But not only for the latent heat it is of importance to keep apart the two cases, viz. space between the molecules greater or smaller than the dimensions of the molecule itself — it is also of importance for other phenomena. For example, friction, diffusion etc. It is not my purpose to demonstrate this myself in detail — the more so because it has already been pointed out by different observers that these phenomena follow different laws for liquids and gases. But I will particularly call attention to a paper by BATSCHINSKY, who already according to a communication in the Journal de Physique 1914 tried to explain the diffusion for liquids not by the thermal movement, but by the attraction of the molecules, hence by the quantity a of the equation of state, and who has given a view which is most probably at bottom essentially analogous to the above [1]).

That in the title I have put the transition of the twin cases at the critical density, holds of course only for normal substances, so without association. But in general the transition case is that for which the distance of the molecules is equal to the dimension of the molecule. Before we have got perfect certainty about this, a thorough investigation also above the critical temperature would be necessary. This would probably be much more difficult, and I hope that also other investigators will show an interest in this research.

Astronomy. — *"Planetary motion and the motion of the moon according to* EINSTEIN'S *theory."* By DR. W. DE SITTER.

(Communicated in the meeting of June 24, 1916).

1. *The gravitational field of the sun.*

In EINSTEIN'S new theory gravitation is determined by 10 quantities g_{ab}, which are given by the differential equations [2])

$$G_{ab} - \tfrac{1}{2}\, g_{ab}\, G = -\varkappa\, T_{ab}. \qquad \qquad (1)$$

These equations are invariant for any arbitrary transformation of the "coordinates" $x_1 \ldots x_4$, by means of which the phenomena are described. It is an essential feature of EINSTEIN'S theory that the

[1]) I have only read a short review in the Journal de Physique, but I think that I may conclude from it that BATSCHINSKY finds back the quantity b of the equation of state as characteristic quantity. This would be a corroboration of my opinion that the attraction of the molecules is only exerted in case of perfect contact.

[2]) EINSTEIN: *Die Feldgleichungen der Gravitation,* Sitzungsber. Berlin, Nov. 1915 page 845, formula (2*a*). It is easily found that $G = \varkappa T$.

equations (1) do not determine the g_{ab} completely. To determine them completely an arbitrary restriction must be added, which can be considered as a definition of the coordinates $x_1 \ldots x_4$. In first approximation we find the ordinary mechanics according to NEWTON's law, in the terms of higher order there remains an undeterminateness.

If we take rectangular coordinates $x_1 = x, x_2 = y, x_3 = z, x_4 = ct$ (c being the velocity of light in a portion of space where there is no matter and no gravitation), then the g_{ab} which determine the field of a sphere at rest at the origin of coordinates can be expressed in terms of three[1]) quantities α, β, γ, which are of the first order of smallness. Thus

$$g_{ii} = -(1+\beta) + \frac{x_i^2}{r^2}(\beta - \alpha)$$
$$g_{ij} = \frac{x_i x_j}{r^2}(\beta - \alpha) \qquad (i, j = 1, 2, 3). \quad . \quad . \quad (2)$$
$$g_{i4} = 0 \;,\; g_{44} = 1 + \gamma$$

If we introduce polar coordinates $x'_1 = r,\ x'_2 = \vartheta,\ x'_3 = \psi$ by the formulae of transformation:

$$x = r \cos\psi \cos\vartheta$$
$$y = r \cos\psi \sin\vartheta$$
$$z = r \sin\psi$$

then we find

$$\begin{aligned} g_{11} &= -(1+\alpha) \\ g'_{22} &= -(1+\beta)\, r^2 \cos^2\psi \\ g'_{33} &= -(1+\beta)\, r^2 \\ g'_{ij} &= 0 \text{ for } i \neq j. \end{aligned} \qquad (i, j = 1, 2, 3) \quad . \quad . \quad . \quad (2')$$

The radial symmetry requires that α, β, γ are functions of r alone.

The differential equations contain the quantities T_{ab}. If we neglect pressures etc. inside the sun, arising from the mutual gravitation of its parts, and if the material constituting the sun is at rest, these are

$$T_{ij} = 0 \qquad T_{i4} = 0 \qquad (i, j = 1, 2, 3)$$
$$T_{44} = \varrho(1+\gamma)$$

Here ϱ is the number of material points contained in the four-dimensional element of volume $dx_1 dx_2 dx_3 dx_4$. We can take $dx_4 = cdt = 1$, and since the matter is at rest ϱ then becomes the ordinary density.

