

Citation:

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from Cabinda and was lighted at night by an acetylene lamp with reflector.

For the coordinates of the signal with respect to the observation-pier I found by the measurement of azimuth and distance

$$\begin{array}{r} \Delta \varphi = - \quad 7''.14 \quad \Delta \lambda = - \quad 0''.24 \\ \text{thus Signal Matuba } \varphi = - 5^{\circ} 17' 6''.2 \quad \lambda = - 12^{\circ} 10' 7''.4 \end{array}$$

On Sept. 25, 1913 at Cabinda I made a time-determination by means of β Ceti in the east and α Ophiuchi in the west. The telescope was twice pointed on each star in each position of the instrument, at mean zenith distances of 59° and 57° respectively. As correction of the HÖRŪ chronometer I found:

$$\begin{array}{r} \text{by } \beta \text{ Ceti} \quad + 0^{\text{h}}54^{\text{m}}35^{\text{s}}19 \\ \quad \alpha \text{ Ophiuchi} \quad \quad \quad 35.46 \\ \hline \text{Mean} \quad + 0^{\text{h}}54^{\text{m}}35^{\text{s}}33 \end{array}$$

On Sept 27 I then determined the azimuth of the signal by means of the greatest digression of ν Ophiuchi (the observation of that of θ Ceti failed) and found, counting from the north through the east etc.

Azimuth Signal Matuba $A = 353^{\circ} 55' 26''.1$.

From this and from $\varphi' - \varphi = - 16' 16'' 1$ I calculated from ALBRICHT and from SCHÖLS in complete agreement:

$$\lambda' - \lambda = - 1' 43''.65$$

from which

$$\text{Longitude Cabinda Obs. P.} = - 12^{\circ} 11' 51''.1 = - 0^{\text{h}}48^{\text{m}}47''.4$$

An error of $30''$ in the azimuth causes in the longitude one of $0''14$ only.

Physics. — “*Note on the model of the hydrogen-molecule of BOHR and DEBIJE*”. By J. M. BURGERS. (Communicated by Prof. H. A. LORENTZ).

(Communicated in the meeting of June 24, 1916.)

Miss H. J. VAN LEEUWEN has recently published a paper containing some notes on DEBIJE's calculation of the dispersion formula of hydrogen, which calculation is founded on the well-known model of the H_2 -molecule¹⁾. In that paper it is demonstrated that some of the vibrations which occur in DEBIJE's calculations are unstable, and methods are discussed by which the stability of the model may be ensured.

¹⁾ These Proc (1916) Vol. XVIII, p. 1071.

One of these methods, viz. that of introducing as kinematical relation the condition that the moment of momentum of each electron must preserve the same value $\left(\frac{h}{2\pi}\right)$, is only touched upon in that paper¹⁾ and not fully worked out. It seems to me, however, that this may be done in the following way.

The equations expressing these conditions can be written as follows :

$$\left. \begin{aligned} mr_1^2 d\psi_1 &= \frac{h}{2\pi} dt \\ mr_2^2 d\psi_2 &= \frac{h}{2\pi} dt \end{aligned} \right\} \dots \dots \dots (1)$$

(m : mass of the electron;
 r : radius of the orbit;
 ψ : angle of position).

By their form they recall the equations between infinitesimal changes of the coordinates in a non-holonomic system; only dt also appears here. Now we may try to form the equations of motion in a way analogous to the treatment of non-holonomic systems by introducing into the formula of D'ALEMBERT'S principle auxiliary forces Q_μ which do not come into play in any virtual displacement. A virtual displacement will be defined as an arbitrary variation of the coordinates, subjected to the relations:

$$\left. \begin{aligned} mr_1^2 \delta\psi_1 &= 0 \\ mr_2^2 \delta\psi_2 &= 0 \end{aligned} \right\} \dots \dots \dots (2)$$

or:

$$\delta\psi_1 = \delta\psi_2 = 0,$$

which are derived from (1) by taking $\delta t = 0$. It appears that in a virtual displacement the *position angles* of the electrons must *not* be varied; from this it follows easily that only *tangential* auxiliary forces Q_1 and Q_2 may be introduced (i.e. forces which act upon the coordinates ψ_1 and ψ_2), which have the task of ensuring the constancy of the moment of momentum.

