Mathematics. — *"Logarithmic Frequency Distribution"*. By Dr. M. J. VAN UVEN. (Communicated by Prof. J. C. KAPTEYN).

(Communicated in the meeting of September 30, 1916).

When the frequency-distribution of some measured quantity x does not follow the normal law of GAUSS:

$$W_{\xi_1}^{\xi_2} = \frac{h}{\sqrt{\pi}} \int_{\xi_1}^{\xi_2} e^{-h^2\xi^2}\, d\xi,$$

$\xi = x - X$ being the deviation from the arithmetical mean X and $W_{\xi_1}^{\xi_2}$ the probability that this deviation is found between the limits ξ_1 and ξ_2 (the measured quantity between $x_1 = X + \xi_1$ and $x_2 = X + \xi_2$), this need not be a reason to drop the law of GAUSS, as Prof. J. C. KAPTEYN has shown.[1]) On the contrary the "skewness" of the frequency-curve may in many cases be explained by merely supposing that, instead of x, another quantity $Z = F(x)$ connected with x is distributed according to this law, so that it is only due to the wrong choice of the quantity measured, that the normal distribution has not come out. Then it is interesting to deduce the normal function $Z = F(x)$ from the given skew frequency-distribution.

Let this normal function Z have the value M for its arithmetical mean, so that the deviations $\zeta = Z - M$ are spread round the mean value zero according to the normal law, and thus satisfy the equation

$$W_{\zeta_1}^{\zeta_2} = \frac{h}{\sqrt{\pi}} \int_{\zeta_1}^{\zeta_2} e^{-h^2\zeta^2}\, d\zeta.$$

Among the quantities $\zeta = F(x) - M$, which apparently are also functions of the observed quantity x, there is one, viz. $z = h\zeta = h\{F(x) - M\} = f(x)$, which answers to the formula

$$W_{z_1}^{z_2} = \frac{1}{\sqrt{\pi}} \int_{z_1}^{z_2} e^{-z^2}\, dz\,.$$

This z has $h = 1$ for its modulus of precision and consequently $\varepsilon_z = \frac{1}{\sqrt{2}}$ for its (quadratic) mean value.

1) J. C. KAPTEYN: Skew Frequency Curves in Biology and Statistics; Groningen, 1903, Noordhoff.

As has been shown by Prof. J. C. KAPTEYN and the author of this paper [1]), the normal function $z = f(x)$ may be determined from the given frequency-distribution, at any rate graphically.

A normally distributed quantity may be considered as the result of growing from an initial value x_l, common to all the individuals, with increments individually different but distributed round the mean increment according to the normal law of error, and independent of the instantaneous value of x.

When spread in a skew distribution, the quantity is built up of elementary increments which contain a factor $\psi(x)$ dependent on the x undergoing the increment. Thus the cause of growing being supposed to be spread purely accidentally, the reaction upon it is proportional to the function $\psi(x)$, which is called the "reaction-function" and is determinate but for a constant factor.

According to the theory of Prof. J. C. KAPTEYN the following relation holds between the reaction-function $\eta = \psi(x)$ and the normal function $z = f(x)$:

$$\eta = \psi(x) = \frac{dx}{dz} = \frac{1}{f'(x)}.$$

Thus far [1]) some normal functions have been examined analytically, viz. that which answers to the normal distribution $z = \lambda(x - x_m)$ with $\eta = \frac{1}{\lambda}$, and those which correspond to the so-called "logarithmic distribution": $z = \lambda \log \frac{x - x_0}{x_m - x_0}$ with $\eta = \frac{x - x_0}{\lambda}$ and $z = \lambda \log \frac{x_n - x_m}{x_n - x}$ with $\eta = \frac{x_n - x}{\lambda}$ $(\lambda > 0, x_0 < x < x_n)$. In the normal distribution the reaction-function η is a constant, in the logarithmic distribution η is a *linear* function of x.

In the present paper we shall treat the also logarithmic case that the reaction-function is a *quadratic* function of x. Then the normal function is of the form:

$$z = \lambda\, {}^{10}\log\left(\frac{x - x_0}{x_n - x} : \frac{x_m - x_0}{x_n - x_m}\right).$$

The general method furnishes the values z_k corresponding to the $n-1$ class-limits x_k. The curve which can be drawn through the points (x_k, z_k) is the graph of the normal function $z = f(x)$.

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[1]) J. C. KAPTEYN and M. J. VAN UVEN: Skew Frequency Curves in Biology and Statistics, 2nd Paper; Groningen, 1916, Hoitsema Br.

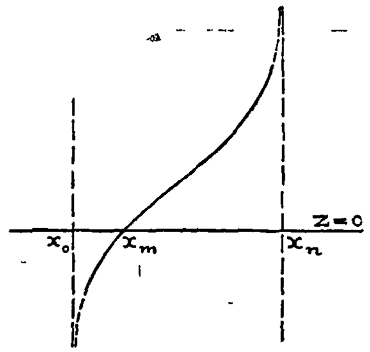


Fig. 1.

