Citation:

W. de Sitter, On the relativity of rotation in Einstein''s theory, in:
KNAW, Proceedings, 19 I, 1917, Amsterdam, 1917, pp. 527-532

of curves at the one and $n+1$ bundles of curves at the other side. The (M)-bundle contains four curves at least.

In following communications we shall apply those rules in order to deduce the types of the P,T-diagram, which may occur in the different systems of n components.

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(*To be continued*).

Mechanics. — *"On the relativity of rotation in* EINSTEIN'S *theory."* By Prof. W. DE SITTER.

(Communicated in the meeting of September 30, 1916).

Observations have taught us that the relative accelerations of material bodies at the surface of the earth differ from those which would be caused by NEWTON's law of gravitation only. The difference is explained by NEWTON's law of inertia, combined with the rotation of the earth relatively to an "absolute space". NEWTON [1]) quite deliberately introduces this absolute space, and also absolute time, as an element of his explanation of observed phenomena. Many objections have been raised against it, all of which are based on the logical claim that a true causal explanation shall involve only observable quantities. It has been tried to replace the absolute space by the fixed stars, by the "Body Alpha", etc. All these substitutes are, however, entirely hypothetical and quite as objectionable, or more so, as the absolute space itself.

EINSTEIN [2]) also rejects the absolute space, but he apparently still clings to the "ferne Massen". It appears to me that EINSTEIN has made a mistake here. The "Allgemeine Relativitätstheorie" is in fact entirely relative, and has no room for anything whatever that would be independent of the system of reference. The need for the introduction of the distant masses arises from the wish to have the gravitational field zero at infinity in *any* system of reference. This wish, however well founded in a theory based on absolute space, is contrary to the spirit of the principle of relativity. The best way to show this clearly is to consider the fundamental tensor g_{ij}. We will, to simplify the argument, neglect the mass of the earth. This does not affect the fundamental question, if only we imagine the

[1]) Principia, Definitiones, Scholium.

[2]) *Die Grundlagen der allgemeinen Relativitätstheorie*, Annalen der Physik, Band 49, p. 772 (separate edition p. 9).

experiments to be made with gyroscopes, instead of with FOUCAULT's pendulum.

Take a system of coordinates $x_1 = r$, $x_2 = \vartheta$, $x_3 = z$, $x_4 = ct$, the axis of z being the axis of rotation of the earth, r and ϑ being polar coordinates in a plane perpendicular to it, and t the time. Now the argument which leads to the introduction of the distant masses is the following: If the earth had no rotation, the values of g_{ij} would be

$$\left.\begin{matrix} -1 & 0 & 0 & 0 \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \end{matrix}\right\} \quad \ldots\ldots\ldots \quad (1)$$

If we transform to rotating axes, by putting $\vartheta' = \vartheta - \omega t$, we find for g'_{ij} in the new system

$$\left.\begin{matrix} -1 & 0 & 0 & 0 \\ 0 & -r^2 & 0 & -r^2\omega \\ 0 & 0 & -1 & 0 \\ 0 & -r^2\omega & 0 & +1-r^2\omega^2 \end{matrix}\right\} \quad \ldots\ldots\ldots \quad (2)$$

It is found that the set (1) does not explain the observed phenomena at the surface of the actual earth correctly, and (2) does, if we take the appropriate value for ω. This value of ω we call the velocity of rotation of the earth. Then relatively to the axes (2) the earth has no rotation, and we should expect the values (1) of g_{ij}. The g'_{24} and the second term of g'_{44} in (2) therefore do not belong to the field of the earth itself, and must be produced by distant masses. [1])

This reasoning is however faulty.

We will here only consider g_{24}. The differential equation determining this quantity, if we neglect the mass of the earth (or if we consider only the field outside the earth), is:

$$\frac{d^2 g_{24}}{dr^2} - \frac{2}{r^2}\, g_{24} = 0,$$

of which the general solution is

$$g_{24} = k_1 r^2 + \frac{k_2}{r},$$

where k_1 and k_2 are constants of integration. The equation as given here is not exact, as it supposes g_{24} to be small, and if k_1 is different

1) Evidently with NEWTON's theory no masses will do this. The hypothesis therefore implies a change of NEWTON's law of gravitation. Perhaps it will be possible with EINSTEIN's theory to imagine masses producing the desired effect. The fixed stars will not do it.

from zero, this is not true for large values of r. It is found, however, that also the rigorous equation [1] is satisfied by

$$g_{24} = k r^2 \, , \qquad (3)$$

k being an arbitrary constant. It will be seen that both (1) and (2) are special cases of the general formula (3). The flaw in the argument used above was that (1) was considered to be *the* solution, instead of (3). EINSTEIN's theory in fact requires that g_{24} shall be of the form (3), but it does not prescribe the constant of integration. NEWTON's theory did, and even therein lay its absolute character. EINSTEIN's theory however is relative: in it the *differential* equation is the fundamental one, and the choice of the constants of integration remains free.

