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to BOLTZMANN's theorem is connected with the probability that, in a system consisting of a solid and a gaseous phase, a greater or a smaller part belongs to the latter. The circumstance that, in considering this latter probability, we must attend to the difference in potential energy of the two phases cannot but increase our doubt, for neither in the determination of S'' nor in the determination of S in the above mentioned way we have had to speak of this difference. If, as we should expect, the difference $S—S''$ depended to a considerable extent on the relative values of the potential energy, we might still put the entropy $S''=0$ for $T=0$, but it would no longer be possible to determine the constant a which occurs in formula (1) for the gaseous state by considering only the phenomena in the gas, as is done in the theories discussed here. We ought rather to derive it from an examination of the equilibrium between the two phases.

I think we may conclude from what precedes that, though the value found for ω, if it be not quite accidental, pleads in favour of the application of the theory of quanta to the problem of vaporisation, yet the way in which this application has been made requires further explanation and justification.

Physics. — *"On* HAMILTON's *principle in* EINSTEIN's *theory of gravitation".* By H. A. LORENTZ.

(Communicated in the meeting of January 30, 1915).

The discussion of some parts of EINSTEIN's theory of gravitation [1]) may perhaps gain in simplicity and clearness, if we base it on a principle similar to that of HAMILTON, so much so indeed that HAMILTON's name may properly be connected with it. Now that we are in possession of EINSTEIN's theory we can easily find how this variation principle must be formulated for systems of different nature and also for the gravitation field itself.

Motion of a material point.

§ 1. Let a material point move under the influence of a force with the components K_1, K_2, K_3. Let us vary every position x, y, z

[1]) EINSTEIN u. GROSSMANN, Entwurf einer verallgemeinerten Relativitätstheorie und einer Theorie der Gravitation. Zeitschr. f. Math. u. Phys. **62**, (1914), p. 225.
EINSTEIN, Die formale Grundlage der allgemeinen Relativitätstheorie, Sitz. Ber. Akad. Berlin, 1914, p. 1030.

occurring in the real motion in the way defined by the infinitely small quantities $\delta x, \delta y, \delta z$. If, in the varied motion, the position $x + \delta x, y + \delta y, z + \delta z$ is reached at the same time t as the position x,y,z in the real motion, we shall have the equation

$$\delta \int L dt + \int (K_1 \delta x + K_2 \delta y + K_3 \delta z) dt = 0, \quad . \quad . \quad . \quad (1)$$

L being the Lagrangian function and the integrals being taken over an arbitrary interval of time, at the beginning and the end of which the variations of the coordinates are zero. K is supposed to be a force acting on the material point beside the forces that are included in the Lagrangian function.

§ 2. We may also suppose the time t to be varied, so that in the varied motion the position $x + \delta x, y + \delta y, z + \delta z$ is reached at the time $t + \delta t$. In the first term of (1) this does not make any difference, if we suppose that for the extreme positions also $\delta t = 0$. As to the second term we remark that the coordinates in the varied motion at the time t may now be taken to be $x + \delta x - v_1 \delta t, y + \delta y - v_2 \delta t, z + \delta z - v_3 \delta t$, if v_1, v_2, v_3 are the velocities in the real motion. In the second term we must therefore replace $\delta x, \delta y, \delta z$ by $\delta x - v_1 \delta t, \delta y - v_2 \delta t, \delta z - v_3 \delta t$. In the equation thus found we shall write x_1, x_2, x_3, x_4 for x,y,z,t. For the sake of uniformity we shall add to the three velocity components a fourth, which, however, necessarily must have the value 1 as we take for it $\frac{dx_4}{dx_4}$. We shall also add to the three components of the force K a fourth component, which we define as

$$K_4 = -(v_1 K_1 + v_2 K_2 + v_3 K_3), \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and which therefore represents the work of the force per unit of time with the negative sign.

Then we have instead of (1)

$$\delta \int L dt + \int \Sigma\,(a)\, K_a\, \delta x_a \,.\, dt = 0, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and for (2) we may write [1])

[1]) In these formulae we have put between parentheses behind the sign of summation the index with respect to which the summation must be effected, which means that the values 1, 2, 3, 4 have to be given to it successively. In the same way two or more indices behind the sign of summation will indicate that in the expression under this sign these values have to be given to each of the indices. $\Sigma(ab)$ f. i. means that each of the four values of a has to be combined with each of the four values of b.

$$\Sigma(a)\, v_a K_a = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

§ 3. In EINSTEIN's theory the gravitation field is determined by certain characteristic quantities g_{ab}, functions of x_1, x_2, x_3, x_4, among which there are 10 different ones, as

$$g_{ba} = g_{ab} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

A point of fundamental importance is the connection between these quantities and the corresponding coefficients g'_{ab}, with which we are concerned, when by an arbitrary substitution $x_1, \ldots x_4$ are changed for other coordinates $x'_1, \ldots x'_4$. This connection is defined by the condition that

$$ds^2 = g_{11}\, dx_1^2 + \ldots + g_{44} dx_4^2 + 2 g_{12} dx_1\, dx_2 + \ldots$$

or shorter

$$ds^2 = \Sigma(ab)\, g_{ab}\, dx_a\, dx_b$$

be an invariant.

