For great values of C_1 the critical current density is therefore, independent of the value of the solubility product, i. e. the value is the same for all halogens, as D_1 is here about equal.

It is different when C_1 has a small value, one that is comparable with $\sqrt{L}$.

The critical current density is $=0$, when $D_1 C_1 = D_2 \frac{L}{C_1}$ or, D_1 and D_2 differing little, when $C_1 = \sqrt{L}$. For silver chloride, for which $L = 10^{-10}$, the critical density will therefore be $=0$ for $C_1 = 10^{-5}$. Already at the smallest possible current density more AgCl will here be deposited in the liquid than on the anode. If on the other hand we work with an iodide, practically all the silver iodide will be deposited on the anode for $C_1 = 10^{-5}$ as $L_{AgI} = 10^{-16}$ and the critical current density is not $=0$ until $C_1 = 10^{-8}$.

By the aid of the above considerations it is now possible to indicate in what way the electro-analytic determination of the halogens can take place most rationally, as will be set forth in the following paper.

Chemical Laboratory of the University.

Amsterdam, June 1916.

Mathematics. — *"Skew Frequency Curves."* By M. J. van Uven. (Communicated by Prof. J. C. Kapteyn).

(Communicated in the meeting of October 28, 1916).

The skewness of a frequency-curve appertaining to some observed quantity x may be explained, as Prof. J. C. Kapteyn [1]) has shown, without dropping the normal Gaussian law of error, namely by supposing that, instead of the observed quantity x, a certain function of x: $Z = F(x)$, is spread according to the normal law.

Denoting the mean value of Z by M and the modulus of precision by h, the quantity

$$z = h(Z - M) = h\{F(x) - M\} = f(x)$$

will be distributed round the mean value zero with the modulus of precision unity, so that the probability that z is found between z_1 and z_2 is represented by

$$W_{z_1}^{z_2} = \frac{1}{\sqrt{\pi}} \int_{z_1}^{z_2} e^{-z^2}\, dz \,.$$

1) J. C. Kapteyn: Skew Frequency Curves in Biology and Statistics; Groningen, 1903, Noordhoff.

Prof. J. C. KAPTEYN and the author of this paper [1]) have developed a method to derive the so-called "normal function" $z = f(x)$ from the given frequency-distribution, by applying the principle corresponding values of x and z are equally probable.

Then it appears that x, as function of z, must be one-valued.

Two simplifications are besides introduced:

1. In the whole real domain, we suppose

$$f'(x) > 0 .$$

hereby we prevent that $\frac{dz}{dx} = f'(x)$ may vanish, and consequently that z can be a many-valued function of x.

2. The lower limit x_0 of the real domain corresponds to $z = -\infty$.

In the following paper we shall expand the thus far developed theory by dropping the two simplifications mentioned.

A. First we drop the latter simplification while retaining the first. So we suppose that the extreme limits x_0 and x_n correspond to the values z_0 and z_n resp. of z, the extreme limits $-\infty$ and $+\infty$ of z being in their turn conjugate to the values $x_{-\infty}$ and x_{+} of x.

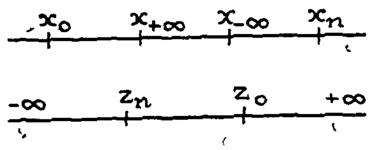

FIG. 1a.

If $x_{+\infty}$ and $x_{-\infty}$ do not coincide, then no part of the real domain is found between these values; so the real domain consists of the partial domains $x_0 \ldots x_{+\infty}$ and $x_{-\infty} \ldots x_n$. In fact because $f'(x)$ must always be > 0, to $z_0 < +\infty$ must correspond $x_0 < x_{+\infty}$, and to $z_n > -\infty$ $x_n > x_{-\infty}$.

The segments $x_n \ldots +\infty$ and $-\infty \ldots x_0$, which do not belong to the real domain, are represented together in the segment between z_n and z_0 of the axis of z, and as x, in passing from x_n to x_0, continually increases (excepting the fall from $+\infty$ to $-\infty$), also z_n will be less than z_0.

Now the quantity z must pass through all values from $-\infty$ to $+\infty$; so on the axis of z no segments are found which do not belong to the real domain. Consequently the points z_n and z_0 must coincide.

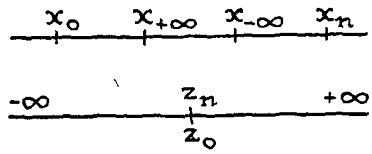

FIG. 1b.

Hence we have this situation (fig. 1b). If the domain between x_0 and x_n has no gap, there is also coincidence of $x_{+\infty}$ and $x_{-\infty}$.

