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Mathematics. — "*On the values which the function $\zeta(s)$ assumes for s positive and odd.*" By J. G. VAN DER CORPUT. (Communicated by Prof. J. C. KLUYVER).

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This article is intended to deduce some formulae that may be used to calculate $\zeta(s)$ for odd values of the argumentum $s > 1$; for this purpose we will express these ζ functions in the quantities $I_1(m, n)$, $I_2(m, n)$ and $I(n, a)$, which have the following significance:

$$I_1(m, n) = \pi \int_0^1 y^m (1-y)^n \cot g\, \pi y \, dy =$$

$$= n! \left(\frac{1}{m(m+1)\ldots(m+n)} - \sum_1^{\infty} \frac{B_\nu \pi^{2\nu}}{(2\varkappa)!(2\varkappa+m)(2\varkappa+m+1)\ldots(2\varkappa+m+n)} \right)$$

for m and n integer and positive,

$$I_2(m, n) = \frac{\pi}{2} \int_0^1 y^m (1-y)^n \cot g\, \frac{\pi y}{2} \, dy =$$

$$= n! \left(\frac{1}{m(m+1)\ldots(m+n)} - \sum_1^{\infty} \frac{B_\nu \left(\frac{\pi}{2}\right)^{2\nu}}{(2\varkappa)!(2\varkappa+m)(2\varkappa+m+1)\ldots(2\varkappa+m+n)} \right)$$

for m and n integer, $m > 0$ and $n \geq 0$,

$$I(n, a) = \int_0^1 (1-y)^n \left(\frac{1}{y} - \pi a \cot g\, \pi a y \right) dy = n! \sum_1^{\infty} \frac{B_\nu (a\pi)^{2\nu}}{(2\varkappa) \cdot (2\varkappa+n)!}$$

for $1 > a > 0$, and n integer ≥ 0 and also for $a = 1$ and n integer > 0.

In order to connect the ζ function with the quantities $I_1(m, n)$, $I_2(m, n)$ and $I(n, a)$ we will use the method indicated by Professor Dr. J. C. KLUYVER in the article: "Sur les valeurs que prend la fonction $\zeta(s)$ de RIEMANN pour s entier positif et impair". (Bulletin des Sciences Mathématiques, 1896).

If $f(y)$ represents a polynomium in y, which becomes zero for $y = 0$ and for $y = 1$, the uniform function

$$\frac{f\left(\frac{z}{2\pi i}\right)}{e^z - 1}$$

is holomorphous in the domain of the rectangle bounded by the lines $x = 0$, $y = 0$, $x = \beta$, $y = 2\pi$. By applying the theorem of CAUCHY and then putting $\beta = \infty$, we therefore find the relation

$$\int_0^\infty \frac{f\left(\frac{z}{2\pi i}\right) - f\left(\frac{z}{2\pi i}+1\right)}{e^z - 1}\, dz = -\pi i \int_0^1 f(y)\, dy + \pi \int_0^1 f(y) \operatorname{cotg} \pi y\, dy \quad . \ (1)$$

By writing in this formula $f(y) = y^m (1-y)^n$, we conclude

$$\int_0^1 \frac{\left(\frac{z}{2\pi i}\right)^m \left(1 - \frac{z}{2\pi i}\right)^n + (-1)^{n+1} \left(\frac{z}{2\pi i}\right)^n \left(1 + \frac{z}{2\pi i}\right)^m}{e^z - 1}\, dz =$$

$$= -\frac{\pi i m! n!}{(m+n+1)!} + I_1(m, n)$$

for m and n integer and positive and by giving definite values to m and n, in this relation we find for $\zeta(3)$

$$\zeta(3) = -\frac{2\pi^2}{3} I_1(2,1) = \frac{2\pi^2}{3} I_1(1,2) = -\frac{2\pi^2}{3} I_1(3,1) = \frac{2\pi^2}{3} I_1(1,3).$$