I now write down the equations (1) of EINSTEIN. Differentials with respect to r are indicated by accents. Then I find

$$-\tfrac{1}{2}\gamma'' - \frac{\gamma'}{r} - \tfrac{1}{2}\gamma'(\beta' - \tfrac{1}{2}\alpha' - \tfrac{1}{2}\gamma') = -\tfrac{1}{2}\varkappa(1+\alpha)T_{44} = -\tfrac{1}{2}\varkappa\varrho(1+\alpha+\gamma) \quad (3)$$

[1]) See DROSTE, these Proceedings XVII (Dec. 1914) page 998.

$$\frac{\beta-\alpha}{r^2} - \tfrac{1}{2}\beta'' - \tfrac{1}{2}\gamma'' + \frac{1}{2r}(\alpha' + \gamma') = 0 \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$\frac{\beta-\alpha}{r^2} + \frac{1}{r}(\beta' + \gamma') = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Exactly the same equations are found from the generalised principle of HAMILTON, as enounced by LORENTZ [1]). The equations as here given are only exact to the required order of accuracy. In a recent communication [2]) Mr. DROSTE has derived the complete equations from the same principle, and by an elegant analysis has succeeded in rigorously integrating them. In the present paper no rigorous solution will be attempted, but only an approximation to the order which is required for practical applications.

It is easily found that

$$\frac{1}{r}\frac{d}{dr}\left[\tfrac{1}{2} r^2 \, . \, (5)\right] + (4) = \tfrac{1}{2} \, . \, (5).$$

Consequently the equations (4) and (5) are not independent of each other. To determine α, β, γ completely we must. as has already been pointed out above, add an arbitrary condition.

EINSTEIN [3]) advises $\sqrt{-g} = 1$, which, to the required order of accuracy, is equivalent to

$$\beta + \tfrac{1}{2}\alpha + \tfrac{1}{2}\gamma = 0$$

This equation, together with (3) and (5) determines α, β and γ. EINSTEIN finds

$$\gamma = -\frac{\mathbf{a}}{r} \qquad \beta = 0 \qquad \alpha = -\gamma.$$

DROSTE in the paper already quoted introduces a condition, which is equivalent to $\beta = 0$. He finds

$$\gamma = -\frac{\mathbf{a}}{r} \qquad \beta = 0 \qquad 1 + \alpha = \frac{1}{1+\gamma}.$$

This result is entirely rigorous, while EINSTEIN's was only approximate. Within the limits of the approximation given by EINSTEIN the two are identical. Both EINSTEIN and DROSTE consider only the field *outside* the sun.

I will take as arbitrary condition

[1]) These Proceedings, Feb. 1916 (Not yet translated into English).

[2]) These Proceedings, Vol. XIX, page 197 (May 1916).

[3]) *Erklarung der Perikelbewegung des Merkur aus der allgemeinen Relativitatstheorie*, Sitzungsber. Berlin, Nov. 1915, page 833. It would be better to say that EINSTEIN's condition is that $\sqrt{-g}$ shall be independent of gravitation. For rectangular coordinates EINSTEIN has indeed $g = -1$, for polar coordinates this becomes $g = -r^4 \cos^2 \psi$.

$$\beta - \alpha = 0.$$

Then (5) gives

$$\beta + \gamma = \text{const.}$$

Since at infinity both β and γ must be zero, the constant is also zero. We have thus

$$\alpha = \beta = -\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The equation (3) now becomes, accurate to the second order

$$r^2 (\gamma'' - \gamma'^2) + 2 r \gamma' = \varkappa r^2 \varrho.$$

We can split up γ in its terms of the first and of the second order, thus

$$\gamma = \gamma_1 + \gamma_2,$$

then we have the two equations

$$r^2 \gamma''_1 + 2r \gamma'_1 = \varkappa r^2 \varrho \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$r^2 \gamma''_2 + 2r \gamma'_2 = r^2 \gamma'^2_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The integration is not difficult. First I will introduce instead of $\varkappa$ the GAUSSIAN constant k. We have

$$\varkappa = 8 \pi \lambda_0^2 \quad , \quad \lambda_0 = \frac{k}{c}.$$

If now we put

$$4\pi \int_0^r r^2 \varrho \, dr = m(r), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

then we find from (7)

$$r^2 \gamma'_1 = 2\lambda_0^2 \, m(r)$$

and then from (8)

$$r^2 \gamma'_2 = -\frac{4\lambda_0^4}{r} m(r)^2 + 8\lambda_0^4 \, q(r),$$

where we have put

$$4\pi \int_0^r r \varrho \, m(r) \, dr = q(r) \, . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

If now we put

$$m'(r) = m(r) + 4\lambda_0^2 \, q(r) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

then

$$\gamma' = \frac{\lambda_0^2}{r^2} m'(r) - \frac{4\lambda_0^4}{r^3} m(r)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

from which we find easily

$$\gamma = -\frac{2\lambda_0^2}{r^2} m'(r) + \frac{2\lambda_0^4}{r^2} m(r)^2 + 8\pi \lambda_0^2 \int_\infty^r [r + 3\lambda_0^2 \, m(r)] \, \varrho \, dr$$

The lower limit of the last integral has been so chosen that at infinity we have $\gamma = 0$.