When the points (x_k, z_k) lie in a curve of the form shown (fig. 1), which appears to tend asymptotically to the ordinate-lines $x = x_0$ and $x = x_n$ of the extreme limits x_0 and x_n of x, a trial with

$$z = \lambda\ {}^{10}log\, p \frac{x - x_0}{x_n - x} \quad . \quad . \quad (1)$$

suggests itself.

Introducing the median x_m (corresponding to $z = 0$) we find

$$p = \frac{x_n - x_m}{x_m - x_0}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and therefore

$$z = \lambda\ {}^{10}log \left(\frac{x - x_0}{x_n - x} : \frac{x_m - x_0}{x_n - x_m} \right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Now we have to determine the constants x_0, x_n, x_m and λ from the given curve. The curve on ordinary squared paper still furnishes the value x_m.

From

$$\frac{dz}{dx} = \lambda M \left[\frac{1}{x - x_0} + \frac{1}{x_n - x} \right] = \frac{\lambda M (x_n - x_0)}{(x - x_0)(x_n - x)} = \frac{\lambda M (x_n - x_0)}{-x^2 + (x_0 + x_n)x - x_0 x_n} \ . \quad (4)$$

$(M = mod = {}^{10}log\, e = 0.434295)$

and

$$\frac{d^2z}{dx^2} = \frac{\lambda M (x_n - x_0)[2x - (x_0 + x_n)]}{(x - x_0)^2 (x_n - x)^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

we find for the coordinates (ξ, ζ) of the point of inflexion $\left(\frac{d^2z}{dx^2} = 0 \right)$

$$\xi = \frac{x_0 + x_n}{2} \quad , \quad \zeta = \lambda\ {}^{10}log\, p = \lambda\ {}^{10}log \frac{x_n - x_m}{x_m - x_0} \quad . \quad . \quad . \quad (6)$$

and for the slope ζ' at this point

$$\zeta' = \left(\frac{dz}{dx} \right)_\xi = \frac{\lambda M (x_n - x_0)}{(\xi - x_0)(x_n - \xi)} = \frac{4 \lambda M}{x_n - x_0} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

In general the position of the point of inflexion itself, situated at equal distances from both the asymptotes, cannot be fixed exactly. On the other hand the position of the inflexion tangent is pretty well determinate; its equation runs

$$z - \zeta = \zeta' (x - \xi) ;$$

The point of intersection with the axis of x has for abscissa:

$$\bar{x} = \xi - \frac{\zeta}{\zeta'} = \frac{x_o + x_n}{2} - \frac{x_n - x_o}{4M} \, {}^{10}log \frac{x_n - x_m}{x_m - x_o}. \quad . \quad . \quad . \quad (8)$$

Putting

$$x_n - x_o = a,$$

$$\xi - x_m = \frac{x_n + x_o}{2} - x_m = \delta,$$

we obtain

$$x_m - \bar{x} = \frac{2x_m - x_o - x_n}{2} + \frac{a}{4M} \, {}^{10}log \frac{x_n - x_m}{x_m - x_o}$$

or

$$x_m - \bar{x} = -\delta + \frac{a}{4M} \, {}^{10}log \frac{\frac{a}{2} + \delta}{\frac{a}{2} - \delta} = -\delta + \frac{a}{4M} \, {}^{10}log \frac{1 + \frac{2\delta}{a}}{1 - \frac{2\delta}{a}} =$$

$$= -\delta + \frac{a}{4} \, {}^{e}log \frac{1 + \frac{2\delta}{a}}{1 - \frac{2\delta}{a}}.$$

Since $2\delta < a$ the logarithm may be expanded in ascending powers of $\frac{2\delta}{a}$:

$$log \frac{1 + \frac{2\delta}{a}}{1 - \frac{2\delta}{a}} = 2\left[\frac{2\delta}{a} + \frac{1}{3}\left(\frac{2\delta}{a}\right)^3 + \frac{1}{5}\left(\frac{2\delta}{a}\right)^5 + ..\right] = \frac{4\delta}{a} + \frac{16\delta^3}{3a^3} + \frac{64\delta^5}{5a^5} + ..$$

Hence

$$x_m - \bar{x} = -\delta + \frac{a}{4}\left[\frac{4\delta}{a} + \frac{16\delta^3}{3a^3} + \frac{64\delta^5}{5a^5} + \,.\right] = \frac{4\delta^3}{3a^2} + \frac{16\delta^5}{5a^4} + .. \quad (9)$$

or approximately

$$x_m - \bar{x} = \frac{4\delta^3}{3a^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9a)$$

If the point of inflexion can be determined, at least by a rough estimate, a provisional value of $\delta = \xi - x_m$ is found. By computing a from the formula (9a) we also obtain provisional values for $x_0 = \xi - \frac{a}{2}$ and $x_n = \xi + \frac{a}{2}$.

Finally λ is found from (7) viz.

$$\lambda = \frac{a\zeta'}{4M}.$$

If the limits x_0 and x_n are known beforehand, it is easy to determine λ and x_m graphically.