If, however, $f(y)$ represents a polynomium in y, which disappears for $y = 1$, but not for $y = 0$, formula (1) will hold good if $f(y)$ is replaced by $f(y) - (1-y) f(0)$; this produces

$$\int_0^\infty \frac{f\left(\frac{z}{2\pi i}\right) - f\left(\frac{z}{2\pi i}+1\right) - f(0)}{e^z - 1}\, dz =$$

$$= -\pi i \int_0^1 \{f(y) - (1-y) f(0)\}\, dy + \pi \int_0^1 \{f(y) - (1-y) f(0)\} \operatorname{cotg} \pi y\, dy$$

$$= -\pi i \int_0^1 f(y)\, dy + \tfrac{1}{2} \pi i f(0) - \int_0^1 f(y) \left(\frac{1}{y} - \pi \operatorname{cotg} \pi y\right) dy +$$

$$+ \int_0^1 \frac{f(y) - f(0)}{y}\, dy - f(0) \int_0^1 (1-y)\left(\pi \operatorname{cotg} \pi y - \frac{1}{y}\right) dy + \int_0^1 f(0)\, dy$$

$$= -\pi i \int_0^1 f(y)\, dy + \tfrac{1}{2}\pi i f(0) - \int_0^1 f(y)\left(\frac{1}{y} - \pi \operatorname{cotg} \pi y\right) dy + \int_0^1 \frac{f(y) - f(0)}{y}\, dy + f(0) \log 2\pi .$$

The particular case $f(y) = (1-y)^n$ produces the formula

$$\int_0^\infty \frac{1 + \left(\frac{zi}{2\pi}\right)^n - \left(1 + \frac{zi}{2\pi}\right)^n}{e^z - 1}\, dz = \frac{\pi i}{n+1} - \frac{\pi i}{2} + I(n, 1) + \sum_1^n \frac{1}{\varkappa} - \log 2\pi ,$$

i.e. for $n = 3$:

$$\zeta(3) = \frac{2\pi^2}{3}\left(\frac{11}{6} - \log 2\pi + I(3, 1)\right)$$

If we reduce the height of the rectangle to half its original height and so take it equal to π, we find in the same way as in which (1) has been deduced, supposing $f(y)$ is a polynomium in y disappearing for $y=0$,

$$\int_0^\infty \frac{f\left(\frac{z}{\pi i}\right)}{e^z-1}\,dz+\int_0^\infty \frac{f\left(\frac{z}{\pi i}+1\right)}{e^z+1}\,dz=-\frac{\pi i}{2}\int_0^1 f(y)\,dy+\frac{\pi}{2}\int_0^1 f(y)\,cotg\,\frac{\pi y}{2}\,dy$$

and as:

$$\int_0^\infty \frac{f\left(\frac{z}{\pi i}+1\right)}{e^z+1}\,dz=f(1)\int_0^\infty \frac{dz}{e^z+1}+\int_0^\infty \frac{f\left(\frac{z}{\pi i}+1\right)-f(1)}{e^z+1}\,dz$$

$$=f(1)\,log\,2+\int_0^\infty \left\{f\left(\frac{z}{\pi i}+1\right)-f(1)\right\}\left\{\frac{1}{e^z-1}-\frac{2}{e^{2z}-1}\right\}dz=$$

$$=f(1)\,log\,2+\int_0^\infty \frac{f\left(\frac{z}{\pi i}+1\right)-f\left(\frac{z}{2\pi i}+1\right)}{e^z-1}\,dz$$

at last:

$$\int_0^\infty \frac{f\left(\frac{z}{\pi i}\right)+f\left(\frac{z}{\pi i}+1\right)-f\left(\frac{z}{2\pi i}+1\right)}{e^z-1}\,dz=$$

$$=-\frac{\pi i}{2}\int_0^1 f(y)\,dy-f(1)\,log\,2+\frac{\pi}{2}\int_0^1 f(y)\,cotg\frac{\pi y}{2}\,dy \quad (2)$$

The substitution $f(y)=y^m(1-y)^n$ produces the relations

$$\int_0^\infty \frac{\left(-\frac{zi}{\pi}\right)^m\left(1+\frac{zi}{\pi}\right)^n+\left(\frac{zi}{\pi}\right)^n\left(1-\frac{zi}{\pi}\right)^m-\left(\frac{zi}{2\pi}\right)^n\left(1-\frac{zi}{2\pi}\right)^m}{e^z-1}\,dz$$

$$=-\frac{\pi i\,m!\,n!}{2(m+n+1)!}+I_2(m,n)$$

for m and n integer and positive and

$$\int_0^\infty \frac{\left(\frac{zi}{\pi}\right)^m+\left(1+\frac{zi}{\pi}\right)^m-\left(1+\frac{zi}{2\pi}\right)^m}{e^z-1}\,dz=\frac{\pi i}{2(m+1)}-log\,2+I_2(m,0)$$

for m integer and positive and consequently in particular

$$\zeta(3) = \frac{2\pi^2}{7} I_2(1,1) = \frac{2\pi^2}{5} I_2(1,2) = \pi^2 I_1(2,1) = \frac{2\pi^2}{7}(\log 2 - I_2(2,0)).$$