Putting

$$dx_a = \Sigma(b)\, p_{ab}\, dx'_b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

we find

$$g'_{ab} = \Sigma(cd) p_{ca}\, p_{db}\, g_{cd} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Instead of (6) we shall also write

$$dx'_a = \Sigma(b)\, \pi_{ba}\, dx_b,$$

so that the set of quantities π_{ba} may be called the inverse of the set p_{ab}. Similarly, we introduce a set of quantities γ_{ba}, the inverse of the set g_{ab}. [1])

We remark here that in virtue of (5) and (7) $g'_{ba} = g'_{ab}$ and that likewise $\gamma_{ba} = \gamma_{ab}$.

Our formulae will also contain the determinant of the quantities g_{ab}, which we shall denote by g, and the determinant p of the coefficients p_{ab} (absolute value: $|p|$). The determinant g is always negative.

We may now, as has been shown by EINSTEIN, deduce the motion of a material point in a gravitation field from the principle expressed by (3) if we take for the Lagrangian function

$$L = -m \frac{ds}{dt} = -m \sqrt{\Sigma(ab)\, g_{ab}\, v_a\, v_b} \quad . \quad . \quad . \quad . \quad (8)$$

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[1]) Suppose

$$x_a = \Sigma(b) \nu_{ba} \xi_b$$

to follow from the equations

$$\xi_a = \Sigma(b) n_{ab} x_b\,;$$

then the set ν_{ab} is the inverse of the set n_{ab}.

Motion of a system of incoherent material points.

§ 4. Let us now, following EINSTEIN, consider a very large number of material points wholly free from each other, which are moving in a gravitation field in such a way that at a definite moment the velocity components of these points are continuous functions of the coordinates. By taking the number very large we may pass to the limiting case of a continuously distributed matter without internal forces.

Evidently the laws of motion for a system of this kind follow immediately from those for a single material point. If ϱ is the density and $dx\, dy\, dz$ an element of volume, we may write instead of (8)

$$-\varrho \sqrt{\Sigma(ab) g_{ab} v_a v_b} \,.\, dx\, dy\, dz \,. \quad \ldots \ldots \quad (9)$$

If now we wish to extend equation (3) to the whole system we must multiply (9) by dt and integrate with respect to x, y, z and t.

In the last term of (3) we shall do so likewise after having replaced the components K_a by $K_a\, dx\, dy\, dz$, so that in what follows K will represent the external force per unit of volume.

If further we replace $dx\, dy\, dz\, dt$ by dS, an element of the four-dimensional extension $x_1, \ldots x_4$, and put

$$\varrho v_a = w_a, \quad \ldots \ldots \ldots \quad (10)$$

$$\mathrm{L} = -\sqrt{\Sigma(ab)\, g_{ab}\, w_a\, w_b} \quad \ldots \ldots \quad (11)$$

we find the following form of the fundamental theorem.

Let a variation of the motion of the system of material points be defined by the infinitely small quantities δx_a, which are arbitrary continuous functions of the coordinates within an arbitrarily chosen finite space S, at the limits of which they vanish. Then we have, if the integrals are taken over the space S, and the quantities g_{ab} are left *unchanged*,

$$\delta \int \mathrm{L}\, dS + \int \Sigma(a)\, K_a \delta x_a \,.\, dS = 0 \quad \ldots \quad (12)$$

For the first term we may write

$$\int \delta \mathrm{L} \,.\, dS,$$

if $\delta\mathrm{L}$ denotes the change of L at a fixed point of the space S.

The quantity $\mathrm{L}dS$ and therefore also the integral $\int \mathrm{L}dS$ is invariant when we pass to another system of coordinates.[1])

[1]) This follows from the invariancy of ds^2, combined with the relations

$$\frac{\varrho'}{dx'_4} = |p|\, \frac{\varrho}{dx_4}, \quad dS' = \frac{1}{|p|}\, dS.$$

§ 5. The equations of motion may be derived from (12) in the following way. When the variations δx_a have been chosen, the varied motion of the matter is perfectly defined, so that the changes of the density and of the velocity components are also known. For the variations at a fixed point of the space S we find

$$\delta w_a = \Sigma(b)\frac{\partial \chi_{ab}}{\partial x_b} \quad \ldots \ldots \ldots \quad (13)$$

where

$$\chi_{ab} = w_b \delta x_a - w_a \delta x_b \quad \ldots \ldots \ldots \quad (14)$$

(Therefore: $\chi_{ba} = -\chi_{ab}$, $\chi_{aa} = 0$).

