu - \frac{D}{2}\right)(\alpha + \beta + \gamma) - \frac{E + 3\mu}{2}\right] (6)$$

We see that in this case X_z cannot be obtained by differentiating φ with respect to ε, as in the preceding one.

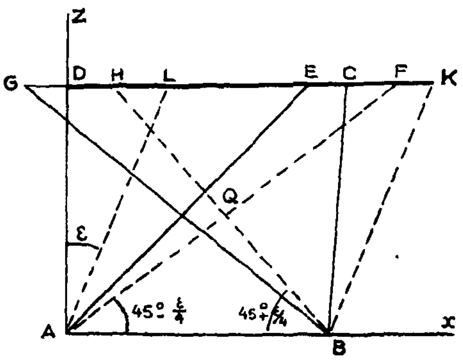
As third application we calculate, accurate down to ε^2, the tensions which occur when a body is deformed out of its natural position according to the equations:

$$x' = x + \varepsilon z \quad y' = y \quad z' = z.$$

We then find:

$$\left.\begin{aligned} X_x &= \tfrac{1}{2}\,\varepsilon^2\left(-\lambda - 3\mu + \frac{D}{2} - \mu\right) \\ Y_y &= \tfrac{1}{2}\,\varepsilon^2\left(-\lambda + \frac{D}{2} + \frac{E}{2}\right) \\ Z_z &= \tfrac{1}{2}\,\varepsilon^2\left(-\lambda - 3\mu + \frac{D}{2} + \mu\right) \\ Y_z &= 0 \qquad Z_x = -\mu\varepsilon \qquad X_y = 0 \end{aligned}\right\} \quad \ldots\ldots \quad (7)$$

§ 3. After POYNTING had first given considerations about the changes of the dimensions in a twisted wire, which he had to relinquish later on, a supposition is made in his more recent consideration, which is *not evident from the standpoint of a third degree potential energy*.



If we transform a cube $ABCD$ by a shear over an angle ε to $ABKL$, the line GB. which undergoes the greatest contraction will get into BH after the deformation, so that angle $ABH = 45° + \frac{\varepsilon}{4}$. The line AF, which has been most stretched, will make an angle $45° - \frac{\varepsilon}{4}$ with AB. This holds for terms up to ε^2 inclusive. POYNTING's supposition now runs that only a normal pressure acts on AQ, and only a normal tension on BQ, and that therefore no tangential stresses exist along AQ and BQ, not of the 2nd order either. POYNTING introduces two new elasticity constants p and q; he does so in the following way. The pressure on AQ will amount to $\mu\varepsilon + p\varepsilon^2$ in 2nd approximation, that on BQ will have a value $-\mu\varepsilon + p\varepsilon^2$, and the pressure normal to the plane of drawing $q\varepsilon^2$.

The problem raised is the following one. A long, thin cylindrical rod is twisted over an angle ϑ without being pressed sideways or on the end planes. Required is the increase in length, the decrease in thickness, and the shortening of the radius at any point in a section. If the three formulae have been found for this, the first two will make it possible to derive the values of p and q from the observed change ot the dimensions. The third formula is a relation that is not practically controllable.

We now first examine what relation there is between the quantities p, q, and those which we have above introduced. POYNTING calculates from his suppositions that a normal pressure $(\frac{1}{2}\mu + p)\,\varepsilon^2$ acts on AB, and a normal pressure $(-\frac{1}{2}\mu + p)\,\varepsilon^2$ on AD, the tangential stresses $\mu\varepsilon$ existing besides. This appears to agree entirely with our equations (7), and the relation between the elasticity constants of POYNTING and ours is expressed by

$$p = \frac{D}{4} - \frac{\lambda}{2} - \frac{3\mu}{2} \qquad q = \frac{D}{4} + \frac{E}{4} - \frac{\lambda}{2}.$$