Such a correspondence is generated by the function

1) J. C. KAPTEYN and M. J. VAN UVEN: Skew Frequency Curves in Biology and Statistics, 2nd Paper; Groningen, 1916 Hoitsema. Br.

$$z = -\frac{1}{x},$$

where $x_0 = -\infty$, $x_n = +\infty, x_{+\infty} = x_{-\infty} = 0$, with $z_0 (\equiv z_n) = 0$.

Another example of a generating function is

$$z = 1 : log \frac{(x - x_0)(x_n - x)}{(x_n \quad x_\infty)(x - x_0)}$$

where $z_0 = 0$.

Giving up the simplification $\left.\begin{matrix} z_0 \\ z_n \end{matrix}\right\} = \mp \infty$ may lead to an easier explanation of frequencies in two ways:

For one thing: to choose a value x_∞ conjugate to $z = \mp \infty$ within the frequency-domain $(x_0 < x_\infty < x_n)$ may be advantageous if the frequencies become exceedingly small somewhere within the domain. In this case the theoretical value of the ordinate of the frequency-curve: $y = \frac{1}{\sqrt{\pi}} f'(x) e^{-[f(x)]^2}$ for $x = x_\infty$ is zero (values of $f'(x_\infty)$ of an excessive order of infinity being excluded).

Moreover to join finite values of z to the limits x_0 and x_n of x may help to make high frequencies at the limits admissible, the factor $e^{-[f(x)]^2}$ not being infinitesimal at the limits.

Next we shall examine what happens if we drop the first simplification also and accordingly suppose the function $z = f(x)$ (as function of x) to be many-valued. So to one value of x several values of z may correspond, and infinite values of $\frac{dz}{dx} = f'(x)$ are admitted for finite values of z.

On this supposition there must be partial domains where $f'(x) < 0$, since $\frac{dz}{dx}$, in passing through ∞, changes its sign.

Thus we seem to come into conflict with the condition that the observed frequency

$$I_{x_1}^{x_2} = \frac{1}{\sqrt{\pi}} \int_{x_1}^{x_2} f'(x) e^{-[f(x)]^2} dx$$

must be positive.

This apparent difficulty is removed by considering that the integral may yet turn out positive, provided that we invert the sense of integrating, so that we proceed along the axis of x in a negative sense in those segments, where $f'(x) < 0$.

We shall now discuss successively two- and three-valued functions.

B. Two-valued functions.

We first consider functions which are two-valued either in the whole real domain or in a part of it.

At the point(s), where the real domain borders on the imaginary one, the two real values of z, which correspond to a single value of x, pass into two imaginary values. So at the limits of the real domain themselves the two values of z coincide. The limits x_0 and x_n of the real domain are the branch-points of the function $z = f(x)$.

Now at the limits of the domain $\frac{dz}{dx} = f'(x) = \infty$; the conjugate value of z may be either finite or infinite. If this value of z is finite, the ordinate

$$y = \frac{1}{\sqrt{\pi}} f'(x)\, e^{-[f(x)]^2}$$

of the ideal frequency-curve is infinite at that point.

If, on the contrary, the corresponding value of z is infinite, this ordinate is likely to be infinitesimal or zero. Only for exceptional forms of $f(x)$ it might be finite.

If now the frequency-table $Y_1, Y_2, \ldots Y_n$ (Y_k individuals lie between the class-limits x_{k-1} and x_k) begins with a very high value Y_1, decreasing till the last frequencies are zero, we may explain this by means of a two-valued function, having a branch-point in x_0 with a finite z, and another in x_n with an infinite z.

Let us take as an example

$$z = f(x) = \pm \sqrt{x}.$$

whence

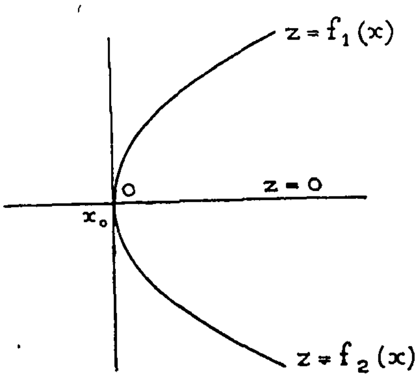

Fig. 2.

$$\frac{dz}{dx} = f'(x) = \pm \frac{1}{2\sqrt{x}}.$$

Here the branch-points are

$$x_0 = 0 \quad \text{with} \quad z_0 = 0,$$

$$x_n = +\infty \quad \text{with} \quad z_n = \pm \infty.$$

The two branches of the function are

$$f_1(x) = +\sqrt{x}, \text{ with } f_1'(x) = +\frac{1}{2\sqrt{x}}$$

$$\text{and } y_1 = \frac{+1}{2\sqrt{\pi x}}\, e^{-x}$$

$$f_2(x) = -\sqrt{x}. \quad \text{with} \quad f_2'(x) = -\frac{1}{2\sqrt{x}} \quad \text{and } y_2 = \frac{-1}{2\sqrt{\pi x}}\, e^{-x}.$$