Put now

$$4\pi \int_0^r [r + 3\lambda^2_0 m(r)]\, \varrho\, dr = n(r) \quad , \quad n(R) = N,$$

R being the sun's radius. Then, since for $r > R$ (i.e. outside the sun) $\varrho = 0$, we find

$$\gamma = -\frac{2\lambda_0^2}{r} m'(r) + \frac{2\lambda_0^4}{r^2} m(r)^2 + 2\lambda_0^2 [n(r) - N] \,. \quad . \quad . \quad (13)$$

These formulae can be used both inside and outside the sun, if we neglect the strains and pressures caused by gravitation inside the sun. Outside we have $n(r) = N$, and $m'(r)$ and $m(r)$ are constants. Since the difference $m' - m$ is of the order of λ^2_0, we can in the term which has λ^4_0 as a factor use m' instead of m. If now we put

$$\lambda^2 = \lambda_0^2\, m'(R) = \frac{k^2 m'}{c^2}$$

then the formulae *outside the sun* become

$$\left.\begin{aligned} \gamma' &= \frac{2\lambda^2}{r^2} - \frac{4\lambda^4}{r^3} \\ \gamma &= -\frac{2\lambda^2}{r} + \frac{2\lambda^4}{r^2} \end{aligned}\right| \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

The quantity $2\lambda^2$, which corresponds to EINSTEIN's **a**, has the dimension of a length. For the sun its value is 2.945 *km*, for an atom of hydrogen 5×10^{-48} microns. For $r = 2\lambda^2$ we have $\gamma' = 0$. The remarkable consequences of this fact have been very completely investigated by DROSTE. In actual problems r is always very much larger[1]) than $2\lambda^2$.

The values of g_{ab} are now, for rectangular coordinates

$$g_{11} = g_{22} = g_{33} = -1 + \gamma \quad , \quad g_{44} = 1 + \gamma \quad . \quad . \quad . \quad (15)$$

and for polar coordinates

$$g'_{11} = -1 + \gamma \quad g'_{22} = -r^2 \cos^2\psi(1 - \gamma) \quad g'_{33} = -r^2(1 - \gamma) \quad g'_{44} = 1 + \gamma \quad (16$$

Those not mentioned are zero.

These g_{ab} are simpler than those of EINSTEIN and DROSTE, since here all $g_{ij} = 0$ for $i \neq j$. Thus e.g. the velocity of light in my system of reference is

$$\frac{d\sigma}{cdt} = 1 + \gamma,$$

1) DROSTE's formulae, like (14), only represent the field *outside* the sun.

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while in EINSTEIN's system it is

$$\frac{d\sigma}{cdt} = 1 + \tfrac{1}{2}\gamma\,(1 + \cos^2 V)$$

Here $d\sigma$ is a line-element in the three-dimensional space (x_1, x_2, x_3) and V is the angle between this element and the radius-vector. The curvature of rays of light of course is the same in both systems.

2. *The equations of motion.*

The world-line of a material point is a geodetic line of which the differential equations are

$$\frac{d^2x_i}{ds^2} + \sum_p \sum_q \left\{ {pq \atop i} \right\} \frac{dx_p}{ds}\frac{dx_q}{ds} = 0 \qquad (i, p, q = 1 \,..\, 4)$$

Here ds is the element of the world-line, which is given by

$$ds^2 = \sum_p \sum_q g_{pq}\, dx_p\, dx_q.$$

All sums are to be taken from 1 to 4.

If now we take $x_4 = ct$ we find easily

$$\frac{d^2x_i}{c^2dt^2} = -\sum_p \sum_q \left[\left\{ {pq \atop i} \right\} - \left\{ {pq \atop 4} \right\} \dot{x}_i \right] \dot{x}_p\, \dot{x}_q \qquad \left({i = 1, 2, 3 \atop p,q = 1 \ldots 4} \right) \quad (17)$$

The points indicate differentials with respect to ct, so that $\dot{x}_4 = 1$.

The brackets $\left\{ {pq \atop i} \right\}$ are easily found from the g_{ab}. To the required order of accuracy we have, for rectangular coordinates:

$$\left.\begin{array}{l} \left\{ {pp \atop i} \right\} = \dfrac{\lambda^2 x_i}{r^3} \;,\quad \left\{ {pi \atop i} \right\} = -\dfrac{\lambda^2 x_p}{r^3} \;,\quad \left\{ {ii \atop i} \right\} = -\dfrac{\lambda^2 x_i}{r^3} \\ \left\{ {44 \atop i} \right\} = \dfrac{\lambda^2 x_i}{r^3} - 4\dfrac{\lambda^4 x_i}{r^4} \qquad \left\{ {p4 \atop 4} \right\} = \dfrac{\lambda^2 x_p}{r^3} \end{array}\right\} . \quad (18)$$