Indeed, putting

$$u = {}^{10}log\, \frac{x - x_0}{x_n - x},$$

and operating with z and the numerus $v = \frac{x - x_0}{x_n - x} = 10^u$ on logarithmic paper, we obtain the graph of the equation:

$$z = \lambda (u - u_m),$$

which is a straight line with the (positive) slope λ; this line cuts the axis of u at the point $u = u_m$ corresponding to the median x_m. At the margin of the logarithmic paper we read at u_m the numerus $v = v_m = 10^{u_m} = \frac{x_m - x_0}{x_n - x_m}$, from which x_m can be calculated $\left(x_m = \frac{x_0 + v_m x_n}{1 + v_m}\right)$, when not yet determined by the figure on ordinary squared paper.

In practice we are obliged to estimate the values of x_0 and x_n and to use, at least provisionally, erroneous values x_0' and x_n' of the limits. So we operate with

$$u' = {}^{10}log\, \frac{x - x_0'}{x_n' - x}$$

instead of u, and thus obtain a set of pairs (u', z) lying in a curve slightly deviating from the true straight line $z = \lambda (u - u_m)$.

Let the errors in the presumed values x_0' and x_n' be σ and τ, so that

$$x_0' - x_0 = \sigma ,$$
$$x_n' - x_n = \tau .$$

then, putting

$$\frac{x - x_0}{x_n - x} = v \quad , \quad \frac{x - x_0'}{x_n' - x} = v'$$

and, accordingly

$$u = {}^{10}log\, v \quad , \quad u' = {}^{10}log\, v' ,$$

we derive

$$x = \frac{x_n' v' + x_0'}{v' + 1}$$

and

$$v = \frac{(x_n' v' + x_0') - (v' + 1) x_0}{(v' + 1) x_n - (x_n' v' + x_0')} = \frac{(x_n' - x_0) v' + (x_0' - x_0)}{-(x_n' - x_n) v' + (x_n - x_0')} = \frac{(a + \tau) v' + \sigma}{-\tau v' + (a - \sigma)} , \quad (10)$$

or, putting

$$\frac{\sigma}{a + \tau} = \beta \quad , \quad \frac{-\tau}{a - \sigma} = \gamma ,$$

$$v = \frac{a+\tau}{a-\sigma} \times \frac{v'+\beta}{\gamma v'+1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Now

$$z = \lambda u + const = \lambda\,{}^{10}log\, v + const = \lambda M\,{}^{e}log\, v + const\,,$$

$$\frac{dz}{du'} = \frac{dz}{dv'}\cdot\frac{dv'}{du'} = \lambda M\left(\frac{1}{v'+\beta} - \frac{\gamma}{\gamma v'+1}\right)\cdot\frac{v'}{M} = \frac{\lambda(1-\beta\gamma)v'}{(v'+\beta)(\gamma v'+1)}\,. \quad (12)$$

and

$$\frac{d^2z}{du'^2} = \frac{d\left(\frac{dz}{du'}\right)}{dv'}\;\frac{dv'}{du'} = \frac{\lambda(1-\beta\gamma)v'}{M}\cdot\frac{-\gamma v'^2+\beta}{(v'+\beta)^2(\gamma v'+1)^2}\,. \quad . \quad . \quad (13)$$

The factor

$$1-\beta\gamma = 1 + \frac{\sigma\tau}{(a+\tau)(a-\sigma)} = \frac{a(a+\tau-\sigma)}{(a+\tau)(a-\sigma)}$$

is positive, provided that σ and τ are sufficiently small.

Also the quantity $v = 10^{u}$ is always positive, as long as u' is real. The quantity v is, $\frac{a+\tau}{a-\sigma}$ being supposed positive, also positive, without restriction when β and γ are both positive, provided that $v' > -\beta$ when β is negative, and provided that $v' < -\frac{1}{\gamma}$ when γ is negative.

The domain of reality for u, and hence also for z, is limited by the values $-\beta$ and $-\frac{1}{\gamma}$ for v', or by the values $u' = log\,(-\beta)$ and $u' = log\,(-\frac{1}{\gamma})$ for u'. These limits really exist if $\beta < 0$ and $\gamma < 0$ resp., or $\sigma < 0$ and $\tau > 0$ resp.

When β is a small negative quantity, the value $u' = log(-\beta) = -B$ is negative and rather large.

When γ is a small negative quantity, $-\frac{1}{\gamma}$ is a large positive quantity and the value $u' = log\,(-\frac{1}{\gamma}) = +\,C$ is positive and rather large. As for $u' = -B$ we have $v = 0$, or $u = -\infty$, hence $z = -\infty$, the ordinate line $u' = -B$ is a vertical asymptote lying to the left at a rather great distance from the centre of the domain; and, as for $u' = +\,C$ we have $v = \infty$, or $u = +\infty$, hence $z = +\infty$, also the ordinate line $u' + C$ is a vertical asymptote lying to the right at a rather great distance from the centre of the domain.

So, when $\beta < 0$ or $\sigma < 0$, the real domain is limited by a vertical asymptote $u' = - B$ on the left, and when $\lambda < 0$, or $\tau < 0$, it is limited by a vertical asymptote $u' = + C$ on the right.

As the real domain never extends beyond $u' = - B$ and $u' = + C$, the slope $\frac{dz}{du'}$ may never become negative, and the quantities u and u' are simultaneously maximum and minimum.