Deduction of the equations of motion.
A. Free vibrations.

Notation. Distance of the nuclei: $2a$ (this is regarded as a constant).

Radius of the orbit of the electrons in the normal state: R . (For H_2 : $R = a\sqrt{3}$; He , to which the calculation likewise applies, has

¹⁾ l. c. p. 1081.

one nucleus at the centre with a double charge, hence for it $a = 0$)

Normal angular velocity: ω .

Distance of electron from centre of orbit: $r_i = R + \rho_i$ (measured parallel to the plane of the orbit).

Deviation of an electron normally to the plane of the orbit: z_i .

Angle of position: $\psi_i = \omega t + \vartheta_i$.

Furthermore we put.

$$W = \sqrt{R^2 + a^2}$$

$$\alpha = \rho_1 + \rho_2 \quad \gamma = z_1 + z_2 \quad \varphi = \vartheta_2 - \vartheta_1 - \pi.$$

$$\beta = \rho_1 - \rho_2 \quad \delta = z_1 - z_2$$

The quantities $\rho_1, \rho_2, z_1, z_2, \alpha, \beta, \gamma, \delta, \varphi$ are considered as infinitesimal, likewise $\dot{\rho}_1$, etc.

Between a, R, ω the relation exists.

$$mR\omega^2 = \frac{2e^2R}{W^3} - \frac{e^2}{4R^2} \dots \dots \dots (3)$$

which expresses the condition of equilibrium in the stationary state.

The kinetic energy is found to be:

$$T = \frac{1}{2}m(\dot{z}_1^2 + \dot{z}_2^2) + \frac{1}{2}m(\dot{\rho}_1^2 + \dot{\rho}_2^2) + \frac{1}{2}mR^2(\dot{\vartheta}_1^2 + \dot{\vartheta}_2^2) +$$

$$+ mR^2\omega(\vartheta_1 + \vartheta_2) + 2mR\omega(\rho_1\dot{\vartheta}_1 + \rho_2\dot{\vartheta}_2) + \frac{1}{2}m\omega^2(\rho_1^2 + \rho_2^2) +$$

$$+ mR\omega^2(\rho_1 + \rho_2) + mR^2\omega^2.$$

Potential Energy:

$$V = -\frac{4e^2}{W} + \frac{e^2}{2R} + \frac{2e^2R(\rho_1 + \rho_2)}{W^3} - \frac{e^2(\rho_1 + \rho_2)}{4R^2} + \frac{e^2(\rho_1 + \rho_2)^2}{8R^3} -$$

$$- \frac{e^2(z_1 - z_2)^2}{16R^3} - \frac{e^2}{W^6} [(2R^2 - a^2)(\rho_1^2 + \rho_2^2) - (R^2 - 2a^2)(z_1^2 + z_2^2)] +$$

$$+ \frac{e^2\varphi^2}{16R}.$$

(In both expressions terms of the 3rd and higher orders in the quantities ρ , etc. have been neglected).

D'ALEMBERT's principle gives the equations.

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_\mu} \right) - \frac{\partial T}{\partial q_\mu} + \frac{\partial V}{\partial q_\mu} - Q_\mu = 0.$$

(the Q_μ are the auxiliary forces mentioned above).

Hence we get:

$$\rho_1 - 2R\omega\dot{\vartheta}_1 - \omega^2\rho_1 = \frac{e^2}{m} \left[\frac{4R^2\rho_1 - 2a^2\rho_1}{W^6} - \frac{\rho_1 + \rho_2}{4R^3} \right] \dots \dots (4)$$

$$\rho_2 - 2R\omega\dot{\vartheta}_2 - \omega^2\rho_2 = \frac{e^2}{m} \left[\frac{4R^2\rho_2 - 2a^2\rho_2}{W^6} - \frac{\rho_1 + \rho_2}{4R^3} \right] \dots \dots (5)$$

$$R\dot{\vartheta}_1 + 2\omega\rho_1 = \frac{e^2\varphi}{8mR^2} + \frac{Q_1}{mR} \dots \dots \dots (6)$$