and for polar coordinates

$$\left.\begin{array}{l} \left\{ {11 \atop 1} \right\}' = -\dfrac{\lambda^2}{r^2} \;,\quad \left\{ {22 \atop 1} \right\}' = -r\cos^2\psi\left(1 - \dfrac{\lambda^2}{r}\right) \;,\quad \left\{ {33 \atop 1} \right\}' = -r\left(1 - \dfrac{\lambda^2}{r}\right) \\ \left\{ {12 \atop 2} \right\}' = \left\{ {13 \atop 3} \right\}' = \dfrac{1}{r}\left(1 - \dfrac{\lambda^2}{r}\right) \;,\quad \left\{ {22 \atop 3} \right\}' = \sin\psi\cos\psi \;, \\ \left\{ {32 \atop 2} \right\}' = -\tan\psi \;,\quad \left\{ {44 \atop 1} \right\}' = \dfrac{\lambda^2}{r^2} - 4\dfrac{\lambda^4}{r^3} \;,\quad \left\{ {14 \atop 4} \right\}' = \dfrac{\lambda^2}{r^2} \end{array}\right\} \quad (19)$$

Those not mentioned are zero. If now we put $\lambda^2 = \frac{k^2}{c^2} m$, (where m is the same as m' of the preceding article), then the equations of motion become, for rectangular coordinates:

$$\frac{d^2 x_i}{dt^2} + k^2 m \frac{x_i}{r^3} = k^2 m \left\{ \frac{4\lambda^2 x_i}{r^4} + 4\frac{\dot{x}_i \dot{r}}{r^2} - \frac{x_i}{r^3}\varphi^2 \right\}, \quad . \quad . \quad . \quad (20)$$

and for polar coordinates

$$\left.\begin{aligned} &\frac{d^2 r}{dt^2} - r\cos^2\psi\left(\frac{d\vartheta}{dt}\right)^2 - r\left(\frac{d\psi}{dt}\right)^2 + \frac{k^2 m}{r^2} = k^2 m\left\{\frac{4\lambda^2}{r^3} + 4\frac{\dot{r}^2}{r^2} - \frac{\varphi^2}{r^2}\right\}, \\ &\frac{d^2\vartheta}{dt^2} + \frac{2}{r}\frac{dr}{dt}\frac{d\vartheta}{dt} - 2\tan\psi\frac{d\vartheta}{dt}\frac{d\psi}{dt} = 4k^2 m\frac{\dot{r}\dot{\vartheta}}{r^2}, \\ &\frac{d^2\psi}{dt^2} + \frac{2}{r}\frac{dr}{dt}\frac{d\psi}{dt} + \sin\psi\cos\psi\left(\frac{d\vartheta}{dt}\right)^2 = 4k^2 m\frac{\dot{r}\dot{\psi}}{r^2}, \end{aligned}\right\} \quad (21)$$

where we have put

$$\varphi^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \dot{r}^2 + r^2\cos^2\psi\dot{\vartheta}^2 + r^2\dot{\psi}^2.$$

The right-hand members are of the second order. The left-hand members put equal to zero give the motion according to NEWTON's law.

3. *Planetary motion.*

From the third equation (21) it follows, that when once $\varphi = 0$ and $\dot{\varphi} = 0$, this always remains true: the orbit is plane. Then we have, accurate to the second order

$$\ddot{\vartheta} + \frac{2}{r}\dot{r}\dot{\vartheta} = \dot{r}\dot{\vartheta}(\gamma' - \beta') \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

This equation is general; if we introduce the special values of β and γ which are here used, it is reduced to the second of (21). We now put

$$r^2\dot{\vartheta} = G.$$

Then the integral of (22) is

$$G = G_0\, e^{\gamma - \beta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

This equation replaces the integral of areas.[1]) I now put

$$G_0 = \lambda\sqrt{p_0}$$

Then, with our values of β and γ we find

[1]) If we put $r^2\frac{d\vartheta}{ds} = \overline{G}$, ds being the element of the world-line, i.e. the *proper-time* of the planet, then we find

$$\overline{G} = G_0\, e^{-\beta}.$$

If we take $\beta = 0$, as in EINSTEIN's system, then the law of areas is exact if the proper-time of the planet is taken as independent variable, as has already been remarked by EINSTEIN (l. c. page 837).

$$r^2\dot{\vartheta} = \lambda\sqrt{p_0}\left(1 - \frac{4\lambda^2}{r}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

or

$$r^2\frac{d\vartheta}{dt} = k\sqrt{m}\,\sqrt{p_0}\left(1 - \frac{4\lambda^2}{r}\right).$$

The first of (21) is

$$\ddot{r} - r\dot{\vartheta}^2 + \frac{\lambda^2}{r^2} = \frac{4\lambda^4}{r^3} + 4\,\lambda^2\,\frac{\dot{r}^2}{r^2} - \lambda^2\,\frac{\varphi^2}{r^2}. \quad . \quad . \quad . \quad (25)$$

If we multiply (25) by $\dot{r}$ and (22) by $r^2\dot{\vartheta}$, and add, we find

$$\frac{d}{cdt}\left(\tfrac{1}{2}\,\varphi^2 - \frac{\lambda^2}{r}\right) = \left(3\,\varphi^2 + \frac{4\lambda^2}{r}\right)\frac{\lambda^2\dot{r}}{r^2} \quad . \quad . \quad . \quad (26)$$