If $f(0) =/= 0$, we replace in (2) $f(y)$ by the polynomium $f(y) - f(0)$ which leads to

$$\int_0^\infty \frac{f\left(\frac{z}{\pi i}\right) + f\left(\frac{z}{\pi i}+1\right) - f\left(\frac{z}{2\pi i}+1\right) - f(0)}{e^z - 1} dz =$$

$$= -\frac{\pi i}{2}\int_0^1 f(y)\,dy + \frac{\pi i f(0)}{2} - (f(1) - f(0)) \log 2 +$$

$$+ f(0)\int_0^1 \left(\frac{1}{y} - \frac{\pi}{2} coty \frac{\pi y}{2}\right) dy + \int_0^1 \frac{f(y) - f(0)}{y} dy - \int_0^1 f(y)\left(\frac{1}{y} - \frac{\pi}{2} cotg \frac{\pi y}{2}\right) dy$$

$$= -\frac{\pi i}{2}\int_0^1 f(y)\,dy + \tfrac{1}{2}\pi i f(0) - f(1) \log 2 + f(0) \log \pi +$$

$$+ \int_0^1 \frac{f(y) - f(0)}{y} dy - \int_0^1 f(y)\left(\frac{1}{y} - \frac{\pi}{2} cotg \frac{\pi y}{2}\right) dy$$

and, in consequence for $f(y) = (1 - y)^n$, in which n represents an integer positive number,

$$\int_0^\infty \frac{\left(1 + \frac{zi}{\pi}\right)^n + \left(1 - \frac{1}{2^n}\right)\left(\frac{zi}{\pi}\right)^n - 1}{e^z - 1} dz = -\frac{\pi i}{2(n+1)} + \frac{\pi i}{2} + \log \pi - \sum_1^n \frac{1}{\varkappa} - I(n, \tfrac{1}{2}).$$

This formula contains the relation

$$\zeta(3) = \frac{2\pi^2}{7}\left(\tfrac{3}{2} - \log \pi + I(2, \tfrac{1}{2})\right)$$

particularly. If we now choose the height of the rectangle $2a\pi$ in which $1 > a > 0$, the method already followed twice before, produces the relation

$$\int_0^\infty \frac{f\left(\frac{z}{2\pi i a}\right)}{e^z - 1} dz - \int_0^\infty \frac{f\left(\frac{z}{2\pi i a} + 1\right)}{e^{z + 2\pi i a} - 1} dz = -\pi i a \int_0^1 f(y)\,dy + \pi a \int_0^1 f(y) cotg\, \pi a y\, dy,$$

if $f(y)$ represents a polynomium in y and $f(0) = 0$; if however $f(0)$ does not disappear, we arrive, by replacing $f(y)$ in this relation by $f(y) - f(0)$, at the formula

$$\int_0^\infty \frac{f\left(\frac{z}{2\pi i a}\right)-f(0)}{e^z-1} - \int_0^\infty \frac{f\left(\frac{z}{2\pi i a}+1\right)}{e^{z+2\pi i a}-1}\,dz =$$

$$= -f(0)\int_0^1 \frac{dz}{e^{z+2\pi i a}-1} + f(0)\int_0^1\left(\frac{1}{y}-\pi a \operatorname{cotg} \pi a y\right)dy - \pi i a\int_0^1 (f(y)-f(0))\,dy\; +$$

$$+\int_0^1 \frac{f(y)-f(0)}{y}\,dy - \int_0^1 f(y)\left(\frac{1}{y}-\pi a \operatorname{cotg} \pi a y\right)dy$$

$$= f(0)\log 2\pi a + \tfrac{1}{2}\pi i f(0) - \pi i a\int_0^1 f(y)dy + \int_0^1 \frac{f(y)-f(0)}{y}\,dy \;-$$

$$- \int_0^1 f(y)\left(\frac{1}{y}-\pi a \operatorname{cotg} \pi a y\right)dy.$$

As for n integer and positive

$$\frac{1}{(n-1)!}\int_0^\infty \frac{z^{n-1}dz}{e^{z+2\pi i a}-1} = \sum_{\varkappa=1}^{\infty}\frac{\cos 2\varkappa\pi a}{\varkappa^n} - i\sum_{\varkappa=1}^{\infty}\frac{\sin 2\varkappa\pi a}{\varkappa^n}$$