When σ has a small positive value, $u' = - \infty$ or $v' = 0$ answers to $v = \frac{\sigma}{a - \sigma}$ or $u = log \frac{\sigma}{a - \sigma} = - S$, which is a rather large negative quantity. So for $\sigma > 0$ or $\beta < 0$ there exists an inferior limit $z = \lambda(- S - u_m)$ for z, corresponding to $u' = - \infty$, consequently a horizontal asymptote below.

When τ has a small negative value, $u' = + \infty$ or $v' = + \infty$ corresponds to $v = \frac{a + \tau}{-\tau}$, or $u = log\left(\frac{a + \tau}{-\tau}\right) = + T$, which is a rather large positive quantity. So for $\tau < 0$ or $\gamma > 0$ there exists a superior limit $z = \lambda(+ T - u_m)$ for z, corresponding to $u' = + \infty$, consequently a horizontal asymptote above.

From (10) we find

$$v - v' = \frac{(a+\tau)v' + \sigma}{-\tau v' + (a - \sigma)} - v' = \frac{\tau v'^2 + (\sigma + \tau)v' + \sigma}{-\tau v' + (a - \sigma)} = \frac{(v'+1)(\tau v' + \sigma)}{(a - \sigma)(\gamma v' + 1)}, \quad (14)$$

and so conclude that in the real domain $v - v'$ has the sign of $\tau v' + \sigma$.

When τ and σ are both positive, we have always $v' < v$ or $u' < u$, so that the erroneous curve (u', z) has for equal z a smaller u than the true straight line. The curve is as it were generated by shifting (and deforming) the true line to the left.

When σ and τ are both negative, we have everywhere $v' < v$ or $u' < u$, so that the erroneous curve (u', z) is as it were generated by shifting to the right.

When σ and τ have opposite signs, there is a real point, $v' = -\frac{\sigma}{\tau}$, for which $v' = v$ and consequently $u' = u$. Then the erroneous curve cuts the true straight line in a point $u' = u = log\left(-\frac{\sigma}{\tau}\right) = A$.

When $\sigma > 0$ and $\tau < 0$, we have $u' < u$, or $v - v' > 0$, for $v' < -\frac{\sigma}{\tau}$, or $u' < A$. Then at the left of $u' = A$ the provisional curve (u', z) is on the left side of the true straight line and to the right of $u' = A$ it is on the right side of this line.

When $\sigma < 0$ and $\tau > 0$, we have $u' < u$, or $v - v' > 0$ or

$\tau v' + \sigma > 0$ for $v' > -\frac{\sigma}{\tau}$ or, $u' > A$. The disposition is then the inverse of that of the last case.

When $\sigma = 0$, or $\beta = 0$, we have $B = \infty, S = \infty, A = -\infty$. So, the left and lower asymptotes being at infinity, also the point of intersection A is at infinite distance to the left.

When $\tau = 0$ or $\gamma = 0$, we have $C = \infty$, $T = \infty$ and $A + \infty$. Now the right and upper asymptotes are at infinity, the point A being at infinite distance to the right.

We now consider the curvature and the point of inflexion. For this latter $\frac{d^2z}{du'^2} = 0$ holds, or $-\gamma v'^2 + \beta = 0$, or $v' = +\sqrt{\frac{\beta}{\gamma}} = \sqrt{-\frac{\sigma(a-\sigma)}{\tau(a+\tau)}}$.

The point of inflexion is real when σ and τ, the errors in x_0' and x_n', have opposite signs.

When $\sigma > 0$, $\tau < 0$, or $\beta > 0$, $\gamma > 0$, we have $\frac{d^2z}{du'^2} > 0$ for $v' < \sqrt{\frac{\beta}{\gamma}}$, so that left of the point of inflexion the curve is convex downward.

For the slope λ' of the inflexion tangent we find

$$\left(\frac{dz}{du'}\right)_{v'=\sqrt{\frac{\beta}{\gamma}}} = \lambda' = \frac{\lambda(1-\beta\gamma)\sqrt{\frac{\beta}{\gamma}}}{\left(\sqrt{\frac{\beta}{\gamma}}+\beta\right)\left(\gamma\sqrt{\frac{\beta}{\gamma}}+1\right)} = \lambda\frac{1-\sqrt{\beta\gamma}}{1+\sqrt{\beta\gamma}} < \lambda .$$

When $\sigma < 0$, $\tau > 0$, or $\beta < 0$, $\gamma < 0$, we have $\frac{d^2z}{du'^2} > 0$ for $v' > \sqrt{\frac{\beta}{\gamma}}$, so that the curve is convex downward to the right of the point of inflexion.

To find the slope λ' of the inflexion tangent we put

$$\beta = -\beta_1 \quad , \quad \gamma = -\gamma_1 ;$$

and so obtain:

$$\left(\frac{dz}{du'}\right)_{v'=\sqrt{\frac{\beta}{\gamma}}} = \lambda' = \frac{\lambda(1-\beta_1\gamma_1)\sqrt{\frac{\beta_1}{\gamma_1}}}{\left(\sqrt{\frac{\beta_1}{\gamma_1}}-\beta_1\right)\left(-\gamma_1\sqrt{\frac{\beta_1}{\gamma_1}}+1\right)} = \lambda\frac{1+\sqrt{\beta_1\gamma_1}}{1-\sqrt{\beta_1\gamma_1}} = \lambda\frac{1+\sqrt{\beta\gamma}}{1-\sqrt{\beta\gamma}} > \lambda.$$

When $\sigma = 0$, or $\beta = 0$, the point of inflexion lies to the left at infinity and its inflexion asymptote is parallel with the true straight line.