The lefthand member put equal to zero gives, as in NEWTON's theory

$$\tfrac{1}{2}\,\varphi^2 - \frac{\lambda^2}{r} = const.$$

The constant I call $-\frac{\lambda_0^2}{2a_0}$. Thus we have the approximation

$$\varphi^2 = \frac{2\lambda^2}{r} - \frac{\lambda^2}{a_0}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

or

$$\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\vartheta}{dt}\right)^2 = k^2m\left(\frac{2}{r} - \frac{1}{a_0}\right),$$

If this is introduced into the right-hand member of (26), this becomes

$$\frac{d}{cdt}\left(\varphi^2 - \frac{2\lambda^2}{r}\right) = \left(\frac{20\lambda^4}{r^3} - \frac{6\lambda^4}{a_0 r^2}\right)\dot{r},$$

from which

$$\varphi^2 - \frac{2\lambda^2}{r} + \frac{\lambda^2}{a_0} = \frac{6\lambda^4}{a_0 r} - \frac{10\lambda^4}{r^2}. \quad . \quad . \quad . \quad (28)$$

The right-hand member is of the second order. If it is neglected (28) becomes the same as (27), i. e. the integral of living forces in NEWTON's theory.

To find the orbit we must eliminate cdt from (24) and (28). This gives

$$\left(\frac{dr}{d\vartheta}\right)^2 + r^2 - \frac{2r^3}{p_0} + \frac{r^4}{a_0 p_0} = \frac{6\lambda^2 r^2}{p_0} - \frac{2\lambda^2 r^3}{a_0 p_0}.$$

If now we put

$$p_1 = p_0\left(1 + \frac{\lambda^2}{a_0}\right)$$

and if we introduce

$$y=\frac{1}{r}$$

then we find

$$\left(\frac{dy}{d\vartheta}\right)^2+y^2-\frac{2}{p_0}\,y+\frac{1}{a_0p_0}=\frac{6\lambda^2}{p_0}\,y^2 \quad . \quad . \quad . \quad (29)$$

EINSTEIN [1]) has the same equation. only his righthand member is $\mathbf{a}y^3$. The difference is caused by the difference in the arbitrary condition introduced to complete the determination of the g_{ab}. The integration of (29) is easier than of the corresponding equation of EINSTEIN, which leads to elliptic functions. We find easily

$$y=\frac{1}{g^2p_1}+\frac{e_1}{p_1}\cos\,(g\vartheta-\omega),$$

where e_1 and ω are constants of integration, and

$$g=1-\frac{3\lambda^2}{p_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

If now we put

$$p=g^2p_1, \qquad e=g^2e_1, \qquad a=g^2\frac{p_0}{p}\,a_0=a_0-\lambda^2,$$

then we have

$$p=a\,(1-e^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

and

$$\frac{1}{r}=\frac{1+e\cos\,(g\vartheta-\omega)}{p}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

The orbit is thus an ellipse of which the perihelion moves in the direction of the orbital motion The displacement of the perihelion during one revolution is $\frac{2\pi}{g}-2\pi=\frac{3\lambda^2}{p}\,2\pi$. This same value has been found by EINSTEIN The numerical value is for the different planets, in one century:

Mercurius	$\delta\bar{\omega}=+42''.9$	$e\delta\bar{\omega}=+8''.82$
Venus	8.6	+ 0.05
Earth	3.8	+ 0.07
Mars	1.3	+ 0.13

If the elements a and p are introduced in (24), this becomes

$$r^2\frac{d\vartheta}{dt}=k\sqrt{m}\sqrt{p}\left(1-\tfrac{1}{2}\frac{\lambda^2}{a}+\frac{3\lambda^2}{p}-\frac{4\lambda^2}{r}\right) \quad . \quad . \quad (33)$$

[1]) l. c page 837, formula (11) DROSTE of course finds the same formula as EINSTEIN, and he integrates it by means of elliptic functions.

If from this we solve $\frac{dt}{d\vartheta}$, multiply by $d\vartheta$, and integrate from 0 to 2π, we find the period of revolution T. The result naturally depends on the point of starting, i.e. on the point on the ellipse where $\vartheta = 0$. In the moving ellipse the true anomaly is

$$v = g\vartheta - \omega.$$

If we take this as independent variable, we must integrate from v_1 to $v_2 = v_1 + 2\pi g$. I find in this way

$$T = \frac{2\pi a^{3/2}}{k\sqrt{m}}\left[1 + \frac{9}{2}\frac{\lambda^2}{a} - \frac{3\lambda^2}{p} + \frac{6\lambda^2}{p} e \cos l_0 + \ldots\right],$$

where l_0 is the mean anomaly corresponding to the true anomaly $v_0 = \frac{1}{2}(v_1 + v_2)$. All neglected terms of the series, as well as the last term which has been included, are periodic. If these are omitted, we find the mean period T_0. If then we put $nT_0 = 2\pi$, we find

$$a^3 n^3 = k^2 m\left[1 - \frac{3\lambda^2}{p}(1-3e^2)\right] \quad \ldots\ldots \quad (34)$$

which replaces KEPLER's third law.