$$R\dot{\vartheta}_2 + 2\omega\dot{q}_2 = -\frac{e^2\varphi}{8mR^2} + \frac{Q_2}{mR} \dots (7)$$

$$z_1 = \frac{e^2}{m} \left[\frac{-2R^2 z_1 + 4a^2 z_1}{W^5} + \frac{z_1 - z_2}{8R^3} \right] \dots (8)$$

$$z_2 = \frac{e^2}{m} \left[\frac{-2R^2 z_2 + 4a^2 z_2}{W^5} - \frac{z_1 - z_2}{8R^3} \right] \dots (9)$$

(in deducing these, eq. (3) has already been used)

In addition to this, eqq. (1) give the relations.

$$m(R+q_1)^2 (\omega + \vartheta_1) = mR^2\omega \left(= \frac{h}{2\pi} \right)$$

$$m(R+q_2)^2 (\omega + \vartheta_2) = mR^2\omega \left(= \frac{h}{2\pi} \right)$$

After development, and with omission of terms of the 2nd and higher orders:

$$R\dot{\vartheta}_1 + 2q_1\omega = 0 \dots (10)$$

$$R\dot{\vartheta}_2 + 2q_2\omega = 0 \dots (11)$$

From eqq. (6), (7), (10) and (11) we get immediately.

$$Q_1 = -Q_2 = -\frac{e^2\varphi}{8R}$$

Further (4) and (5) can be simplified by means of (10) and (11) into.

$$q_1 + 3\omega^2 q_1 = \frac{e^2}{m} [\dots] \dots (4a)$$

$$q_2 + 3\omega^2 q_2 = \frac{e^2}{m} [\dots] \dots (5a)$$

Finally, by addition and subtraction of the foregoing equations:

$$\alpha + 3\omega^2\alpha = \alpha \cdot \frac{e^2}{m} \cdot \left[\frac{4R^2 - 2a^2}{W^5} - \frac{1}{2R^3} \right] \dots (12)$$

$$\beta + 3\omega^2\beta = \beta \cdot \frac{e^2}{m} \cdot \frac{4R^2 - 2a^2}{W^5} \dots (13)$$

$$\gamma = \gamma \cdot \frac{e^2}{m} \cdot \frac{-2R^2 + 4a^2}{W^5} \dots (14)$$

$$\delta = \delta \cdot \frac{e^2}{m} \cdot \left[\frac{-2R^2 + 4a^2}{W^5} + \frac{1}{4R^3} \right] \dots (15)$$

Result for *Hydrogen*.

$$\ddot{\alpha} + 1.9288 \omega^2 \alpha = 0 \dots (I)$$

$$\ddot{\beta} + 1.4522 \omega^2 \beta = 0 \dots (II)$$

$$\ddot{\gamma} + 0.3096 \omega^2 \gamma = 0 \dots (III)$$

$$\ddot{\delta} + 0.0713 \omega^2 \delta = 0 \dots (IV)$$

If α and β , and thus ϱ_1 and ϱ_2 are solved from (I) and (II), eqq. (10) and (11) give ϑ_1 and ϑ_2 .

It appears from equations (I)–(IV) that we get 4 *stable* modes of vibration; the unstable vibration D_2 and the indifferent one C_1 , which Miss v. LEEUWEN gave (l. c. p. 1074) have disappeared.

Besides some of the frequencies have changed a little.

Helium:

$$\ddot{\alpha} + \omega^2 \alpha = 0 \quad \dots \dots \dots (I)$$

$$\ddot{\beta} + \frac{5}{7} \omega^2 \beta = 0 \quad \dots \dots \dots (II)$$

$$\ddot{\gamma} + \frac{8}{7} \omega^2 \gamma = 0 \quad \dots \dots \dots (III)$$

$$\ddot{\delta} + \omega^2 \delta = 0 \quad \dots \dots \dots (IV)$$

(N.B. In the He-atom the plane of the orbit of the electrons can turn freely about the nucleus; hence the δ -motion is here not a true vibration).

Note.