In the first branch x ranges from 0 to $+\infty$. The contribution to the frequency in the segment pq between $x = x_p$ and $x = x_q (> x_p)$ is then

$$\Delta_1 I = \frac{1}{\sqrt{\pi}} \int_{x_p}^{x_q} f_1'(x)\, e^{-[f_1(x)]^2}\, dx = \int_{x_p}^{x_q} \frac{+1}{2\sqrt{\pi x}}\, e^{-x}\, dx.$$

In the second branch x comes back from $+\infty$ to 0. So the contribution to the frequency in the same segment pq equals

$$\Delta_2 I = \frac{1}{\sqrt{\pi}} \int_{x_q}^{x_p} f_2'(x)\, e^{-[f_2(x)]^2}\, dx = \int_{x_q}^{x_p} \frac{-1}{2\sqrt{\pi x}}\, e^{-x}\, dx.$$

The total frequency in the domain in question is therefore found to be:

$$I_p^q = \Delta_1 I + \Delta_2 I = \int_{x_p}^{x_q} \frac{+1}{2\sqrt{\pi x}}\, e^{-x}\, dx + \int_{x_q}^{x_p} \frac{-1}{2\sqrt{\pi x}}\, e^{-x}\, dx = 2 \int_{x_p}^{x_q} \frac{+1}{2\sqrt{\pi x}}\, e^{-x}\, dx$$

The ideal continuous frequency-curve bounds the area, which equals the total frequency. Its equation obviously is

$$y = y_1 - y_2 = 2\, y_1 = \frac{1}{\sqrt{\pi x}}\, e^{-x}.$$

Evidently the axis of y $(x = 0)$ and the axis of x $(y = 0)$ are asymptotes.

The rough frequency-figure accordingly has a summit at the limit $x = x_0 = 0$ of the domain, and descends towards the other limit $(x = x_n = +\infty)$.

A frequency-series, which, starting with a very high value, gradually diminishes further on, can evidently be explained by a discontinuity in the ideal frequency-curve, which in its turn is a consequence of the many-valuedness of the function $z = f(x)$.

For convenience' sake we may suppose, that the two branches of the function $z = f(x)$ consist of equal and opposite values, so that $f_2(x) = -f_1(x)$, these values coinciding at the limits x_0 and x_n.

If the frequency-series $Y_1, Y_2, \ldots, Y_n$ gradually descends from the highest value Y_1, it is natural to join to x_0 a finite value z_0 of z and to x_n an infinite value of z. The two branches being symmetrical (by agreement) we must take $z_0 = 0$.

The curve $z = f(x)$ has then a shape as in the adjoining sketch (fig. 3).

The frequency in a certain segment pq between $x = x_p$ and $x = x_q$ consists of two equal parts:

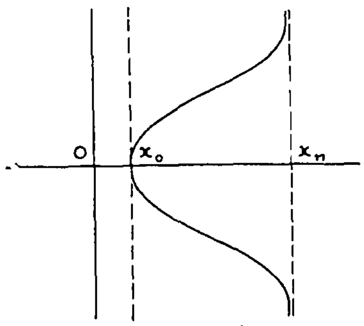

Fig. 3.

$$\Delta_1 I = \frac{1}{\sqrt{\pi}} \int_{x_p}^{x_q} f_1'(x)\, e^{-[f_1(x)]^2}\, dx$$

and $$\Delta_2 I = \frac{1}{\sqrt{\pi}} \int_{x_q}^{x_p} f_2'(x)\, e^{-[f_2(x)]^2}\, dx =$$

$$= \frac{1}{\sqrt{\pi}} \int_{x_q}^{x_p} - f_1'(x)\, e^{-[f_1(x)]^2}\, dx = \Delta_1 I.$$

In order to construct the branch $f_1(x)$ we join x_0 to $z = 0$, or $I = \frac{1}{2}$, and a point $x = x_k$ to the value z_k which satisfies

$$\Theta(z_k) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{z_k} e^{-z^2} dz = \tfrac{1}{2} + \tfrac{1}{2} I(x_k),$$

where $I(x_k)$ represents the quotient $\frac{Y_1 + Y_2 + \ldots Y_k}{N}$ of the number of individuals between x_0 and x_k (thus smaller than x_k), divided by the total number $N = \Sigma\, Y_k$.

In this way we obtain $n-1$ points (x_k, z_k) $(k = 1, \ldots n-1)$ of the positive branch of the curve. Evidently the negative branch will be the reflected image of the positive one.