many relations containing the series $\sum_1^\infty \frac{\cos 2\varkappa\pi a}{\varkappa^n}$ and $\sum_1^\infty \frac{\sin 2\varkappa\pi a}{\varkappa^n}$ are to be deduced from this formula. But we will restrict ourselves to the ζ-function and therefore write $f(y) = (1-y)^{2n}$, in which n represents an integer positive number; if the real part of a complex number γ is indicated by $R(\gamma)$ it ensues from what preceded that

$$\int_0^\infty \frac{R\left(1+\frac{zi}{2\pi a}\right)^{2n}-1}{e^z-1}\,dz + \frac{(-1)^{n+1}}{(2\pi a)^{2n}}\int_0^\infty z^{2n} R\left(\frac{1}{e^{z+2\pi i a}-1}\right)dz$$

$$= \log 2\pi a - \sum_1^{2n}\frac{1}{\varkappa} - I(2n,a) \quad . \; . \; . \; . \; . \; . \; . \quad (3)$$

In order to find relations for the ζ-function by means of this formula, the following auxiliary proposition may be used:

If a represents an integer number > 2 and ϱ describes half the reduced rest system, modulo a, between 0 and a, and that in such a way that the series of the numbers, of which the values are successively assumed by ϱ, does not contain two numbers, the sum of which is equal to a, then:

$$\sum_{\rho} R\left(\frac{1}{e^{z+\frac{2\pi i\rho}{a}}-1}\right)=\tfrac{1}{2}\sum_{d|a}\frac{\mu\left(\frac{a}{d}\right)d}{e^{dz}-1}.$$

The proof of this auxiliary proposition is simple; for if. in the second member of the relation

$$\sum_{d|a}\frac{\mu\left(\frac{a}{d}\right)d}{e^{dz}-1}=\sum_{d|a}\mu\left(\frac{a}{d}\right)\sum_{d_1=1}^{d}\frac{1}{e^{z+\frac{2\pi i d_1}{d}}-1}$$

$a=dd_2$ and $d_1d_2=d_3$ and consequently $\frac{d_1}{d}=\frac{d_1d_2}{dd_2}=\frac{d_3}{a}$ is written, so that d_2 is a divisor of the greatest common divisor (a, d_3) of a and d_3, the relation takes this shape,

$$\sum_{d|a}\frac{\mu\left(\frac{a}{d}\right)d}{e^{dz}-1}=\sum_{d_3=1}^{a}\frac{1}{e^{z+\frac{2\pi i d_3}{a}}-1}\sum_{d_2|(a,d_3)}\mu(d_2)=\sum_{\lambda}\frac{1}{e^{z+\frac{2\pi i\lambda}{a}}-1},$$

in which λ describes the reduced rest system, modulo a, between 0 and a, and so

$$\tfrac{1}{2}\sum_{d|a}\frac{\mu\left(\frac{a}{d}\right)d}{e^{dz}-1}=\tfrac{1}{2}\sum_{\rho}\left(\frac{1}{e^{z+\frac{2\pi i\rho}{a}}-1}+\frac{1}{e^{z+\frac{2\pi i(a-\rho)}{a}}-1}\right)=\sum_{\rho}R\left(\frac{1}{e^{z+\frac{2\pi i\rho}{a}}-1}\right),$$

with which the auxiliary proposition has been proved.

From this proposition follows for $n>0$

$$\sum_{\rho}\int_0^{\infty}z^{2n}R\left(\frac{1}{e^{z+\frac{2\pi i\rho}{a}}-1}\right)dz=\tfrac{1}{2}\sum_{d_1|a}\mu\left(\frac{a}{d_1}\right)d_1\int_0^{\infty}\frac{z^{2n}dz}{e^{d_1z}-1}=\frac{(2n)!\zeta(2n+1)\sum_{d|a}\mu(d)d^{2n}}{2\,.\,a^{2n}}$$