Let the excentric and the mean anomaly in the moving ellipse, corresponding to the true anomaly v, be called u and l respectively. Then

$$\left.\begin{aligned} u - e \sin u &= l \\ \frac{dl}{dt} &= n\left[1 + \frac{\lambda^2}{p}(1-4e^2) - \frac{4\lambda^2}{r}\right]. \end{aligned}\right\} \quad \ldots\ldots \quad (35)$$

The mean value of the expression within the square bracket over a complete revolution in the (moving) ellipse is $1 - \frac{3\lambda^2}{p} = g$.

4. *The motion of the moon.*

The moon will be considered as a material point, of which we will investigate the motion in the gravitational field of the sun and the earth.

We take an arbitrary system of rectangular coordinates, in which

x_i , Δ are the heliocentric coordinates of the moon,
ξ_i , ϱ „ „ „ „ „ earth,
x_i , r „ „ geocentric „ „ „ moon.

Thus $x_i = \mathrm{x}_i - \xi_i$, and we put

$$\lambda^2 = \lambda_0^2\, m \quad , \quad \lambda_1^2 = \lambda_0^2\, m_1,$$

m being the mass of the sun, and m_1 of the earth. We can neglect the excentricity of the earth, ϱ is then a constant. The equations of motion are (17), in which the brackets $\left\{ {pq \atop i} \right\}$ and $\left\{ {pq \atop 4} \right\}$ must be derived by means of the usual formulae from the g_{ij}, which are determined by (1). The right-hand members of (1), i. e. the quantities T_{ij}, are zero except in those parts of the four-dimensional time-space, where the sun and the earth are. Since these two bodies never co-incide, we have always only one T_{ij}, either $(T_{ij})_0$ or $(T_{ij})_1$. We can suppose the sun to be at rest. Then $(T_{ij})_0$ has the same value as above. For the earth, which moves, we can put

$$(T_{ij})_1 = (T_{ij})_1{}^0 + \delta(T_{ij})_1,$$

the first term on the right being the value which we should find if the earth were at rest.

I will restrict the determination of the g_{ij} to an approximation, which is sufficient to give all *secular* terms of the order of magnitude of observable quantities.

We can conceive the g_{ij} to be made up of several parts, thus

$$g_{ij} = (g_{ij})_0 + (g_{ij})_1 + \overline{g}_{ij}.$$

The first two terms taken separately need not correspond to a real problem, they are only mathematically defined as follows: $(g_{ij})_0$ is what we find if in (1) we take account of $(T_{ij})_0$ only, and similarly $(g_{ij})_1$ arises from $(T_{ij})_1$. If now the equations (1) were linear in the g_{ij} and their differential coefficients, then $(g_{ij})_0 + (g_{ij})_1$ would be the complete solution. It follows that $\overline{g}_{ij}$ is of the second order, and need therefore only be computed for $i = j = 4$.

Let us first consider the others, of which $(g_{ij})_0$ is the same as before. We can take $(g_{ij})_1 = (g_{ij})_1{}^0 + \delta(g_{ij})_1$, where the two parts arise from the two parts of $(T_{ij})_1$. The term $\delta(T_{ij})_1$ is at least of the order $\frac{1}{2}$, i.e. of the order of the velocities, and need only be taken into account in the determination of g_{i4}. It there produces a term of the order $\lambda^2{}_1 \xi$ which contains odd functions of the angle Moon—Earth—Sun, and therefore only gives rise to periodic terms of the order ξx, which we neglect. It will appear that even the secular terms of this order are entirely negligible.

By a similar reasoning we find that the term $\delta(g_{44})_1$ will be of the order $\lambda_1{}^2\xi^2$, and the secular terms which may perhaps result from this term will be far beyond the limit of observability.

There remains the term $\overline{g}_{44}$. This will be of the order $\lambda^2\lambda_1{}^2$, and will also contain the angle already mentioned. It can consequently also be neglected.