In reality neither of the limiting points x_0 and x_n is exactly determined by the rough frequency-figure. By tracing a continuous line through the $n-1$ points (x_k, z_k) of the positive branch and another through their reflected images, and uniting these curves as smoothly as possible, we may fix pretty sharply the most probable situation of the point of intersection with the axis of x $(x = x_0,\ z = 0)$.

In the same manner the asymptote $x = x_n$ must be determined by estimating. If it seems to lie very far away, x_n may often be put $= \infty$, as i.a. in the case of the above example $z = \pm \sqrt{x}$.

In general the form of the equation will be

$$z = f(x) = \pm \sqrt{g(x)},$$

where $g(x)$ is a one-valued function of x, which in the real domain only vanishes at $x = x_0$ and becomes infinite at $x = x_n$ (c.q. $x_n = \infty$).

If we had applied the original method, founded on the two simplifications, x_0 would have been joined to $z = -\infty$ and x_n to

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$z = +\infty$. The first frequency-number Y_1 being large, the value of z now rises in a very short interval $x_0 \dots x_1$ from $-\infty$ to a value slightly below zero, or perhaps above zero.

Although $f'(x_0)$ becomes infinite, yet in the ideal frequency-curve

$$\underset{x=x_0}{Lim}\, y = \underset{x=x_0}{Lim} \frac{1}{\sqrt{\pi}} f'(x)\, e^{-[f(x)]^2} = 0$$

unless an improbable form is assumed for $f(x)$. But also, very peculiar forms of $f(x)$ being excluded,

$$\frac{dy}{dx} = \frac{1}{\sqrt{\pi}} \{f''(x) - 2f(x) \cdot [f'(x)]^2\}\, e^{-[f(x)]^2}$$

will approach to zero, that is: the ideal frequency-curve will touch the axis of x in $x = x_0$. Then neither y nor $\frac{dy}{dx}$ shows a discontinuity in x_0.

Since however the area must already assume a considerable value in the first interval, not only the slope but also the ordinate must increase rapidly (fig. 4).

FIG. 4

This case is realised for instance with

$$z = log\, x$$

So the discontinuity of the rough frequency-curve appears as an accumulation of elementary frequencies which are finite, continuous and descending to zero.

Whether the original simplified or the extended method is preferable, is difficult to decide a priori. Perhaps it is possible to refine the data in the first interval and to obtain frequency-numbers for portions of the first class-interval. If these numbers, after continued subdivision, at last begin to decrease as we approach x_0, then the original method (of the continuous frequency-figure) is to be preferred. If on the other hand even by the finest subdivision an increase of the frequencies towards x_0 is found, then it is necessary to admit discontinuity, and the extended method must be applied.

Of the integral-curve

$$I = \int_{x_0}^{x} y\,dx,$$

observation furnishes the ordinates

$$I_1 = \frac{Y_1}{N}\ I_2 = \frac{Y_1 + Y_2}{N}, \dots I_k = \frac{Y_1 + Y_2 + \dots Y_k}{N}, \dots I_{n-1} = \frac{Y_1 + Y_2 + \dots Y_{n-1}}{N},$$

the initial ordinate I_0 being zero of course, and the final ordinate I_n being certainly unity.

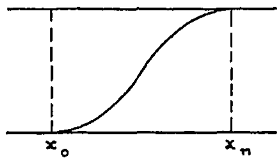

FIG. 5.

In the common case, where the frequencies decrease towards both the extremities, the integral-curve has a course like fig. 5. But if the first frequency I_1 is still great, so that we are inclined to admit a discontinuity in $y = \frac{dI}{dx}$, the curve $I(x)$ has the aspect of fig. 6.

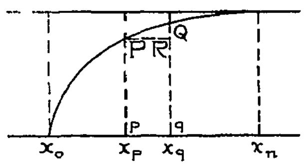

FIG. 6

Now the extended method joins to a single value of x two opposite values of $y = \frac{dI}{dx}$ in this manner we also obtain two branches $I = \Phi_1(x)$ and $I = \Phi_2(x)$ of the integral-curve, satisfying the relation

$$y_2 = \frac{d\Phi_2(x)}{dx} = -y_1 = -\frac{d\Phi_1(x)}{dx}.$$

whence

$$\Phi_1(x) + \Phi_2(x) = \textit{constant},$$

and, x_n being assumed conjugate as well with $z = -\infty$ as with $z = +\infty$, the constant is unity.

So we replace the last-given figure by another of the shape of fig. 7, which is symmetrical with regard to the line $I = \frac{1}{2}$.