The constants of integration must, in a given system of reference, be so determined that the observed relative motions are correctly represented. In a true theory of relativity not only does the transformed general solution satisfy the invariant differential equation, but the particular solution which agrees with observed phenomena in one system must by the transformation give the particular solution which does so in the new system. Consequently the constants of integration must also be transformed and will generally be different in different systems.

Suppose that we have originally taken a system of coordinates relatively to which the earth has a rotation ω_1. This ω_1 is, of course, entirely arbitrary and, as our coordinate axes cannot be observed, it must in a true theory of relativity disappear from the final formulae. Now we have $g_{24} = kr^2$, and to determine k, we transform to axes relatively to which the earth has no rotation by $\vartheta' = \vartheta - \omega_1 t$.

Then in the new system

$$g'_{24} = (k - \omega_1) \, r^2 .$$

Observation shows that the correct value of g'_{24} in this system is $-\omega \, r^2$, therefore

$$k = \omega_1 - \omega \, . \qquad (4)$$

If we use this value of k, the final formulas for the motion of bodies relatively to the earth contain only the observable quantity ω. The relativity of the theory is thus seen to be based on the free choice of the constant of integration k. No value of k is a priori

[1]) The second term, which is small also if r is large, has a more complicated form in the rigorous solution. It does not interest us here, as it is of the order of the mass of the earth. By the "rigorous equation" is meant the complete equation for g_{24}, not reduced to its linear terms, but in which the other g_{ij} are supposed to be known functions of the coordinates.

more probable than any other. In NEWTON's theory on the other hand the value $k = 0$ is prescribed, and this has led to the belief that zero is the "true" value of k, and that any other value it may have in some system of reference must be "explained" by hypothetical masses.

If we inquire how the conviction that $k = 0$ is the "true" value has arisen, we find the following. Observations made during long centuries befores those referred to above had shown that all celestial bodies have, superposed upon their mutual relative motions, a rotation $-\omega$ relatively to the earth. Already long before NEWTON it had become usual to regard this apparent revolution of the stars round the earth as unreal and as caused by a real rotation $+\omega$ of the earth round its axis. One of the reasons why it was considered unreal is the difficulty encountered in conceiving real linear velocities of many million times the velocity of light. Another is that the hypothesis that it were real would, *in an absolute space,* assign a quite exceptional place to the earth in the universe. Both reasons of course have no force in a theory of relativity.

Relatively to the system of coordinates which has just been considered, an average star has a rotation $\omega_1 - \omega = \omega_2$. If we transform to axes relatively to which this star has no rotation, we find

$$g''_{24} = (k - \omega_2)\, r^2 .$$

Observations of stars [1]) require very approximately the value (1), i. e. $g''_{24} = 0$. It is assumed that zero is the exact value, and consequently

$$k = \omega_2 = \omega_1 - \omega \,, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

i. e. we find the same value as above in (4). This is certainly a remarkable fact, and a confirmation of EINSTEIN's theory [2]). Now if we accept an absolute space, and if we also assume that the system of the stars has no real rotation, i.e. no rotation relatively to the absolute space, then the "true" value of ω_2 is zero,

[1]) We need not consider the fixed stars at all. The moon will give exactly the same result through KEPLER's third law, if we do not neglect the mass of the earth.

[2]) Or of NEWTON's theory of inertia, which up to the approximation here used (and also in the next) is equivalent to EINSTEIN's theory. It may be pointed out that in NEWTON's theory inertia and gravitation are two entirely different things, while in EINSTEIN's theory they are one, and may be called by either name. EINSTEIN's theory is generally called a theory of gravitation, and this of course is correct. But it might also be called a theory of inertia, denying the separate existence of gravitation, and this would also be correct. To the approximation here used, what NEWTON called gravitation is not involved at all, it only comes in in the next approximation through the mass of the earth.

and (5) gives $k=0$. But the conviction that this is the true value, and has any preference over any other values, is based on the belief in an absolute space, and must be abandoned if the latter is abandoned. We must do one of two things. Either we must believe in an absolute [1]) space, to which we may impart some substantiality by calling it "Ether". Then $k=0$ is the true value, and that this is so is a property of space or of the ether. Or we can believe that there is no absolute space. Then we must regard the differential equations as the fundamental ones, and be prepared to have different constants of integration in different systems of reference.