We thus come to the conclusion that we shall reach a sufficient approximation by simply superposing the fields of the earth and of the sun, both computed as if these bodies were at rest, i.e. by taking throughout [1])

$$g_{ij} = (g_{ij})_0 + (g_{ij})_1^0.$$

Then we have rigorously

$$\frac{d^2 x_i}{dt^2} = \left(\frac{d^2 x_i}{dt^2}\right)_1^0 + \left(\frac{d^2 \mathrm{x}_i}{dt^2}\right)_0 - \left(\frac{d^2 \xi_i}{dt^2}\right)_0,$$

where the meaning of the different suffixes will be easily understood. The first term gives the same result that has already been found above for the planetary motion, viz. a secular motion of the perigee amounting to $\frac{3\lambda_1^2}{p} . nt$. In one century this is 0″.06, which is entirely negligible. The other terms are found by writing the equation (20) for the moon and for the earth and subtracting the latter from the former. The left-hand members then give the perturbing function as used in the current lunar theory. This contains the factor k^2m/ρ^3, which is by KEPLER's third law replaced by n'^2, n' being the mean motion of the sun. Now, however this law must be replaced by (34) and we have consequently

$$\frac{k^2 m}{\rho^3} = n'^2 \left[1 + \frac{3\lambda^2}{\rho}\right].$$

We must therefore apply a correction to the ordinary perturbing function.

The right-hand members of (20) give

$$\delta \frac{d^2 x_i}{dt^2} = k^2 m \left\{ \frac{4\lambda^2 \mathrm{x}_i}{\Delta^3} - \frac{4\lambda^2 \xi_i}{\rho^3} + \frac{4\mathrm{x}_i \dot{\Delta}}{\Delta^2} - \frac{4\xi_i \dot{\rho}}{\rho^2} - \frac{\mathrm{x}_i}{\Delta^3}\varphi_1^2 + \frac{\xi_i}{\rho^3}\varphi_2^2 \right\}, \quad (36)$$

where we have put

$$\varphi_1^2 = \Sigma \dot{\mathrm{x}}_i^2 \quad , \quad \varphi_2^2 = \Sigma \dot{\xi}_i^2 \qquad (\Sigma \text{ from 1 to 3}).$$

Further we have $\mathrm{x}_i = \xi_i + x_i$ etc., and

$$\Delta^2 = \rho^2 \left[1 + 2\frac{\Sigma \xi_i x_i}{\rho^2} + \frac{r^2}{\rho^2}\right].$$

We develop in powers of r/ρ and we neglect the square and higher powers of this small quantity. We also take, as has already been remarked, $\dot{\rho} = 0$, and we neglect all periodic terms. I then find the following radial, transversal and orthogonal perturbing forces

[1]) Simultaneously with the present communication Mr. DROSTE has published (these Proceedings, June 1916) the complete values of g_{ij} for n moving bodies. His results applied to the lunar theory entirely confirm the conclusion which was reached above.

$$S = -3\mu \frac{v}{\varrho} r\dot{\vartheta} - \frac{3}{2}\mu \frac{v^2}{\varrho} r,$$

$$T = +3\mu \frac{v}{\varrho} \dot{r}, \quad \ldots \ldots \ldots \ldots \quad (37)$$

$$W = +3\mu \frac{v}{\varrho} z\dot{\vartheta} + \frac{9}{2} \frac{v^2}{\varrho} z.$$

In these formulae r and ϑ are the coordinates of the moon in its orbit, and z is the coordinate perpendicular to the ecliptic. Further μ is the ratio m/m_1 of the masses of the sun and the earth and v is the mean motion of the sun divided by the velocity of light, $v = n'/c$. It appears that the second terms in S and W are exactly cancelled by the correction to the ordinary perturbing force, which was mentioned above, and need thus not be computed. The other terms give a secular motion of the perigee and the node, both of the same amount, viz.:

$$\delta\tilde{\omega} = \delta\Omega = \frac{3}{2}\mu \frac{a^{3}/_{2}\lambda_1 v}{\varrho} nt. \quad \ldots \ldots \quad (38)$$

The motion in one century is

$$+1''.91.$$

Beyond these motions of the perigee and the node there are no new secular terms in the motion of the moon.

5. *Comparison with the observations.*

The observed values have been taken from NEWCOMB [1]). I have, however, reduced them to the value 5024".90 of the precessional constant (for 1850·0). To the theoretical values as given by NEWCOMB I have added the motions of the perihelia which have been found above. We then find, for one century

$e\,d\tilde{\omega}$	*Observed*	*Theory*	*Difference*
Mercury	+ 118"00 ± 0"40	+ 118"58 ± 0"16	— 0"58 ± 0"43
Venus	+ 0·28 ± ·20	+ 0·39 ± ·15	— 0·11 ± ·25
Earth	+ 19·46 ± ·12	+ 19·45 ± ·05	+ 0·01 ± ·13
Mars	+ 149·44 ± ·35	+ 148·93 ± ·04	+ 0·49 ± ·35
sin i $d\Omega$			
Mercury	— 92·03 ± 0·45	— 92·50 ± 0·16	+ 0·47 ± 0·48
Venus	— 105·47 ± ·12	— 106·00 ± ·12	+ 0·53 ± ·17
Mars	— 72·64 ± ·20	— 72·63 ± ·09	— 0·01 ± ·22

1) *Astronomical constants*, page 109.