If for shortness we put

$$P = \sqrt{\Sigma(ab) g_{ab} w_a w_b}, \quad \ldots \ldots \ldots \quad (15)$$

so that $L = -P$, and

$$\Sigma(b) g_{ab} w_b = u_a\ , \quad \ldots \ldots \ldots \quad (16)$$

we have

$$\delta L = -\Sigma(a)\frac{u_a}{P}\delta w_a = -\Sigma(ab)\frac{u_a}{P}\frac{\partial \chi_{ab}}{\partial x_b} =$$

$$= -\Sigma(ab)\frac{\partial}{\partial x_b}\left(\frac{u_a}{P}\chi_{ab}\right) + \Sigma(ab)\,\chi_{ab}\frac{\partial}{\partial x_b}\left(\frac{u_a}{P}\right),$$

so that, with regard to (14),

$$\left.\begin{aligned} \delta L + \Sigma(a) K_a \delta x_a = &-\Sigma(ab)\frac{\partial}{\partial x_b}\left(\frac{u_a}{P}\chi_{ab}\right) + \\ &+ \Sigma(ab)(w_b\delta x_a - w_a\delta x_b)\frac{\partial}{\partial x_b}\left(\frac{u_a}{P}\right) + \Sigma(a) K_a \delta x_a \end{aligned}\right\} \quad \ldots \quad (17)$$

If after multiplication by dS this expression is integrated over the space S the first term on the right hand side vanishes, χ_{ab} being 0 at the limits. In the last two terms only the variations δx_a occur, but not their differential coefficients, so that according to our fundamental theorem, when these terms are taken together, the coefficient of each δx_a must vanish. This gives the equations of motion [1])

$$K_a = \Sigma(b) w_b\left[\frac{\partial}{\partial x_a}\left(\frac{u_b}{P}\right) - \frac{\partial}{\partial x_b}\left(\frac{u_a}{P}\right)\right], \quad \ldots \ldots \quad (18)$$

which evidently agree with (4), or what comes to the same, with

$$\Sigma(a) w_a K_a = 0 \quad \ldots \ldots \ldots \quad (19)$$

In virtue of (18) the general equation (17), which holds for

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[1]) In the term

$$-\Sigma(ab) w_a \delta x_b \frac{\partial}{\partial x_b}\left(\frac{u_a}{P}\right)$$

the indices a and b must first be interchanged.

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arbitrary variations that need not vanish at the limits of S, becomes

$$\delta \mathrm{L} + \Sigma(a) K_a \delta x_a = - \Sigma(ab) \frac{\partial}{\partial x_b}\left(\frac{u_a}{P}\chi_{ab}\right) \quad . \quad . \quad . \quad (20)$$

§ 6. We can derive from this the equations for the momenta and the energy.

Let us suppose that only one of the four variations δx_a differs from 0 and let this one, say δx_c, have a *constant* value. Then (14) shows that for each value of a that is not equal to c

$$\chi_{ac} = - w_a \delta x_c, \quad \chi_{ca} = w_a \delta x_c, \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

while all χ's without an index c vanish.

Putting first $b = c$ and then $a = c$, and replacing at the same time in the latter case b by a, we find for the right hand side of (20)

$$\Sigma(a) \frac{\partial}{\partial x_c}\left(\frac{u_a w_a}{P}\right) \delta x_c - \Sigma(a) \frac{\partial}{\partial x_a}\left(\frac{u_c w_a}{P}\right) \delta x_c\,.$$ [1]

But, according to (15) and (16),

$$\Sigma(a) \frac{u_a w_a}{P} = P = - \mathrm{L},$$

so that (20) becomes

$$\delta \mathrm{L} + K_c \delta x_c = - \frac{\partial \mathrm{L}}{\partial x_c} \delta x_c - \Sigma(a) \frac{\partial}{\partial x_a}\left(\frac{u_c w_a}{P}\right) \delta x_c. \quad . \quad . \quad (22)$$

It remains to find the value of $\delta \mathrm{L}$.

The material system together with its state of motion has been shifted in the direction of the coordinate x_c over a distance δx_c. If the gravitation field had participated in this shift, $\delta \mathrm{L}$ would have been equal to $-\frac{\partial \mathrm{L}}{\partial x_c}\delta x_c$. As, however, the gravitation field has been left unchanged, $\frac{\partial \mathrm{L}}{\partial x_c}$ in this last expression must be diminished by $\left(\frac{\partial \mathrm{L}}{\partial x_c}\right)_w$, the index w meaning that we must keep constant the quantities w_a and consider only the variability of the coefficients g_{ab}. Hence

$$\delta \mathrm{L} = \left\{ - \frac{\partial \mathrm{L}}{\partial x_c} + \left(\frac{\partial \mathrm{L}}{\partial x_c}\right)_w \right\} \delta x_c\,.$$

Substituting this in (22) and putting

[1] The circumstance that (21) does not hold for $a = c$ might lead us to exclude this value from the two sums. We need not, however, introduce this restriction, as the two terms that are now written down too much, annul each other.

$$\frac{u_c w_a}{P} = T_{ac}, \quad \ldots \ldots \ldots \quad (23)$$

we find

$$K_c + \left(\frac{\partial L}{\partial x_c}\right)_w = -\Sigma(a)\frac{\partial T_{ac}}{\partial x_a} \quad \ldots \ldots \quad (24)$$

The first three of these equations ($c = 1, 2, 3$) refer to the momenta; the fourth ($c = 4$) is the equation of energy. As we know already the meaning of $K_1, \ldots K_4$ we can easily see that of the other quantities. The stresses $X_x, X_y, X_z, Y_x \ldots$ are $T_{11}, T_{21}, T_{31}, T_{12} \ldots$; the components of the momentum per unit of volume $-T_{41}, -T_{42}, -T_{43}$; the components of the flow of energy T_{14}, T_{24}, T_{34}. Further T_{44} is the energy per unit of volume. The quantities

$$\left(\frac{\partial L}{\partial x_1}\right)_w, \quad \left(\frac{\partial L}{\partial x_2}\right)_w, \quad \left(\frac{\partial L}{\partial x_3}\right)_w$$

are the momenta which the gravitation field imparts to the material system per unit of time and unit of volume, while the energy which the system draws from that field is given by $-\left(\frac{\partial L}{\partial x_4}\right)_w$.