Replace in (3) α by $\frac{\mu}{a}$ multiply then both members by ϱ^{2n} and add further all relations, which are acquired in this way by making ϱ assume the values mentioned before; the result is then

$$\left.\begin{array}{l}\sum_{\rho}\int_0^{\infty}\frac{R\left(\varrho+\frac{azi}{2\pi}\right)^{2n}-\varrho^{2n}}{e^{z}-1}dz+\frac{(-1)^{n+1}(2n)!}{2\,.\,(2\pi)^{2n}}\zeta(2n+1)\sum_{d|a}\mu(d)\,d^{2n}\\ \qquad=\sum_{\rho}\varrho^{2n}\left\{\log\frac{2\pi\varrho}{a}-\sum_{1}^{2n}\frac{1}{\varkappa}-I\left(2n,\frac{\varrho}{a}\right)\right\}\end{array}\right\}\quad(4)$$

and consequently for $n=1$

$$(2a^3 - \sum_{d|a} \mu(d)\, d^2)\, \zeta(3) = 4\pi^2 \sum_{\rho} \rho^2 \left(\frac{3}{2} - \log\frac{2\pi\rho}{a} + I\left(2, \frac{\rho}{a}\right)\right).$$

By supposing a to be respectively equal to 3, 4 and 6, we find particularly:

$$\zeta(3) = \frac{2\pi^2}{13}\left(\frac{3}{2} - \log\frac{2\pi}{3} + I\left(2,\frac{1}{3}\right)\right) = \frac{8\pi^2}{13}\left(\frac{3}{2} - \log\frac{4\pi}{3} + I\left(2,\frac{2}{3}\right)\right)$$

$$= \frac{4\pi^2}{47}\left(\frac{3}{2} - \log\frac{\pi}{2} + I\left(2,\frac{1}{4}\right)\right) = \frac{36\pi^2}{47}\left(\frac{3}{2} - \log\frac{3\pi}{4} + I\left(2,\frac{3}{4}\right)\right)$$

$$= \frac{\pi^2}{12}\left(\frac{3}{2} - \log\frac{\pi}{3} + I\left(2,\frac{1}{6}\right)\right) = \frac{25\pi^2}{12}\left(\frac{3}{2} - \log\frac{5\pi}{6} + I\left(2,\frac{5}{6}\right)\right).$$

It is evident that the relations found also produce many formulae for the calculation of $\zeta(5)$, $\zeta(7)$, ... etc.; this we will work out further for the case that the terms of the series acquired, diminish quickest; this happens by writing in (4) $a=6$ and $\mu=1$, by means of which the relation

$$\int_0^\infty \frac{R\left(1+\frac{6zi}{2\pi}\right)^{2n} - 1}{e^z - 1}\, dz + \frac{(-1)^{n+1}(2n)!}{2.(2\pi)^{2n}} (6^{2n} - 3^{2n} - 2^{2n} + 1)\, \zeta(2n+1)$$

$$= \log\frac{\pi}{3} - \sum_1^{2n} \frac{1}{\varkappa} - I(2n, \tfrac{1}{6})$$

is found, i.e.

$$\frac{6^{2n}+3^{2n}+2^{2n}-1}{2.(2\pi)^{2n}} \zeta(2n+1) = \sum^{n-1} (-1)^{n-1-\varkappa} \left(\frac{3}{\pi}\right)^{2\varkappa} \frac{\zeta(2\varkappa+1)}{(2n-2\varkappa)!} +$$

$$+ \frac{(-1)^{n+1}}{(2n)!}\left(\sum_1^{2n} \frac{1}{\varkappa} - \log\frac{\pi}{3} + I(2n, \tfrac{1}{6})\right),$$

consequently

$$\zeta(3) = \frac{\pi^2}{12}\left(\tfrac{1}{2} - \log\frac{\pi}{3} + I(2, \tfrac{1}{6})\right),$$

$$29\, \zeta(5) = 3\pi^2 \zeta(3) - \frac{\pi^4}{36}\left(\tfrac{25}{12} - \log\frac{\pi}{3} + I(4, \tfrac{1}{6})\right),$$

$$659\, \zeta(7) = 72\, \pi^2 \zeta(5) - \frac{2\pi^4}{3} \zeta(3) + \frac{\pi^6}{5.3^4}\left(\tfrac{49}{20} - \log\frac{\pi}{3} + I(6, \tfrac{1}{6})\right), \ldots \text{ etc.}$$