The mean errors have been taken from NEWCOMB. For the moon we have [1]):

		Observed	*Theory*	*Difference*
$d\tilde{\omega}$	BROWN, COWELL	$+14643536''\pm 2''$	$+14643534''\pm 2''$	$+2''\pm 3''$
	NEWCOMB, DE VOS	$14643530\ \pm 2$		$-4\ \pm 3$
d☊	NEWCOMB, BROWN	$-\ 6967944\ \pm 2$	$-\ 6967939\ \pm 2$	$-5\ \pm 3$

With respect to the perihelion of Mercury we may remark that the residual, which without the new term, resulting from EINSTEIN's theory (but with the improved constant of precession) would be $+\ 8''.24$, has now become negative. The matter within the orbit of Mercury, by the attraction of which SEELIGER explained the anomalous motion of the perihelion, must thus have an exceedingly small density, certainly less than say $^1/_{200}$th of the value adopted by SEELIGER.

The residuals now show no preference for either the positive or the negative sign, there is thus no reason to suppose a rotation of the empirical system of coordinates with respect to the inertial system, as was done by ANDING and SEELIGER. In other words the precessional constant as found from motions within the solar system is the same as that determined from the fixed stars.

The residual of the node of Venus remains large. We might perhaps still be inclined to ascribe this deviation to the attraction of the masses reflecting the zodiacal light (SEELIGER's second ellipsoid). Since the rotation cannot help us, the density of this ellipsoid would then have to be 3 or 4 times the value assumed by SEELIGER. From the computations by Mr. WOLTJER [2]) it has appeared that this density can certainly not exceed SEELIGER's value, because a larger density would give values for the secular variation of the inclination of the ecliptic and for the planetary precession, which are absolutely contradictory to the results of observations. SEELIGER's second ellipsoid can thus not explain the observed discrepancies [3]). Corrections

[1]) *The motion of the lunar perigee and node,* these Proceedings XVII (April 1915) page 1309.

[2]) J. WOLTJER. *On* SEELIGER's *hypothesis,* these Proceedings XVII (April 1914) page 23; W DE SITTER, *Remarks on Mr.* WOLTJER's *paper*, ibid. page 33.

[3]) If SEELIGER's second ellipsoid is adopted with the density ascribed to it by SEELIGER, we would find the following residuals.

	$ed\tilde{\omega}$	$\sin i\, d$☊	di
Mercury	$-0''.47 \pm 0''.43$	$+0''.49 \pm 0''.48$	$+0''.44 \pm 0''.80$
Venus	$-\ 0.10 \pm\ .25$	$+\ 0.39 \pm\ .17$	$+\ 0.37 \pm\ .33$
Earth	$+\ 0.05 \pm\ .13$		$+\ 0.23 \pm\ .27$
Mars	$+\ 0.46 \pm\ .35$	$-\ 0.04 \pm\ .22$	$+\ 0.01 \pm\ .20$

The value of di for the earth is the secular variation of the inclination of the to the adopted masses also cannot help us. If the mass of Mercury were multiplied by 3, which of course in itself is outside all limits of probability, the node of Venus would be put right, but we should then have a still larger discrepancy e.g. in the perihelion of Venus. It is not possible to find a system of masses which will reduce all residuals to within their mean errors.

Chemistry. — "*Investigations on the Temperature-Coefficient of the Free Molecular Surface-Energy of Liquids between* — 80° *and* 1650° *C.*" **XV**. "*The Determination of the Specific Gravity of molten Salts, and of the Temperature-Coefficient of their Molecular-Surface-Energy*". By Prof. Dr. F. M. JAEGER and Dr. JUL. KAHN.

(Communicated in the meeting of June 24, 1916).

§ **1.** For the calculation of the molecular free surface-energy of the molten salts and other compounds, about which we have previously communicated [1]), it is necessary to know the specific gravity of the investigated liquids at temperatures ranging from — 80° up to 1650° C.

As far as organic liquids are concerned, the usual and generally known methods can be applied, — at least if the temperatures of measurement are not too far apart from the range usually considered in laboratory-experiments. In those cases we used in the first place the *pycnometer*. Commonly this consisted in a double-walled vessel, the space between the glass-walls being carefully evacuated; it was closed by means of a ground thermometer. In most cases the densities were determined in thermostats at 25°, 50° and 75° C. In the work with liquids of low boilingpoint such measurements had to be made also at the temperature of melting ice, or in refrigerant mixtures of salt and ice, or in those of solid carbondioxide and alcool; in these cases the pycnometer is evidently not a suitable instrument, and the pycnometrical method appears for many reasons much less adapted than the *volumetrical* method.

More particularly in the determination of the specific gravities of the low-boiling aliphatic *amines*, which moreover will absorb readily the carbondioxide and the water-vapour from the atmosphere, the *volumeter* appeared to be the only applicable instrument, while

ecliptic. For the planetary precession we should have a correction amounting to +0.″30 per century. These residuals are not appreciably better than those given above in the text. (Note added in the English translation).

[1]) F. M JAEGER and Collaborators, these Proceedings, 1914, '15 and '16.