An electromagnetic system in the gravitation field.

§ 7. We shall now consider charges moving under the influence of external forces in a gravitation field; we shall determine the motion of these charges and the electromagnetic field belonging to them. The density ϱ of the charge will be supposed to be a continuous function of the coordinates; the force per unit of volume will be denoted by K and the velocity of the point of a charge by v. Further we shall again introduce the notation (10).

In EINSTEIN's theory the electromagnetic field is determined by two sets, each of four equations, corresponding to well known equations in the theory of electrons. We shall consider one of these sets as the mathematical description of the system to which we have to apply HAMILTON's principle; the second set will be found by means of this application.

The first set, the fundamental equations, may be written in the form

$$\Sigma(b)\frac{\partial \psi_{ab}}{\partial x_b} = w_a, \quad \ldots \ldots \ldots \quad (25)$$

the quantities ψ_{ab} [1]) on the left hand side being subject to the conditions

$$\psi_{aa} = 0, \quad \psi_{ba} = -\psi_{ab}, \quad \ldots \ldots \quad (26)$$

so that they represent 6 mutually independent numerical values. These are the components of the electric force **E** and the magnetic force **H**. We have indeed

$$\left.\begin{array}{lll} \psi_{41} = \mathsf{E}_x, & \psi_{42} = \mathsf{E}_y, & \psi_{43} = \mathsf{E}_z, \\ \psi_{23} = \mathsf{H}_x, & \psi_{31} = \mathsf{H}_y, & \psi_{12} = \mathsf{H}_z, \end{array}\right\} \quad \ldots \quad (27)$$

and it is thus seen that the first three of the formulae (25) express the connection between the magnetic field and the electric current. The fourth shows how the electric field is connected with the charge.

On passing to another system of coordinates we have for w_a the transformation formula

$$w'_a = |p| \, \Sigma\,(b)\, \pi_{ba}\, w_b,$$

which can easily be deduced, while for ψ_{ab} we shall assume the formula

$$\psi'_{ab} = |p| \, \Sigma\,(cd)\, \tau_{ca}\, \pi_{db}\, \psi_{cd} \, . \quad \ldots \ldots \quad (28)$$

In virtue of this assumption the equations (25) are covariant for any change of coordinates.

§ 8. Beside ψ_{ab} we shall introduce certain other quantities $\overline{\psi}_{ab}$ which we define by

$$\overline{\psi}_{ab} = \frac{1}{\sqrt{-g}}\, \Sigma\,(cd)\, g_{ca}\, g_{db}\, \psi_{cd} \quad \ldots \ldots \quad (29)$$

or with regard to (26)

$$\overline{\psi}_{ab} = \frac{1}{\sqrt{-g}}\, \Sigma\,(\overline{cd})\, (g_{ca}\, g_{db} - g_{da}\, g_{cb})\, \psi_{cd}, \quad \ldots \ldots \quad (30)$$

in which last equation the bar over cd means that in the sum each combination of two numbers occurs only once.

As a consequence of this definition we have

$$\overline{\psi}_{aa} = 0, \; \overline{\psi}_{ba} = -\overline{\psi}_{ab}, \quad \ldots \ldots \quad (31)$$

and we find by inversion [2])

$$\psi_{ab} = \sqrt{-g}\, \Sigma\,(cd)\, \gamma_{ac}\, \gamma_{bd}\, \overline{\psi}_{cd} \quad \ldots \ldots \quad (32)$$

[1]) The quantities ψ_{ab} are connected with the components φ_{ab} of the tensor introduced by EINSTEIN by the equations $\psi_{ab} = \sqrt{-g} \, . \, \varphi_{ab}$.

[2]) By the definition of the quantities γ (§ 3) we have

$$\Sigma\,(a)\, g_{ab}\, \gamma_{ab} = 1 \quad \ldots \ldots \quad (\alpha)$$

and for $b \neq c$

$$\Sigma\,(a)\, g_{ab}\, \gamma_{ac} = 0, \text{ or } \Sigma\,(a)\, g_{ba}\, \gamma_{ca} = 0. \quad \ldots \ldots \quad (\beta)$$

Substituting for $\overline{\psi}_{cd}$ an expression similar to (29) with other letters as indices, To these equations we add the transformation formula for $\overline{\psi}_{ab}$, which may be derived from (28) [1])

$$\overline{\psi}'_{ab} = \Sigma\,(cd)\, p_{ca}\, p_{db}\, \overline{\psi}_{cd}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

§ 9. We shall now consider the 6 quantities (27) which we shall especially call "the quantities ψ" and the corresponding quantities $\overline{\psi}$, viz. $\overline{\psi}_{41} \ldots \overline{\psi}_{12}$.