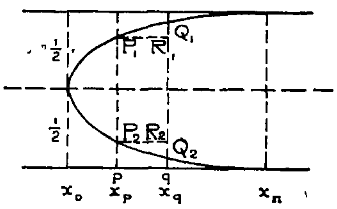

FIG. 7.

The value of I at the point p $(x = x_p)$, which was formerly (fig. 6) represented by the ordinate pP, is now given by the difference

$$pP_1 - pP_2 = P_1P_2 = pP.$$

The increase of I in the interval pq was formerly equal to the rise of the ordinate, viz. $qQ - pP = RQ$. At present it is the sum of the increase $qQ_1 - pP_1 = R_1Q_1$ and the increase $pP_2 - qQ_2 = Q_2R_2$ (fig. 7); this latter corresponds to the second branch, in which the axis of x is assumed to be travelled along in a negative sense.

The area, which was formerly bounded by the curve $I(x)$, the axis of x and the final ordinate-line $x = x_n$, is now found again in the area inclosed between the two branches Φ_1 and Φ_2 and the final ordinate-line $x = x_n$. So it is as if the area of the original figure is so far lifted up as to be symmetrically divided by the line $I = \frac{1}{2}$.

Now we might begin with this latter operation and derive the two-branched curve $I = \Phi_1(x)$, $I = \Phi_2(x)$ from the curve $I = \Phi(x)$. Then, by fixing the values of z, conjugate to the different values of I, according to the relation

$$\Theta(z) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{z} e^{-t^2}\, dt = I,$$

we obtain the required symmetrical curve $z = f(x)$, which for a single value of x gives two opposite values of z.

Also the reaction-function [1])

$$\eta = \frac{1}{f'(x)}$$

has two opposite values for each value of x. There are two domains (overlapping the same segment of the axis of x), one of positive, the other of negative growth. It is exactly this negative growth which explains the accumulation at the lower limit.

We next consider the case that the frequency-domain ends at both sides with rising frequency-numbers, so that the smaller frequencies lie inside.

Now the two limits x_0 and x_n must correspond to finite values of z, being at the same time branch-points of the function $z = f(x)$ which is assumed two-valued.

Thus the curve $z = f(x)$ must have either the form of fig. 8*a*, or that of fig. 8*b*. The former represents a function, which is two-valued in the whole domain, with a single asymptote $x = x_\infty$. The

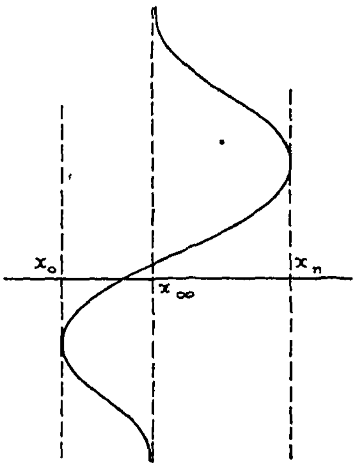

Fig. 8a.

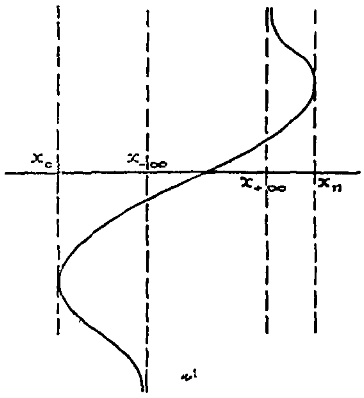

Fig. 8b.

[1]) J. C. Kapteyn and M. J. van Uven: Skew Frequency Curves in Biology and Statistics, 2nd Paper; Groningen, 1916, Hoitsema Br.

latter belongs to a function, which is two-valued in the zones $x_0 \dots x_{-\infty}$ and $x_{+\infty} \dots x_n$, but one-valued in the part $x_{-\infty} \dots x_{+\infty}$; it has two asymptotes, $x = x_{-\infty}$ for $z = -\infty$ and $x = x_{+\infty}$, for $z = +\infty$. It is usually difficult, from a mathematical point of view, to decide between these two forms.

The branches of the function where $f'(x) < 0$ correspond to negative values of the reaction-function, hence with negative growth. It may be desirable, for biological reasons, to suppose such domains of negative growth as small as possible. In this case the second form is to be preferred.

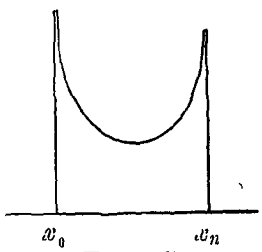

FIG. 9.

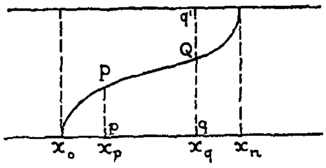

FIG. 10.