With these values of p and q we can follow the reasoning of POYNTING. The result can be represented as follows in another notation. A rod of a length l and a section with the radius R, on being twisted over an angle θ, and not subjected to any external pressure, becomes longer in the ratio of 1 to $1+\gamma$, its radius changes in ratio of 1 to $1+\sigma$. A point at a distance r from the axis will get at a distance $r(1+s)$ from it. The quantities γ, σ, and s are found from:

$$\left.\begin{aligned} 2\lambda\gamma + 4(\lambda+\mu)\,\sigma &= -\frac{\theta^2 R^2}{4l^2}(\mu - 2p - 2q) \\ (\lambda + 2\mu)\,\gamma + 2\lambda\sigma &= \frac{\theta^2 R^2}{4l^2}(\mu + 2p) \\ s &= \frac{\theta^2}{16l^2}\,\frac{\mu - 2p + 6q}{\lambda + 2\mu}(r^2 - R^2) + \sigma \end{aligned}\right\} \quad . \; . \; . \quad (8)$$

(formulae (8), (9), and (10) in the cited paper by POYNTING).

Observed were σ and γ. The two first formulae gave the possibility to find the quantities p and q for a definite steel wire. For that wire $\lambda = 9{,}77 \times 10^{11}$; $\mu = 8{,}35 \times 10^{11}$. The values for p and q were then $p = 1{,}67 \times 10^{12}$; $q = -0{,}70 \times 10^{12}$, hence $D = 13{,}6 \times 10^{12}$, $E = -14{,}5 \times 10^{12}$, all this expressed in C, G, S unities.

Prof. LORENTZ treats the same problem, for which other constants of elasticity a and b are introduced. The three equations (29), (30), and (28) in his paper can, however, not be made to agree all at the same time with the equations (8) by a suitable connection between p, q on one side, and a, b on the other side. The coefficients a and b introduced by LORENTZ occur as follows. When a body which has undergone the dilatations α, β, γ in the directions of the coordinates, by which the point x, y, z has got into x', y', z', then undergoes a shear ε in the x,z-plane, which causes that point to be displaced to $\varepsilon x''$, y'', z'', Prof. LORENTZ puts $X_z = \mu' \left(\frac{\partial z'}{\partial x'} + \frac{\partial x''}{\partial z'}\right)$ and the increase of the density of energy $= \frac{1}{2} \mu' \left(\frac{\partial z''}{\partial x'} + \frac{\partial x''}{\partial z'}\right)^2$, in which μ' is a coefficient of rigidity changed by the preceding dilatations α, β, γ. Prof. LORENTZ puts for this $\mu' = \mu + a(\alpha + \gamma) + b\beta$. That α and γ only occur here in the combination $\alpha + \gamma$ is proved on the supposition that the tension X_z depends really only on $\frac{\partial z''}{\partial x'} + \frac{\partial x''}{\partial z'}$, which does not appear to be the case, when we compare formulae (3) and (6), which give us the tensions for two deformations with equal $\frac{\partial z''}{\partial x'} + \frac{\partial x''}{\partial z'} = \varepsilon$. The proof remains valid when we effect the shear according to (4); then, however, the increase of energy is not given by $\frac{1}{2} \mu' \varepsilon^2$, which we see by (5) and (6). In our problem, we must however shear according to (1). Then α and γ do not occur any longer only in the combination $\alpha + \gamma$, and for the increase of energy we must use formula (2).

Starting from this and following for the rest Prof. LORENTZ's reasoning, we come to the same result as POYNTING.

Appendix by H. A. LORENTZ.

Mr. TRESLING is right. On account of an error in the reasoning of which I have made use in the note on p. 673 of my communication, my formula (21) is not correct; it should run:

$$\mu' = \mu(1 + 2\mathbf{z}) + a(\mathbf{x} + \mathbf{z}) + b\mathbf{y}.$$

Consequently in the expression derived from it for the change of the free energy per volume unity $\mu(1 + 2q + 2s)$ should be substituted for $\mu(1 + 2s)$, and in the second integral (22) $+q$ for $-q$. Then $\mu + a$ comes in the place of $-\mu + a$, in the expression for Q on p. 675, and equation (30) becomes:

$$(\lambda + 2\mu) q + 2\lambda\mathbf{s} = -\frac{\Theta^2 R^2}{4l^2}(\mu + a).$$

If we replace in this, and also in (28) and (29):

	q	, $\mathbf{s}$	, a	and	b
by	γ	, σ	, $-2\mu - 2p$	,,	$-\mu - 2q$,

we get exactly the above formulae (8).

I shall communicate on a later occasion what modifications my further calculations now must undergo.