As the coordinates ψ_1 and ψ_2 do not appear themselves in T , we might, in deducing the equations of motion, treat them as cyclic coordinates, eliminating $\dot{\psi}_i$ ($= \omega + \dot{\vartheta}_i$) by means of

$$\dot{\psi}_i = \frac{h}{2\pi m r_i^2}$$

and forming the kinetic potential according to ROUTH and HELMHOLTZ.

This method of eliminating the $\dot{\psi}_i$ is used by L. FÖPPL in an investigation on the stability of BOHR's model¹⁾. If we applied it here, the term

$$\frac{e^2 \varphi^2}{16R} = \frac{e^2 (\psi_2 - \psi_1 - \pi)^2}{16R}$$

would remain in V , so that we should be obliged to omit it altogether, whereas in the calculation given above its influence is annihilated by the forces Q_1 and Q_2 .

B. Forced vibrations. — Dispersion formula.

The course of the following calculation is for the greater part the same as that followed by DEBIJE²⁾.

We will make use of two systems of coordinates: One system is invariably attached to the molecule, the axis of z is laid along the line which joins the nuclei; the axis of x along the line which joins the electrons; the axis of y perpendicularly to the latter in

¹⁾ Phys. Z. S. 15 (1914) S. 707.

²⁾ Sitz. Ber. Bayr. Akad. 1915, S. 1.

the plane of the orbit. The other one (the $x'y'z'$ -system) is fixed in space, and coincides with the former for $t = 0$.

Let the incident electric vibration be $E \cdot e^{ist}$, its components along $x'y'z'$ -axes :

$$P \cdot e^{ist}, \quad Q e^{ist}, \quad R e^{ist};$$

where :

$$P = E \cos \alpha, \quad Q = E \cos \beta, \quad R = E \cos \gamma$$

Putting

$$\begin{aligned} P + i Q &= p, \\ P - i Q &= q, \end{aligned}$$

we find for the components along the rotating axes :

$$\left. \begin{aligned} X &= \frac{1}{2} p e^{i(s-\omega)t} + \frac{1}{2} q e^{i(s+\omega)t} \\ Y &= -\frac{i}{2} p e^{i(s-\omega)t} + \frac{i}{2} q e^{i(s+\omega)t} \\ Z &= R e^{ist} \end{aligned} \right\} \dots \dots \dots (16)$$

The equations for the forced vibrations can be deduced from (4)–(9) by adding to the members on the right hand side :

$$-\frac{Xe}{m}, \quad +\frac{Xe}{m}, \quad -\frac{Ye}{mR}, \quad \text{etc.}$$

As only the β -vibration with the φ -vibration (which is coupled to the former by the equations (10) and (11), to which we adhere in this calculation too), and the γ -motion give an electric moment, we obtain, denoting the frequencies of the free β - and γ -motion by n_1 and n_2 :

$$\beta + n_1^2 \beta = -\frac{2Xe}{m} \dots \dots \dots (17)$$

$$\gamma + n_2^2 \gamma = -\frac{2Ze}{m} \dots \dots \dots (18)$$

$$R\varphi - 2\omega\beta = 0 \dots \dots \dots (19)$$

Eq. (19) is deduced by subtracting eq. (10) from eq. (11). The auxiliary forces Q_1 , and Q_2 have now different values, as they have also to annihilate the Y -component of the incident electric vibration.

The components of the electric moment are found to be :

$$\left. \begin{aligned} M_x &= -e\beta = \frac{e^2}{m} \left[\frac{p e^{i(s-\omega)t}}{n_1^2 - (s-\omega)^2} + \frac{q e^{i(s+\omega)t}}{n_1^2 - (s+\omega)^2} \right] \\ M_y &= Re\varphi = 2i \frac{e^2}{m} \left\{ \frac{p \frac{\omega}{s-\omega} e^{i(s-\omega)t}}{n_1^2 - (s-\omega)^2} + \frac{q \frac{\omega}{s+\omega} e^{i(s+\omega)t}}{n_1^2 - (s+\omega)^2} \right\} \\ M_z &= -e\gamma = 2 \frac{e^2}{m} \frac{R e^{ist}}{n_2^2 - s^2} \end{aligned} \right\} \dots \dots \dots (20)$$