The ideal frequency-curve (fig. 9) has now two asymptotes, viz $x = x_0$ and $x = x_n$.

The integral-curve $I = \Phi(x)$ (fig. 10), directly derived from the observations, touches the line $x = x_0$ in $(x = x_0, I = 0)$ and the line $x = x_n$ in $(x = x_n, I = 1)$.

If we construct the curves $z = f(x)$ by means of such simple ground-forms, the two values of $f'(x)$ corresponding to the same value of x in the two-valued domain have opposite signs. Accordingly in the two branches we travel along the axis of x in opposite directions.

The contribution to the frequency in a segment between p and q then consists of two parts:

$$\Delta_1 I = \frac{1}{\sqrt{\pi}} \int_{x_p}^{x_q} f_1'(x)\, e^{-[f_1(x)]^2} dx \quad \text{and} \quad \Delta_2 I = \frac{1}{\sqrt{\pi}} \int_{x_q}^{x_p} f_2'(x)\, e^{-[f_2(x)]^2} dx,$$

both of which are positive; they must be added to obtain the total frequency.

For the integral-curve this means that the ordinate $I = \Phi(x)$ is considered as the difference of the two ordinates

$$\Phi_1(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{x} f_1'(x)\, e^{-[f_1(x)]^2} dx \quad \text{and} \quad \Phi_2(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{x} f_2'(x)\, e^{-[f_2(x)]^2} dx.$$

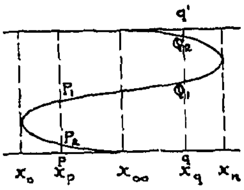

FIG. 11 a.

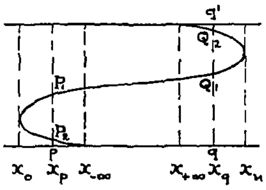

FIG. 11 b.

The integral-curve now assumes either the form of fig. 11*a*, or that of fig. 11*b*, depending on the function being two-valued in the whole domain or only in a part of it. We therefore have to construct the curve in such a way that

$$pP_1 - pP_2 = P_1P_2 = pP \quad \text{(fig. 11 and 10)}$$

and

$$Q_1q' - Q_2q' = Q_1Q_2 = Qq' \quad (\text{,, ,, ,, ,,})$$

or

$$qQ_1 + Q_2q' = qQ \quad (\text{,, ,, ,, ,,})$$

Obviously it is still possible to satisfy the above conditions in a great many ways On one hand the problem becomes theoretically indeterminate (as a necessary consequence of giving up the provisional simplifications). On the other hand we gain the possibility of simplifying the curve $I = \Phi(a)$ and with it the curve $z = f(x)$ as much as possible. Once the form of the curve (I, x) has been determined, the curve $z = f(x)$ may be drawn with the aid of the table (I, z).

When the frequency-figure is symmetrical with regard to the median, so that $x_m = \frac{x_0 + x_n}{2}$, it is natural to construct the curve $I = \Phi(a)$ in such a manner, that $(x = a_m, \; I = \frac{1}{2})$ becomes the centre. The curve $z = f(x)$ will also have a centre, viz. in the point $(x = x_m, \; z = 0)$.

If the frequency-curve is not symmetrical but higher in the left half of the domain than in the right, then in the figure $z = f(x)$ the point of intersection with the axis of a $(z = 0)$ will lie in the left half of the figure.

The reaction-function has now the form of fig. 12*a* or that of fig. 12*b*.

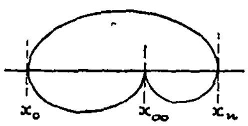

FIG. 12a

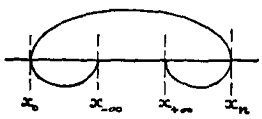

FIG. 12b.

In $x = x_0$ we have $f'(x) = \infty$. Assuming the radius of curvature

$$\varrho = \frac{\{1 + [f'(x)]^2\}^{3/2}}{f''(x)}$$

to be finite, $f''(x)$ must be infinite of the same order as $[f'(x)]^3$. For the reaction-function

$$\eta = \psi(x) = \frac{1}{f'(x)}$$

we have

$$\frac{d\eta}{dx} = \psi'(x) = \frac{-f''(x)}{[f'(x)]^2}.$$

Hence in the point $x = x_0$ also $\psi'(x)$ will have an infinite value, so that the reaction-curve touches the ordinate-lines $x = x_0$ and $x = x_n$ in the points $(x = x_0, \eta = 0)$ and $(x = x_n, \eta = 0)$ resp.

C. Three-valued functions $z = f(x)$.

A frequency-series with a summit at only one or at both extremities may, as we have seen, be rather easily connected with a two-valued function $z = f(x)$. Likewise a function, which is three-valued in a part of the real domain, may be used for examining a frequency-series with two discontinuities *within* the limits.