When $\tau = 0$ or $\gamma = 0$, the point of inflexion lies to the right at infinity and its inflexion asymptote is parallel with the true straight line.

After this preparatory study we may distinguish the following eight cases.

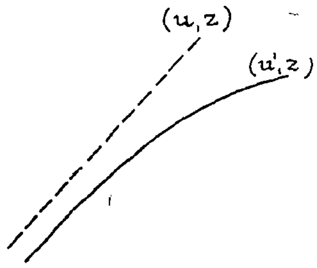


Fig. 2a.

1. x_0' right, x_n' too low;

i.e. $\sigma = 0$, $\tau < 0$, or $\beta = 0$, $\gamma > 0$;

$$\frac{d^2z}{du'^2} < 0 \quad , \quad Lim\left(\frac{dz}{du'}\right)_{u=-\infty} = \lambda \ ,$$

$$Lim\left(\frac{dz}{du'}\right)_{u'=+\infty} = 0.$$

$u' > u$ from $u' = -\infty$ to $u' = +\infty$.

(fig. 2a).

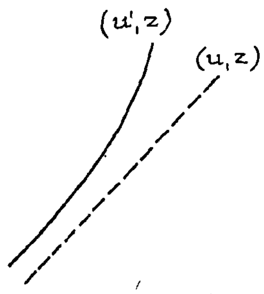


Fig. 2b

2. x_0' right, x_n' too high;

i.e. $\sigma = 0$, $\tau > 0$, or $\beta = 0$, $\gamma < 0$;

$$\frac{d^2z}{du'^2} > 0 \quad , \quad Lim\left(\frac{dz}{du'}\right)_{u'=-\infty} - \lambda \ ,$$

$$\left(\frac{dz}{du'}\right)_{u'=+C} = \infty.$$

$u' < u$ from $u' = -\infty$ to $u' = +C$

(fig. 2b).

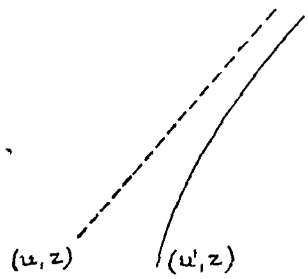


Fig. 2c

3. x_0' too low, x_n' right;

i.e. $\sigma < 0$, $\tau = 0$, or $\beta < 0$, $\gamma = 0$;

$$\frac{d^2z}{du'^2} < 0 \quad , \quad \left(\frac{dz}{du'}\right)_{u'=-B} = \infty \ ,$$

$$Lim\left(\frac{dz}{du'}\right)_{u'=+\infty} = \lambda.$$

$u' > u$ from $u' = -B$ to $u' = +\infty$.

(fig. 2c).

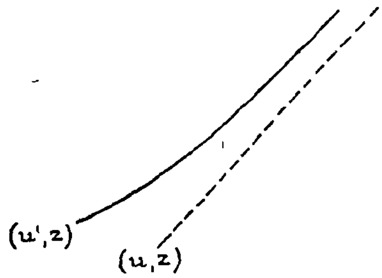


Fig. 2d.

4. x_0' too high, x_n' right;

i.e. $\sigma > 0$, $\tau = 0$, or $\beta > 0$, $\gamma = 0$;

$$\frac{d^2z}{du'^2} > 0 \quad , \quad Lim\left(\frac{dz}{du'}\right)_{u'=-\infty} = 0 \ ,$$

$$Lim\left(\frac{dz}{du'}\right)_{u'=+\infty} = \lambda.$$

$u' < u$ from $u' = -\infty$ to $u' = +\infty$.

(fig. 2d).

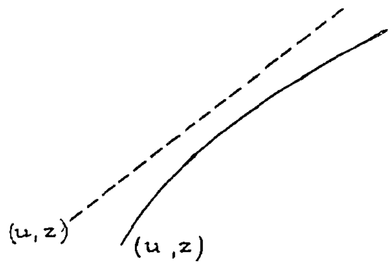


Fig. 2e

5. x_0' too low, x_n' too low;

i.e. $\sigma < 0$, $\tau < 0$, or $\beta < 0$, $\gamma > 0$;

$$\frac{d^2z}{du'^2} < 0, \quad \left(\frac{dz}{du'}\right)_{u'=-B} = -\infty, \quad -$$

$$Lim \left(\frac{dz}{du'}\right)_{u'=+\infty} = 0.$$

$u' > u$ from $u' = -B$ to $u' = +\infty$.

(fig. 2e)

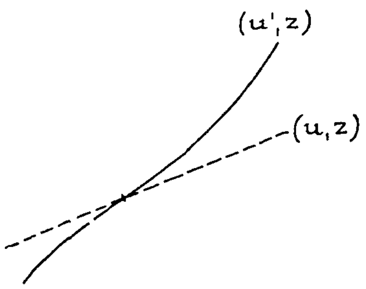


Fig. 2f.