From these quantities we must derive the components along the fixed system of axes :

$$M_x' = M_x \cos \omega t - M_y \sin \omega t = \\ = e^{ist} \cdot \frac{e^2}{m} \left[p \cdot \frac{(\omega-s) + (\omega+s) i \sin \omega t e^{-i\omega t}}{(\omega-s)[n_1^2 - (\omega-s)^2]} + q \cdot \frac{(\omega+s) - (\omega-s) i \sin \omega t e^{i\omega t}}{(\omega+s)[n_1^2 - (\omega+s)^2]} \right]$$

Part of the function between the [] is independent of t , part of it depends periodically on t with frequency ω . As ω has nothing to do with s (usually ω is much greater than s) we can take the mean of this part with respect to the time¹⁾; the formula then reduces to:

$$M_x' = e^{ist} \cdot \frac{e^2}{m} \left[\frac{\frac{3\omega-s}{2} \cdot p}{(\omega-s)[n_1^2 - (\omega-s)^2]} + \frac{\frac{3\omega+s}{2} \cdot q}{(\omega+s)[n_1^2 - (\omega+s)^2]} \right]$$

In the same way we find:

$$M_y' = e^{ist} \cdot \frac{e^2}{m} \left[\frac{-\frac{3\omega-s}{2} \cdot ip}{(\omega-s)[n_1^2 - (\omega-s)^2]} + \frac{\frac{3\omega+s}{2} \cdot iq}{(\omega+s)[n_1^2 - (\omega+s)^2]} \right] \quad (21)$$

Finally:

$$M_z' = e^{ist} \cdot \frac{e^2}{m} \frac{2R}{n_2^2 - s^2}$$

Resulting moment in the direction of E :

$$M = M_x' \cdot \cos \alpha + M_y' \cdot \cos \beta + M_z' \cos \gamma = \\ = e^{ist} \cdot \frac{Ee^2}{m} \left[\frac{\frac{3\omega-s}{2} (\cos^2 \alpha + \cos^2 \beta)}{(\omega-s)[n_1^2 - (\omega-s)^2]} + \frac{\frac{3\omega+s}{2} (\cos^2 \alpha + \cos^2 \beta)}{(\omega+s)[n_1^2 - (\omega+s)^2]} + \frac{2\cos^2 \gamma}{n_2^2 - s^2} \right]$$

The mean of this quantity for all possible directions of the $x'y'z'$ system is:

$$\overline{M} = e^{ist} \cdot \frac{Ee^2}{3m} \left[\frac{3\omega-s}{(\omega-s)[n_1^2 - (\omega-s)^2]} + \frac{3\omega+s}{(\omega+s)[n_1^2 - (\omega+s)^2]} + \frac{2}{n_2^2 - s^2} \right] \quad (22)$$

(as:

$$\overline{\cos^2 \alpha} = \overline{\cos^2 \beta} = \overline{\cos^2 \gamma} = \frac{1}{3})$$

For the index of refraction we have the formula:

$$n^2 - 1 = 4\pi \frac{N_1 \cdot \overline{M}}{E \cdot e^{ist}}$$

(N_1 : number of molecules per cc.)

Hence:

$$n^2 - 1 = \frac{4\pi N_1 e^2}{3m} \left[\frac{3\omega-s}{(\omega-s)[n_1^2 - (\omega-s)^2]} + \frac{3\omega+s}{(\omega+s)[n_1^2 - (\omega+s)^2]} + \frac{2}{n_2^2 - s^2} \right] \quad (23)$$

¹⁾ DEBIJE (l. c. S. 15) expresses this in a somewhat different way: in his formulae the axes of x and x' make the angle α for $t=0$, hence he has everywhere $\omega t + \alpha$ instead of ωt . Then the mean is taken for all values of the phase-angle α , which gives the same result.

As s is small as compared to ω , we may develop according to powers of $\frac{s}{\omega}$; this gives, when s^4, \dots is neglected:

$$n^2 - 1 = \frac{4\pi N_1 e^2}{3m} \left[\frac{6}{n_1^2 - \omega^2} + \frac{2}{n_2^2} + s^2 \left(\frac{4}{\omega^2(n_1^2 - \omega^2)} - \frac{2}{(n_1^2 - \omega^2)^2} + \frac{24\omega^2}{(n_1^2 - \omega^2)^3} + \frac{2}{n_2^4} \right) \right] \quad (24)$$

Hydrogen.