According to (30) these latter are homogeneous and linear functions of the former and as (because of (5)) the coefficient of ψ_{cd} in $\overline{\psi}_{ab}$ is equal to the coefficient of ψ_{ab} in $\overline{\psi}_{cd}$, there exists a homogeneous quadratic function L of $\psi_{41}, \ldots \psi_{12}$, which, when differentiated with respect to these quantities, gives $\overline{\psi}_{41}, \ldots \overline{\psi}_{12}$. Therefore

$$\frac{\partial L}{\partial \psi_{ab}} = \overline{\psi}_{ab} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

and

$$L = \tfrac{1}{2}\, \Sigma(ab)\, \psi_{ab}\, \overline{\psi}_{ab} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

If, as in (34), we have to consider derivatives of L, this quantity will be regarded as a quadratic function of the quantities ψ.

The quantity L can now play the same part as the quantity that is represented by the same letter in §§ 4—6. Again $L\,dS$ is invariant when the coordinates are changed. [2])

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we have

$$\sqrt{-g}\, \Sigma\,(cd)\, \gamma_{ac}\, \gamma_{bd}\, \overline{\psi}_{cd} = \Sigma\,(cdef)\, \gamma_{ac}\, \gamma_{bd}\, g_{ec}\, g_{fd}\, \psi_{ef} = \Sigma\,(df)\, \gamma_{bd}\, g_{fd}\, \psi_{af} = \psi_{ab}.$$

The last two steps of this transformation, which rest on (α) and (β), will need no further explanation. In a similar way we may proceed (comp. the following notes) in many other cases, using also the relations $\Sigma\,(a)\, p_{ba}\, \pi_{ba} = 1$ and $\Sigma\,(a)\, p_{ba}\, \pi_{ca} = 0$ (the latter for $b \neq c$), which are similar to (α) and (β).

[1]) If we start from the equation for $\overline{\psi}'_{ab}$ that corresponds to (29) and if we use (7) and (28), attending to $\sqrt{-g'} = |p|\sqrt{-g}$, we find

$$\overline{\psi}'_{ab} = \frac{1}{\sqrt{-g'}}\, \Sigma\,(cd)\, g'_{ca}\, g'_{db}\, \psi'_{cd} =$$

$$= \frac{1}{\sqrt{-g}}\, \Sigma\,(c\,d\,e\,f\,h\,i\,j\,k)\, p_{ec}\, p_{fa}\, p_{hd}\, p_{ib}\, \pi_{jc}\, \pi_{kd}\, g_{ef}\, g_{hi}\, \psi_{jk}.$$

This may be transformed in two steps (comp. the preceeding note) to

$$\frac{1}{\sqrt{-g}}\, \Sigma\,(e\,f\,h\,i)\, p_{fa}\, p_{ib}\, g_{ef}\, g_{hi}\, \psi_{eh}.$$

In this way we may proceed further, after first expressing ψ_{eh} as a function of $\overline{\psi}_{lm}$ by means of (32).

[2]) Instead of (35) we may write $L = \tfrac{1}{4}\Sigma\,(ab)\, \psi_{ab}\, \overline{\psi}_{ab}$ and now we have according to (28) and (33)

§ 10. We shall define a varied motion of the electric charges by the quantities δx_a and we shall also vary the quantities ψ_{ab}, so far as can be done without violating the connections (25) and (26). The possible variations $\delta\psi_{ab}$ may be expressed in δx_a and four other infinitesimal quantities q_a which we shall presently introduce. Our condition will be that equation (12) shall be true if, leaving the gravitation field unchanged, we take for δx_a and q_a any continuous functions of the coordinates which vanish at the limits of the domain of integration. We shall understand by δw_a, $\delta\psi_{ab}$, δL the variations at a fixed point of this space. The variations δw_a are again determined by (13) and (14), and we have, in virtue of (26) and (25);

$$\delta\psi_{aa} = 0,\ \delta\psi_{ba} = -\delta\psi_{ab},\ \Sigma(b)\frac{\partial\delta\psi_{ab}}{\partial x_b} = \delta w_a = \Sigma(b)\frac{\partial\chi_{ab}}{\partial x_b}.$$

If therefore we put

$$\delta\psi_{ab} = \chi_{ab} + \vartheta_{ab}, \quad \ldots \ldots \ldots \quad (36)$$

we must have

$$\vartheta_{aa} = 0,\ \vartheta_{ba} = -\vartheta_{ab},\ \Sigma(b)\frac{\partial\vartheta_{ab}}{\partial x_b} = 0.$$

It can be shown that quantities ϑ_{ab} satisfying these conditions may be expressed in terms of four quantities q_a by means of the formulae

$$\vartheta_{ab} = \frac{\partial q_{b'}}{\partial x_a} - \frac{\partial q_{a'}}{\partial x_{b'}}\ (a \neq b). \quad \ldots \ldots \ldots \quad (37)$$

Here a' and b' are the numbers that remain when of 1, 2, 3, 4 we omit a and b, the choice of the value of a' and that of b' being such that the order a, b, a', b' can be derived from the order 1, 2, 3, 4 by an even number of permutations each of two numbers.