6. x_0' too low, x_n' too high;

i.e. $\sigma < 0$, $\tau > 0$, or $\beta < 0$, $\gamma < 0$;

$\frac{d^2z}{du'^2} < 0$ on the left, $\frac{d^2z}{du'^2} > 0$ on the right of the point of inflexion;

$$\left(\frac{dz}{du'}\right)_{u'=-B} = \infty, \quad \left(\frac{dz}{du'}\right)_{u'=+C} = \infty.$$

$u' > u$ for $u' < A$, $u' < u$ for $u' > A$.

(fig. 2f)

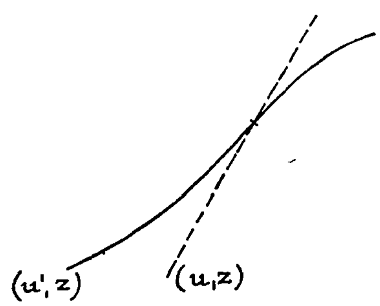


Fig. 2g.

7. x_0' too high, x_n' too low;

i.e. $\sigma > 0$, $\tau < 0$, of $\beta > 0$, $\gamma > 0$;

$\frac{d^2z}{du'^2} > 0$ on the left, $\frac{d^2z}{du'^2} < 0$ on the right of the point of inflexion;

$$Lim \left(\frac{dz}{du'}\right)_{u'=-\infty} = 0, \quad Lim \left(\frac{dz}{du'}\right)_{u'=+\infty} = 0.$$

$u' < u$ for $u' < A$, $u' > u$ for $u' > A$.

(fig. 2g)

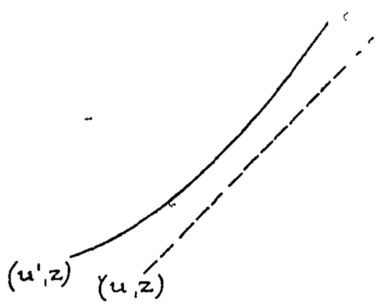


Fig 2h

8. x_0' too high, x_n' too high;

i.e. $\sigma > 0$, $\tau > 0$, of $\beta > 0$, $\gamma < 0$;

$$\frac{d^2z}{du'^2} > 0, \quad Lim \left(\frac{dz}{du'}\right)_{u'=-\infty} = 0,$$

$$\left(\frac{dz}{du'}\right)_{u'=+C} = \infty.$$

$u' < u$ from $u' = -\infty$ to $u' = +C$.

(fig. 2h)

If in the cases 6 and 7 we have $\tau = -\sigma$, or $\gamma = \beta$, we find for the point of intersection A $v' = 1$ or $u' = 0$, hence $v = 1$ or $u = 0$. This point therefore coincides with the point of inflexion of the erroneous curve. This point of inflexion corresponding to $v = 1$ or $\frac{x - x_0}{x_n - x} = 1$ or $x = \frac{x_0 + x_n}{2}$, it is conjugate to the point of inflexion of the original curve traced on ordinary squared paper.

6a.
$\tau = -\sigma > 0$ or $\beta = \gamma < 0$
(fig. 3*a*)

7a.
$\tau = -\sigma < 0$, or $\beta = \gamma > 0$
(fig. 3*b*)

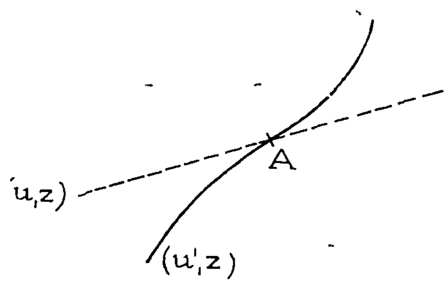


FIG 3a

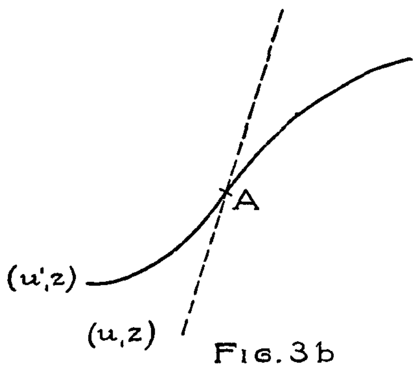


FIG. 3b

From (4) we find for the reaction-function

$$\eta = \frac{1}{f'(x)} = \frac{dx}{dz} = \frac{-x^2 + (x_0 + x_n)\,x - x_0\,x_n}{\lambda M\,(x_n - x_0)} = -Px^2 + Qx + R.$$

the reaction consists of a (positive or negative) constant, of a (probably positive) term proportional to x and of a negative term proportional to x^2.

Example·

Length of oat-stalks (data procured by Dr. E. GILTAY, Wageningen). Unit of x: 1 cm.; class-range = 1 unit = 1 cm.

The following table contains the observed values of x with the corresponding numbers Y of individuals ($\Sigma Y = N = 1008$); moreover the values of z appertaining to the class-limits are given.