The quantity $I(2n, \frac{1}{6})$ occurring in this formula may be determined from its definition

$$I\left(2n, \frac{1}{6}\right) = \int_0^1 (1-y)^{2n}\left(\frac{1}{y} - \frac{\pi}{6}\, cotg\, \frac{\pi y}{6}\right) dy = (2n)! \sum_{\varkappa=1}^{\infty} \frac{B_\varkappa \left(\frac{\pi}{6}\right)^{2\varkappa}}{(2\varkappa)\,.\,(2\varkappa + 2n)!}$$

but also by means of other series of which the terms diminish more quickly, for, if q represents an arbitrary integer positive number,

$$\frac{1}{y}-\frac{\pi}{6}\cot g\frac{\pi y}{6}=\frac{2}{y}\sum_{\rho=1}^{\infty}\left(\frac{y}{6}\right)^{2\rho}\zeta(2\rho)=$$

$$=\frac{2}{y}\sum_{\rho=1}^{\infty}\left(\frac{y}{6}\right)^{2\rho}\left\{\zeta(2\rho)-\sum_{\varkappa=1}^{q}\frac{1}{\varkappa^{2\rho}}\right\}+\frac{2}{y}\sum_{\rho=1}^{\infty}\sum_{\varkappa=1}^{q}\left(\frac{y}{6\varkappa}\right)^{2\rho}$$

$$=\frac{2}{y}\sum_{\rho=1}^{\infty}\left(\frac{y}{6}\right)^{2\rho}\left\{\zeta(2\rho)-\sum_{\varkappa=1}^{q}\frac{1}{\varkappa^{2\rho}}\right\}+\sum_{\varkappa=1}^{q}\frac{1}{6\varkappa-y}-\sum_{\varkappa=1}^{q}\frac{1}{6\varkappa+y}$$

and therefore

$$I(2n.\tfrac{1}{6})=2\int_0^1(1-y)^{2n}\sum_{\rho=1}^{\infty}\frac{y^{2\rho-1}}{6^{2\rho}}\left\{\zeta(2\rho)-\sum_{\varkappa=1}^{q}\frac{1}{\varkappa^{2\rho}}\right\}dy+$$

$$+\sum_{\varkappa=1}^{q}\int_0^1\frac{(1-y)^{2n}}{6\varkappa-y}dy-\sum_{\varkappa=1}^{q}\int_0^1\frac{(1-y)^{2n}}{6\varkappa+y}dy$$

Now is

$$\int_0^1\frac{(1-y)^{2n}}{6\varkappa-y}dy=\int_0^1\frac{(6\varkappa-1)^{2n}}{6\varkappa-y}dy-\int_0^1\frac{(6\varkappa-1)^{2n}-(1-y)^{2n}}{(6\varkappa-1)+(1-y)}dy=$$

$$=-(6\varkappa-)^{2n}\log\left(1-\frac{1}{6\varkappa}\right)+\sum_{\rho=1}^{2n}(-1)^{\rho}.\frac{1}{\rho}(6\varkappa-1)^{2n-\rho};$$

and

$$\int_0^1\frac{(1-y)^{2n}}{6\varkappa+y}dy=\int_0^1\frac{(6\varkappa+1)^{2n}}{6\varkappa+y}dy-\int_0^1\frac{(6\varkappa+1)^{2n}-(1-y)^{2n}}{(6\varkappa+1)-(1-y)}dy=$$

$$=(6\varkappa+1)^{2n}\log\left(1+\frac{1}{6\varkappa}\right)-\sum_{\rho=1}^{2n}\frac{1}{\rho}.(6\varkappa+1)^{2n-\rho};$$

so the calculation of $I(2n,\frac{1}{6})$ is to be reduced to the calculation of the integral

$$2\int_0^1(1-y)^{2n}\sum_{\rho=1}^{\infty}\frac{y^{2\rho-1}}{6^{2\rho}}\left\{\zeta(2\rho)-\sum_{\varkappa=1}^{q}\frac{1}{\varkappa^{2\rho}}\right\}dy=2.(2n)!\sum_{\rho=1}^{\infty}\frac{\zeta(2\rho)-\sum_{\varkappa=1}^{q}\frac{1}{\varkappa^{2\rho}}}{(2\rho)(2\rho+1)\ldots(2\rho+2n).6^{2\rho}}$$

and the ratio of two consecutive terms of this series is smaller than $\frac{1}{6^2(q+1)^2}$, so that, if there is a breaking off in an arbitrary place, the rest-term is smaller than the term last used divided by $6^2(q+1)^2-1$.