§ 11. By (34), (36) and (37) we have

$$\delta \mathrm{L} + \Sigma(a) K_a \delta x_a = \Sigma(\overline{ab})\,\overline{\psi}_{ab}\left(\frac{\partial q_{b'}}{\partial x_{a'}} - \frac{\partial q_{a'}}{\partial x_{b'}}\right) +$$
$$+ \Sigma(\overline{ab})\,\overline{\psi}_{ab}\,\chi_{ab} + \Sigma(a) K_a \delta x_a \quad \ldots \ldots \quad (38)$$

Here, in the transformation of the first term on the right hand side it is found convenient to introduce a new notation for the quantities $\overline{\psi}_{ab}$. We shall put

$$\overline{\psi}_{ab} = \psi^*_{a'b'},$$

$$\mathrm{L}' = \tfrac{1}{4}\Sigma(ab)\,\psi'_{ab}\,\overline{\psi}'_{ab} = \tfrac{1}{4}|p|\,\Sigma(abcdef)\,\pi_{ca}\,\pi_{db}\,p_{ea}\,p_{fb}\,\psi_{cd}\,\overline{\psi}_{ef} =$$
$$= \tfrac{1}{4}|p|\,\Sigma(cd)\,\psi_{cd}\,\overline{\psi}_{cd} = |p|\,\mathrm{L},$$

while

$$|p|\,dS' = dS.$$

a consequence of which is

$$\psi^*_{ba} = -\psi^*_{ab}$$

and we shall complete our definition by[1])

$$\psi^*_{aa} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

The term we are considering then becomes

$$\Sigma\,(\overline{ab})\,\psi^*_{a'b'}\left(\frac{\partial q_{b'}}{\partial x_{a'}} - \frac{\partial q_{a'}}{\partial x_{b'}}\right) = \Sigma\,(\overline{ab})\,\psi^*_{ab}\left(\frac{\partial q_b}{\partial x_a} - \frac{\partial q_a}{\partial x_b}\right) =$$

$$= \tfrac{1}{2}\,\Sigma\,(ab)\,\psi^*_{ab}\left(\frac{\partial q_b}{\partial x_a} - \frac{\partial q_a}{\partial x_b}\right) = -\,\Sigma\,(ab)\,\psi^*_{ab}\,\frac{\partial q_a}{\partial x_b} =$$

$$= -\,\Sigma\,(ab)\,\frac{\partial(\psi^*_{ab}\,q_a)}{\partial x_b} + \Sigma\,(ab)\,\frac{\partial \psi^*_{ab}}{\partial x_b}\,q_a,$$

so that, using (14), we obtain for (38)

$$\delta L + \Sigma\,(a)\,K_a\,\delta x_a = -\,\Sigma\,(ab)\,\frac{\partial(\psi^*_{ab} q_a)}{\partial x_b} + \Sigma\,(ab)\,\frac{\partial \psi^*_{ab}}{\partial x_b}\,q_a +$$

$$+\,\Sigma\,(ab)\,\overline{\psi_{ab}}\,w_b\,\delta x_a + \Sigma\,(a)\,K_a\,\delta x_a, \quad . \quad . \quad . \quad . \quad (40)$$

where we have taken into consideration that

$$\Sigma\,\overline{(ab)}\,\overline{\psi_{ab}}\,(w_b\,\delta x_a - w_a\,\delta x_b) = \Sigma\,(ab)\,\overline{\psi_{ab}}\,w_b\,\delta x_a.$$

If we multiply (40) by dS and integrate over the space S the first term on the right hand side vanishes. Therefore (12) requires that in the subsequent terms the coefficient of each q_a and of each δx_a be 0. Therefore

$$\Sigma\,(b)\,\frac{\partial \psi^*_{ab}}{\partial x_b} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

and

$$K_a = -\,\Sigma\,(b)\,\overline{\psi_{ab}}\,w_b, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

by which (40) becomes

$$\delta L + \Sigma\,(a)\,K_a\,\delta x_a = -\,\Sigma\,(ab)\,\frac{\partial(\psi^*_{ab} q_a)}{\partial x_b} \quad . \quad . \quad . \quad . \quad (43)$$

In (41) we have the second set of four electromagnetic equations, while (42) determines the forces exerted by the field on the charges. We see that (42) agrees with (19) (namely in virtue of (31)).