In the figure on ordinary squared paper the frequency-curve is represented by x—x—x, the normal function by ·—·, the reaction-curve by - - - -.

The figure on logarithmic paper contains the line $z = \lambda\,(u - u_m) = 3{,}36\,(u - \log 1{,}318) = 3{,}36\,u - 0{,}403$.

Since $x_m - \bar{x}$ (see p. 536) has an uncommonly small value, the formula (9*a*) is not suitable for the computation of a. Therefore we started by estimating $x_0 = 0$ and $x_n = 2\,\xi = 2 \times 47{,}5 = 95$, which

x	Y	z	x	Y	z	x	Y	z
25	2		45	28		65	22	
		— 2.035			— 0.502			+ 0.771
26	3		46	24		66	20	
		— 1.820			— 0.449			+ 0.837
27	3		47	41		67	19	
		— 1.707			— 0.363			+ 0.910
28	1		48	25		68	18	
		— 1.675			— 0.314			+ 0.987
29	4		49	20		69	13	
		— 1.576			— 0.276			+ 1.052
30	5		50	30		70	22	
		— 1.485			— 0.220			+ 1.187
31	5		51	32		71	10	
		— 1.414			— 0.165			+ 1.266
32	6		52	30		72	7	
		— 1.343			— 0.108			+ 1.332
33	10		53	35		73	3	
		— 1.249			— 0.046			+ 1.364
34	6		54	32		74	7	
		— 1.202			+ 0.011			+ 1.456
35	11		55	47		75	8	
		— 1.126			+ 0.094			+ 1.598
36	7		56	37		76	3	
		— 1.085			+ 0.160			+ 1.670
37	11		57	42		77	3	
		— 1.026			+ 0.237			+ 1.777
38	16		58	24		78	2	
		— 0.951			+ 0.282			+ 1.875
39	17		59	32		79	2	
		— 0.882			+ 0.344			+ 2.035
40	22		60	41		80	0	
		— 0.803			+ 0.427			+ 2.035
41	20		61	34		81	0	
		— 0.740			+ 0.502			+ 2.035
42	23		62	33		82	0	
		— 0.673			+ 0.579			+ 2.035
43	11		63	29		83	1	
		— 0.643			+ 0.654			+ 2.185
44	30		64	18		84	0	
		— 0.567			+ 0.704			+ 2.185
45	28		65	22		85	1	

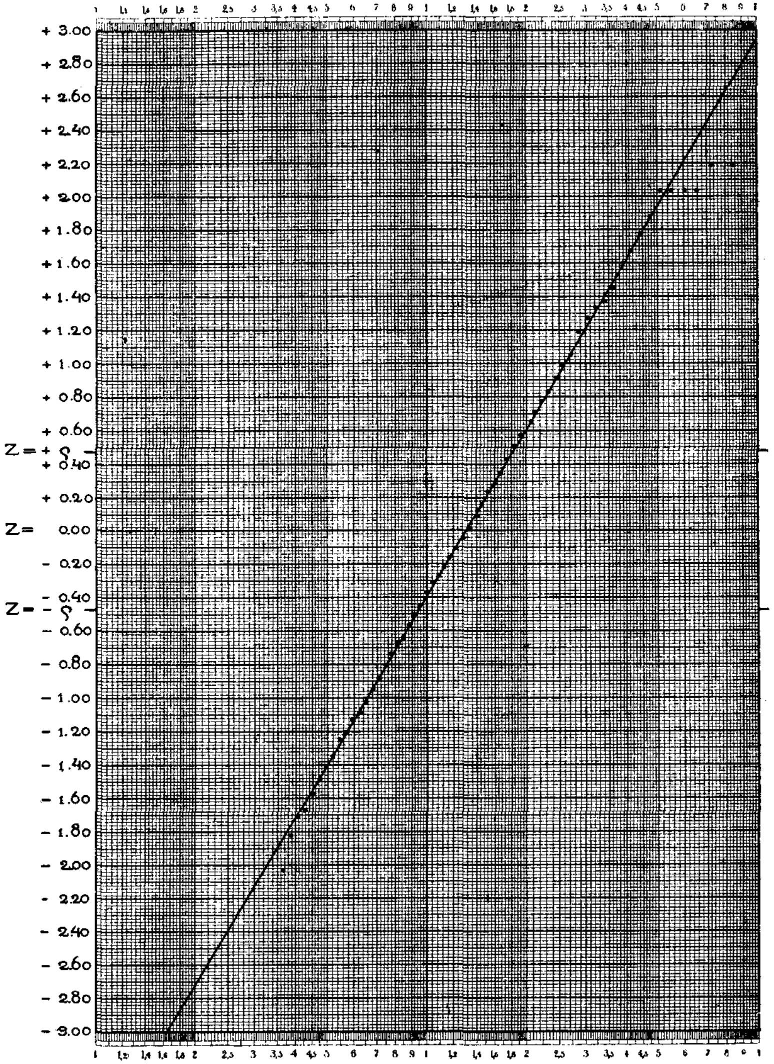
values appeared to be pretty satisfactory. Besides we found $x_m = 54$ and $\lambda = 3{,}36$. So the normal function runs:

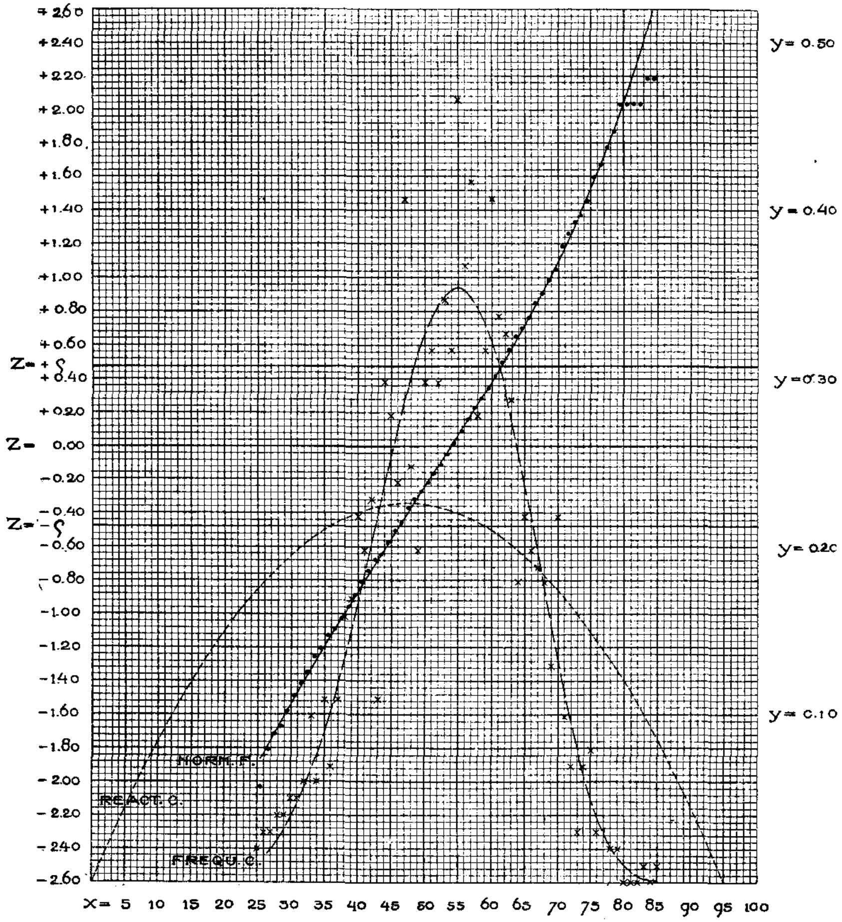
$$z = 3{,}36\ {}^{10}log\left(\frac{x-0}{95-x} : \frac{54-0}{95-54}\right) = 3{,}36\ {}^{10}log\ \frac{x}{95-x} - 0{,}403.$$

The reaction-function is

$$\eta = 95\,x - x^2.$$

The elementary increment consists of a positive part proportional to the length already reached and of a negative part, corresponding to a shrinking, proportional to the 2^{nd} power of the length. Supposing the growing stalk to retain the same shape, this stunting part is


Z = + ϱ
Z =
Z = − ϱ


NORM. F.
REACT. C.
FREQU. C.
y = 0.50
y = 0.40
y = 0.30
y = 0.20
y = 0.10

proportional to the surface. Thus we may say that the factors dependent on the 1st power of the length are preponderatingly favourable to growth, whereas those factors, which are connected with the surface, are chiefly disadvantageous.

Physics. — *"Note on* P. SCHERRER's *calculation of the entropy-constant."* [1]) By J. M. BURGERS. Supplement N°. 41*b* to the Communications from the Physical Laboratory at Leiden. (Communicated by Prof. H. KAMERLINGH ONNES).

(Communicated in the meeting of Sept. 30, 1916).

The object of this note, which is suggested by a remark made by Dr. W. H. KEESOM, is to point out that:

(1). If a model of a monatomic gas constructed according to the theory of quanta on cooling at constant volume ceases to conform to the classical theory at temperatures which are too low (i.e. lower than is indicated by experiment), it will also give values for the entropy-constant which are too high, unless the entropy is not taken as zero at the absolute zero of temperatures for ideal gases

(2). The model suggested by SCHERRER remains ideal to temperatures far below the allowable limit.

§ 1. If for $T=0$ the entropy S is taken as 0, the absolute value of the entropy of one grammolecule of a gas for a given temperature T and a given volume V may be found by the following integration from 0 to T, the volume being kept constant at V,

$$S=\int_0^T dT\,\frac{C_v}{T}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This integral may be divided into two parts as follows

$$S=\int_0^{T_0} dT\,\frac{C_v}{T}+\int_{T_0}^T dT\,\frac{C_v}{T}. \quad . \quad . \quad . \quad . \quad . \quad (1^*)$$

and the temperature T_0 may be chosen such, that above T_0 the deviations from the ideal gaseous state are to be neglected. (In general T_0 will depend on the value of the volume; to begin with we will take as a special case $V=1$ cc.. For $V \gtrless 1$ cc. see below). The first part of equation (1*) is certainly positive; we shall call

[1]) P. SCHERRER, Gött. Nachr. 1916.