From eqq. (II) and (III) we get:

$$n_1^2 = 1.4522 \omega^2; \quad n_2^2 = 0.3096 \omega^2.$$

hence:

$$n - 1 = \frac{2\pi N_1 e^2}{3m \omega^2} \left(19.73 + 280.4 \frac{s^2}{\omega^2} \right)$$

If we introduce the values:

$$N_1 = \frac{6.15 \times 10^{23}}{22400}; \quad \frac{e}{m} = 5.31 \times 10^{17}; \quad e = 4.78 \times 10^{-10};$$

$$h = 6.415 \times 10^{-27};$$

formula (3) in connection with

$$mR^2\omega = \frac{h}{2\pi}$$

gives

$$\omega = 4.856 \times 10^{16};$$

further:

$$\frac{2\pi N_1 e^2}{3m} = 1.461 \times 10^{28}$$

Thus the dispersion formula becomes:

$$n - 1 = 1.22 \times 10^{-4} + 7.35 \times 10^{-37} s^2 \quad (25)$$

DEBIJE's formula: ¹⁾

$$n - 1 = \frac{2\pi N_1 e^2}{3\omega^2} \left(19.26 + 75.3 \frac{s^2}{\omega^2} \right)$$

gives with the same values:

$$n - 1 = 1.193 \times 10^{-4} + 1.98 \times 10^{-37} s^2.$$

The experimental formulae are (cf. DEBIJE, l.c. pag. 20, 22), that of J. KOCH:

$$n - 1 = 1.361 \times 10^{-4} + 2.908 \times 10^{-37} s^2$$

and that of C. and M. CUTHBERTSON:

$$n - 1 = 1.362 \times 10^{-4} + 2.780 \times 10^{-37} s^2.$$

In formula (25) the coefficient of s^2 appears to be *much too large*. Partly this is due to the frequency n_1 of the β -vibration, which is *smaller* than the value of the corresponding frequency in DEBIJE's

¹⁾ l. c. p. 20.

calculation, this makes one of the resonance frequencies lie much nearer to the visible spectrum.

We have found:

$$n_1 = 1.205 \omega$$

while in the original model:

$$n_1 = 1.412 \omega$$

(cf. the paper by Miss H. J. v. LEEUWEN. p. 1074).

Resonance occurs if the incident wave motion has one of the values:

$$s_1 = n_1 + \omega = 2.205 \omega$$

$$s_2 = n_1 - \omega = 0.205 \omega$$

$$s_3 = n_2 = 0.556 \omega$$

λ_2 is the smallest; to it corresponds a wave-length of about 1890 Å units).

Helium.

For Helium: $n_1^2 = \frac{5}{7} \omega^2 = 0.7143 \omega^2$, and so $n_1^2 - \omega^2 < 0$. Hence in formula (25) the principal terms become negative, and for values of s not too high

$$n < 1.$$

This is in contradiction with the experimental values. (Cf. C. and M. CUTHBERTSON, Proc. Roy. Soc. (A) LXXXIV, p. 13).

S U M M A R Y.

1. In continuation of the investigation by Miss H. J. VAN LEEUWEN on the instability of BOHR and DEBIJÉ's model of the hydrogen molecule, a new hypothesis is examined by which the system may be made stable.

2. The model made stable in this way gives neither for H_2 , nor for He a dispersion formula which agrees with the formula experimentally found.

Note added in the English translation

Since this was written a paper has appeared by C. DAVISSON on the *Dispersion of Hydrogen and Helium* (Phys. Rev. (22) VIII, p. 20, July 1916).

Mr. DAVISSON uses the same method to ensure the stability of the model, but he arrives at a somewhat different formula (it seems to me that he overlooks the influence of the conditions (10) and (11) and of the auxiliary forces Q_1 and Q_2 necessitated by them on the -vibration). As Mr. DAVISSON points out himself, his formula too gives for both gases results which are in conflict with the experimental ones.