§ 12. To deduce also the equations for the momenta and the energy we proceed as in § 6. Leaving the gravitation field unchanged we shift the electromagnetic field, i. e. the values of w_a and ψ_{ab} in the direction of one of the coordinates, say, of x_c, over a distance defined by the constant variation δx_c so that we have

[1]) The quantities ψ^*_{ab} are connected with the quantities φ^*_{ab} introduced by Einstein by the equation $\psi^*_{ab} = \sqrt{-g}\,.\,\varphi^*_{ab}$.

$$\delta\psi_{ab} = -\frac{\partial\psi_{ab}}{\partial x_c}\,\delta x_c$$

From (36), (14) and (37) we can infer what values must then be given to the quantities q_a. We must put $q_c = 0$ and for $a \neq c$[1])

$$q_a = \psi_{a\,c'}\,\delta x_c.$$

For δL we must substitute the expression (cf. § 6)

$$\left\{ -\frac{\partial L}{\partial x_c} + \left(\frac{\partial L}{\partial x_c}\right)_\psi \right\} \delta x_c,$$

where the index ψ attached to the second derivative indicates that only the variability of the coefficients (depending on g_{ab}) in the quadratic function L must be taken into consideration. The equation for the component K_c which we finally find from (43) may be written in the form

$$K_c + \left(\frac{\partial L}{\partial x_c}\right)_\psi = -\Sigma(b)\,\frac{\partial T_{bc}}{\partial x_b}, \quad . \quad . \quad . \quad . \quad . \quad (44)$$

where

$$T_{cc} = -L + \sum_{a \neq c} (a)\,\psi^*_{ac}\,\psi_{a'c'} . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

and for $b \neq c$

$$T_{bc} = \sum_{a \neq c} (a)\,\psi^*_{ab}\,\psi_{a'c'} . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

Equations (44) correspond exactly to (24). The quantities T have the same meaning as in these latter formulae and the influence of gravitation is determined by $\left(\frac{\partial L}{\partial x_c}\right)_\psi$ in the same way as it was formerly by $\left(\frac{\partial L}{\partial x_c}\right)_w$.

We may remark here that the sum in (45) consists of three and that in (46) (on account of (39)) of two terms.

Referring to (35), we find f.i. from (45)

$$T_{11} = \tfrac{1}{2}\left(\psi_{43}\overline{\psi}_{43} + \psi_{42}\overline{\psi}_{42} - \psi_{41}\overline{\psi}_{41} + \psi_{23}\overline{\psi}_{23} - \psi_{31}\overline{\psi}_{31} - \psi_{12}\overline{\psi}_{12}\right),$$

while (46) gives

$$T_{12} = \psi_{31}\overline{\psi}_{23} - \psi_{41}\overline{\psi}_{42}.$$

The differential equations of the gravitation field.

§ 13. The equations which, for a given material or electromagnetic system, determine the gravitation field caused by it can also be derived from a variation principle. EINSTEIN has prepared the way

[1]) To understand this we must attend to equations (25).

for this in his last paper by introducing a quantity which he calls H and which is a function of the quantities g_{ab} and their derivatives, without further containing anything that is connected with the material or the electromagnetic system. All we have to do now is to add to the left hand side of equation (12) a term depending on that quantity H. We shall write for it the variation of

$$\frac{1}{\varkappa}\int Q dS,$$

where $\varkappa$ is a universal constant, while Q is what EINSTEIN calls $H\sqrt{-g}$, with the same or the opposite sign [1]). We shall now require that

$$\delta\int \mathrm{L} dS + \frac{1}{\varkappa}\,\delta\int Q dS + \int \Sigma(a) K_a \delta x_a \,.\, dS = 0\,, \quad . \quad . \quad . \quad (47)$$

not only for the variations considered above but also for variations of the gravitation field defined by δg_{ab}, if these too vanish at the limits of the field of integration.

To obtain now

$$\delta \mathrm{L} + \frac{1}{\varkappa}\,\delta Q + \Sigma(a) K_a \delta x_a$$

we have to add to the right hand side of (17) or (40), first the change of L caused by the variation of the quantities g, viz.

$$\Sigma(\overline{ab})\frac{\partial \mathrm{L}}{\partial g_{ab}}\,\delta g_{ab},$$

and secondly the change of Q multiplied by $\frac{1}{\varkappa}$. This latter change is

$$\Sigma(\overline{ab})\frac{\partial Q}{\partial g_{ab}}\,\delta g_{ab} + \Sigma(\overline{ab}\,e)\frac{\partial Q}{\partial g_{ab,e}}\,\delta g_{ab,e},$$

where $g_{ab,e}$ has been written for the derivative $\frac{\partial g_{ab}}{\partial x_e}$.

As

$$\delta g_{ab,e} = \frac{\partial \delta g_{ab}}{\partial x_e}$$

we may replace the last term by

$$\Sigma(\overline{ab}\,e)\frac{\partial}{\partial x_e}\left(\frac{\partial Q}{\partial g_{ab,e}}\,\delta g_{ab}\right) - \Sigma(\overline{ab}\,e)\frac{\partial}{\partial x_e}\left(\frac{\partial Q}{\partial g_{ab,e}}\right)\delta g_{ab}.$$

§ 14. As we have to proceed now in the same way in the case

1) I have not yet made out which sign must be taken to get a perfect conformity to EINSTEIN's formulae.