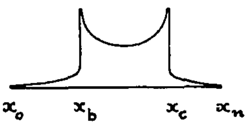

FIG. 13.

Here we suppose a frequency-figure of the form of fig. 13. If the discontinuities (infinite ordinates in the ideal frequency-curve) are found at $x = x_b$ and $x = x_c$, the function $z = f(x)$ must have branch-points there, so that $f'(x_b) = \infty$ and $f'(x_c) = \infty$.

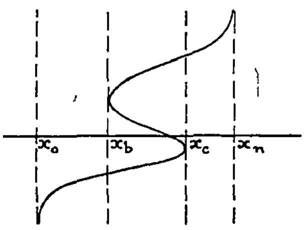

FIG. 14

Besides the curve $z = f(x)$ must also extend to the left of x_b and to the right of x_c. So we arrive, intricate forms being not considered, at a curve of the shape of fig. 14. The function has three branches $f_1(x)$, $f_2(x)$ and $f_3(x)$. The lower branch ranges from x_0 (corresponding to $z = -\infty$) to x_c; In this branch $f_1'(x)$ is always > 0.

The middle branch extends from x_c to $x_b (< x_c)$ in a negative direction along the axis of x; in this branch $f_2'(x) < 0$.

The upper branch extends from x_b to x_n (corresponding to $z = + \infty$); in it $f_3'(x)$ is always > 0.

So the function is three-valued in the segment $x_b \ldots x_c$.

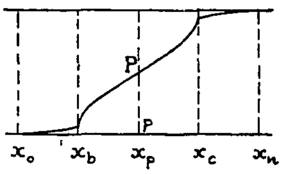

FIG. 15.

In the integral-curve $I = \Phi(x)$, as it follows directly from the observations (see fig. 15), the ordinate in the interval $x_0 \ldots x_b$ slowly increases from 0 to I_b (which is a small value). Then the ordinate suddenly rises, so that in the integral-curve (x_b, I_b) is a corner, of which the left hand tangent is nearly horizontal, the right hand one vertical Thereupon the slope decreases to its minimum in the point of inflexion to rise again to an exceedingly high value at x_c. Here the ordinate attains the value I_c, which differs but little from unity. Finally the ordinate still slightly increases from I_c to unity. So the integral-curve has another corner at (x_c, I_c), the tangent being vertical on the negative side, nearly horizontal on the positive.

Now the frequency in the interval $p \dots q$, belonging to the three-valued zone, consists of three parts, viz.:

$$\Delta_1 I = \frac{1}{\sqrt{\pi}} \int_{x_p}^{x_q} f_1'(x)\, e^{-[f_1(x)]^2} dx \quad , \quad \Delta_2 I = \frac{1}{\sqrt{\pi}} \int_{x_q}^{x_p} f_2'(x)\, e^{-[f_2(x)]^2} dx \ ,$$

$$\Delta_3 I = \frac{1}{\sqrt{\pi}} \int_{x_p}^{x_q} f_3'(x)\, e^{-[f_3(x)]^2} dx \ ,$$

which are all positive, because we travel along the axis of x in a negative sense in the part ($\Delta_2 I$), where $f'(x) < 0$. Now for $x_b < x < x_c$ the integral

$$\Phi(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{x} f'(x)\, e^{-[f(x)]^2} dx$$

may be broken-up into three parts:

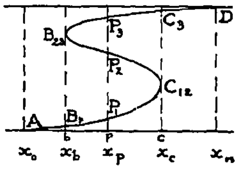

Fig. 16

$$\Phi_1(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{x} f_1'(x)\, e^{-[f_1(x)]^2} dx \quad , \quad \Phi_2(x) = \frac{1}{\sqrt{\pi}} \int_{x}^{x_b} f_2'(x)\, e^{-[f_2(x)]^2} dx \ ,$$

$$\Phi_3(x) = \frac{1}{\sqrt{\pi}} \int_{x_b}^{x} f_3'(x)\, e^{-[f_3(x)]^2} dx$$

The function $\Phi_1(x)$ is represented by the part AB_1C_{12}, the function $-\Phi_2(x) = \frac{1}{\sqrt{\pi}} \int_{x_b}^{x} f_2'(x) e^{-[f_2(x)]^2} dx$ by the part $C_{12}B_{23}$, the function $\Phi_3(x)$ by the part $B_{23}C_3D$.

The total frequency $I = \Phi(x_p)$ in the point $x = x_p$ is therefore represented by

$I = pP_1 - pP_2 + pP_3 = pP_1 + P_2P_3 = pP_3 - P_1P_2 = pP$ (see fig. 15 and 16).