The difference between the two points of view is shown very clearly by the values of g_{24} for $r=\infty$. In the absolute space we have $g_{24}=0$ at infinity. In EINSTEIN's theory the value of g_{24} at infinity is different in different systems of coordinates. However, no observation has ever taught us anything about infinity and no observation ever will. The condition that the gravitational field shall be zero at infinity forms part of the conception of an absolute space, and in a theory of relativity it has no foundation. [2])

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[1]) It should be noted that, owing to the indeterminateness of EINSTEIN's field-equations, this "absolute" space is *not* completely determined by the condition that the fixed stars shall have no rotation in it, or that at an infinite distance from any material body the gravitational field shall be zero. There are an infinity of systems satisfying these conditions. We can limit the choice e g. by putting $g=-1$, as EINSTEIN generally does, but even this does not fix the system of reference, and it is also entirely arbitrary.

[2]) We could imagine that there was a system of degenerated values towards which the g_{ij} could converge in infinity, and which were invariant for all transformations, or at least for a group of transformations of so wide extent that the restriction of the allowable transformations to this group would not be equivalent to giving up the principle of relativity. Prof. EINSTEIN has actually found such a set of values. They are

$$\left.\begin{matrix} 0 & 0 & 0 & \infty \\ 0 & 0 & 0 & \infty \\ 0 & 0 & 0 & \infty \\ \infty & \infty & \infty & \infty^2, \end{matrix}\right\} \quad \ldots\ldots\ldots, \ldots \quad (\alpha)$$

and we must limit ourselves to such transformations in which at infinity x'_4 is a function of x_4 alone. Consequently the hypothesis that the g_{ij} actually have these values at infinity, and that at finite, though very large, distances from all known masses there are other unknown masses which cause them to have these values, is not contrary to the formal principle of relativity. But also denying the hypothesis is not contrary to this principle. The hypothesis has arisen from the wish to explain not only a small portion of the g_{ij} (i.e. of inertia) by the influence of material bodies, but to ascribe *the whole of the* g_{ij} [or rather the whole of the difference of the actual g_{ij} from the standard values (α)] to this influence. Theoretically it is certainly important that thus the *possibility* has been shown

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Rotation is thus relative in EINSTEIN's theory. Does this mean that it is physically equivalent to translation, which is also relative (and was relative in classical mechanics)? Evidently not. The fundamental difference between a uniform translation and a rotation is that the former is an *orthogonal* transformation (LORENTZ-transformation) of the four coordinates, or world-parameters, and the latter is not. Now orthogonal transformations are the only ones that leave the line-element invariant *in the coordinates*, i.e. that do not affect the g_{ij}, and are therefore without influence on the gravitational field. Consequently we can by a LORENTZ-transformation "transform away" linear velocity. We can always find a system of reference relatively to which a given body has no rotation, as we can find a system in which the acceleration produced by a given body at a given point is zero, but we cannot transform away rotation, no more than mass. This is a fact, independent of all theories. Of course the fact is differently represented in different theories. NEWTON "explains" it by his law of inertia and the absolute space. For EINSTEIN, who makes no difference between inertia and gravitation, and knows no absolute space, the accelerations which the classical mechanics ascribed to centrifugal forces are of exactly the same nature and require no more and no less explanation, than those which in classical mechanics are due to gravitational attraction.

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of an entirely material origin of inertia. But practically it makes no difference whether we explain a thing by an uncontrollable hypothesis invented for the purpose, or not explain it at all. The hypothesis implies the finiteness of the physical world, it assigns to it a priori a limit, however large, beyond which there is *nothing* but the field of the g_{ij} which at infinity degenerate into the values (α). This field, in which also the fourdimensional time space is separated into a threedimensional space and a onedimensional time, undoubtedly has some of the characteristics of the old absolute space and absolute time. The hypothesis can thus be said to make space and time absolute at infinity, although arbitrary transformations of three-dimensional space are still allowed. If we wish to have complete four-dimensional relativity for the actual world, this world must of necessity be finite [Note added (29 Sept.) after a conversation with Prof. EINSTEIN].