of a material and in that of an electromagnetic system we need consider only the latter. The conclusions drawn in § 11 evidently remain valid, so that we may start from the equation which we obtain by adding the new terms to (43). We therefore have

$$\delta \mathrm{L} + \frac{1}{\varkappa}\delta Q + \Sigma(a) K_a \delta x_a = -\Sigma(ab)\frac{\partial(\psi^*_{ab} q_a)}{\partial x_b} + \frac{1}{\varkappa}\Sigma(\overline{abe})\frac{\partial}{\partial x_e}\left(\frac{\partial Q}{\partial g_{ab,e}}\delta g_{ab}\right) +$$

$$+ \Sigma(\overline{ab})\left(\frac{\partial \mathrm{L}}{\partial g_{ab}} + \frac{1}{\varkappa}\frac{\partial Q}{\partial g_{ab}}\right)\delta g_{ab} - \frac{1}{\varkappa}\Sigma(\overline{abe})\frac{\partial}{\partial x_e}\left(\frac{\partial Q}{\partial g_{ab,e}}\right)\delta g_{ab} \quad . \quad . \quad . \quad (48)$$

When we integrate over S, the first two terms on the right hand side vanish. In the terms following them the coefficient of each δg_{ab} must be 0, so that we find

$$\frac{\partial Q}{\partial g_{ab}} - \Sigma(e)\frac{\partial}{\partial x_e}\left(\frac{\partial Q}{\partial g_{ab,e}}\right) = -\varkappa\frac{\partial \mathrm{L}}{\partial g_{ab}} \quad . \quad . \quad . \quad . \quad (49)$$

These are the differential equations we sought for. At the same time (48) becomes

$$\delta \mathrm{L} + \frac{1}{\varkappa}\delta Q + \Sigma(a) K_a \delta x_a = -\Sigma(ab)\frac{\partial(\psi^*_{ab} q_a)}{\partial x_b} + \frac{1}{\varkappa}\Sigma(\overline{abe})\frac{\partial}{\partial x_e}\left(\frac{\partial Q}{\partial g_{ab,e}}\delta g_{ab}\right) \quad (50)$$

§ 15. Finally we can derive from this the equations for the momenta and the energy of the gravitation field. For this purpose we impart a virtual displacement δx_c to this field only (comp. §§ 6 and 12). Thus we put $\delta x_a = 0$, $q_a = 0$ and

$$\delta g_{ab} = - g_{ab,c}\,\delta x_c.$$

Evidently

$$\delta Q = -\frac{\partial Q}{\partial x_c}\delta x_c$$

and (comp. § 12)

$$\delta \mathrm{L} = -\left(\frac{\partial \mathrm{L}}{\partial x_c}\right)_b \delta x_c.$$

After having substituted these values in equation (50) we can deduce from it the value of $\left(\frac{\partial \mathrm{L}}{\partial x_c}\right)_b$.

If we put

$$T^g_{cc} = -\frac{1}{\varkappa}Q - \frac{1}{\varkappa}\Sigma(\overline{ab})\frac{\partial Q}{\partial g_{ab,c}} g_{ab,c} \quad . \quad . \quad . \quad (51)$$

and for $e \neq c$

$$T^g_{ec} = -\frac{1}{\varkappa}\Sigma(\overline{ab})\frac{\partial Q}{\partial g_{ab,e}} g_{ab,c} \quad . \quad . \quad . \quad . \quad . \quad (52)$$

the result takes the following form

$$-\left(\frac{\partial L}{\partial x}\right)_b = -\Sigma(e)\frac{\partial T^g_{ec}}{\partial x_e} \quad \ldots \ldots \quad (53)$$

Remembering what has been said in § 12 about the meaning of $\left(\frac{\partial L}{\partial x_c}\right)_b$, we may now conclude that the quantities T^g_{ab} have the same meaning for the gravitation field as the quantities T_{ab} for the electromagnetic field (stresses, momenta etc.). The index g denotes that T^g_{ab} belongs to the gravitation field.

If we add to (53) the equations (44), after having replaced in them b by e, we obtain

$$K_c = -\Sigma(e)\frac{\partial T^t_{ec}}{\partial x_e}, \quad \ldots \ldots \quad (54)$$

where

$$T^t_{ec} = T_{ec} + T^g_{ec}.$$

The quantities T^t_{ec} represent the *total* stresses etc. existing in the system, and equations (54) show that in the absence of external forces the resulting momentum and the total energy will remain constant.

We could have found directly equations (54) by applying formula (50) to the case of a common virtual displacement δx_c imparted both to the electromagnetic system and to the gravitation field.

Finally the differential equations of the gravitation field and the formulae derived from them will be quite conform to those given by EINSTEIN, if in Q we substitute for H the function he has chosen.

§ 16. The equations that have been deduced here, though mostly of a different form, correspond to those of EINSTEIN. As to the covariancy, it exists in the case of equations (18), (24), (41), (42) and (44) for any change of coordinates. We can be sure of it because LdS is an invariant.

On the contrary the formulae (49), (53) and (54) have this property only when we confine ourselves to the systems of coordinates adapted to the gravitation field, which EINSTEIN has recently considered. For these the covariancy of the formulae in question is a consequence of the invariancy of $\delta\int HdS$ which EINSTEIN has proved by an ingenious mode of reasoning.