So we may put the following problem:

To transform the discontinuous integral-curve (fig. 15) furnished directly by observation into a continuous curve (fig. 16), which in the range $x_b \dots x_c$ has three branches, such that

$$pP_1 + P_2P_3 = pP,$$

or what comes to the same,

$$pP_3 - P_1P_2 = pP,$$

where pP is the ordinate in the given discontinuous curve.

Also this construction is in a great measure indeterminate. This vagueness may be utilised to satisfy conditions of a non-mathematical character.

The construction having been carried out in some way or other, for each value of I the conjugate value of z may be determined and the curve $z = f(x)$ may thus be constructed, which will then show the shape of fig. 14.

The reaction function looks as fig. 17.

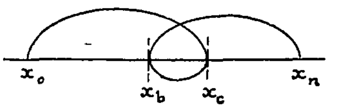

FIG. 17.

The transformation of the observed integral-curve into the theoretical one may also be interpreted in this way:

In the point $x = x_p$ the observed integral-curve has for ordinate the total frequency of the values $x < x_p$, i.e. the quotient of the whole number of individuals for which $x < x_p$, divided by the number N of all the individuals.

The theoretical ordinate pP_1 in the branch Φ_1 represents the frequency of the values $x < x_p$ as far as these are due to the branch f_1 of the function f, i.e. to the branch $\eta_1 = \frac{1}{f_1'}$, of the reaction-function. The final ordinate cC_{12} then represents the quotient of the whole number of values $x < x_c$, that is of all values due to the first branch, divided by N.

From C_{12} onwards the second branch of the reaction-function begins to play a part. New values of x now appear between x_c and x_b ($< x_c$). The number of values of x between x_p and x_c, which are thus added, amounts to N times the increase of the ordinate from C_{12} to P_2. On the other hand the increase from P_2 to B_{23} represents the quotient of the number of values x ($x_b < x < x_p$) as far as due to the second branch, divided by N. Accordingly the increase from B_{23} to P_3 represents the N^{th} part of the number of values of x, which, lying between x_b and x_p, are due to the third branch. Hence the increase from P_2 to P_3, or the segment P_2P_3 represents the N^{th} part of the number of values of x lying between x_b and x_p, as far as due to both the second and the third branch. Adding to this the ordinate pP_1, we obtain the N^{th} part of the total number of values of x that are inferior to x_p. Now in the observed integral-figure this number was represented by the ordinate pP. Hence the relation

$$pP = pP_1 + P_2P_3,$$

which may also be written:

$$pP = pP_3 - P_1P_2.$$

If the two points of discontinuity $x = x_b$ and $x = x_c$ are very near each other, they may appear as a single accumulation in the rough frequency-curve.

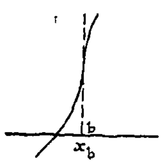

Fig 18a.

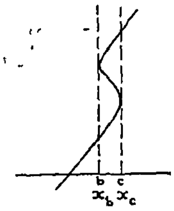

Fig. 18b.

The former (simplified) analysis then required a very steep slope in the curve $z = f(x)$ at this point, by which the smooth character of the curve is often disturbed (fig. 18*a*).

Considering however this accumulation as a fusion of two discontinuities, we may assume that the function is three-valued in the immediate vicinity of $x = x_b$ (fig. 18*b*) Usually the smooth transition may be obtained by freehand drawing. Care must however be taken that the three-valued zone remains as narrow as possible.

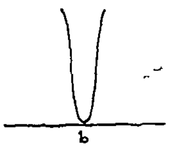

Fig. 19a.

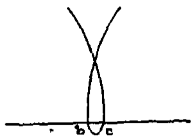

Fig. 19b

The reaction-curve must now be modified in such a way that in the point *b* the reaction becomes neither very small and positive, nor zero, but negative.

So instead of the shape of fig. 19*a* the reaction-curve obtains the shape of fig. 19*b*.

Astronomy. — *"Calculation of Dates in the Babylonian Tables of Planets"*. By Dr. A. Pannekoek. (Communicated by E. F. van de Sande Bakhuyzen).

(Communicated in the meeting of September 30, 1916)

By the researches of F. X. Kugler S. J. in Valkenburg we have for some years been acquainted with the methods and results of Babylonian astronomy during the period of its highest development. The material for this was provided by a number of more or less damaged fragments of clay tablets covered with cuneiform writing, which are preserved in the British Museum, and which have been very carefully copied by Strassmaier. They contain observations and calculations made in advance of the places of the moon and planets from the 5 centuries before the Christian Era, the complete deciphering and explanation of which is given by Kugler in his work "Die babylonische Mondrechnung" (1902) and in Vol. I of his larger work "Sternkunde und Sterndienst in Babel" (1907).